Fuzzy Stochastic Probability of the Solution of Single Stationary Non- homogeneous Linear Fuzzy Random Differential Equations

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Abstract

The aim of this paper is to find the fuzzy probability characteristics (the fuzzy correlation function and the fuzzy spectral density function) of the fuzzy solutions of given kind of non-homogenous fuzzy random linear differential equations with constant coefficients, which is a single having on it's right hand-side a stationary fuzzy random functions.

Key Words: Fuzzy stochastic probability, Fuzzy stationary random linear differential equations

1- Introduction

A fuzzy random differential equation is a fuzzy differential equation in which one or more of the terms is a random function, thus resulting in a solution is itself a random function, [1]. The concept of a fuzzy random function is not included in the framework of classical probability theory, and to study fuzzy random function, a new mathematical apparatus has to be created. The mathematical theory of fuzzy random function has been an area of considerable activity in recent years. The earliest work on fuzzy random differential equations was done to describe Brownian motion in our work. Fuzzy random differential equations arise in modelling a variety of fuzzy random dynamic phenomena in the physical, biological, engineering, probability and social sciences. The Numerical solution of fuzzy random differential equations and especially fuzzy random partial differential equations is a young field relatively speaking.

2- Fuzzy Stochastic Number

We begin this section by defining the notation we will use in the article, write A(x), a number in [0,1], for the membership function of A evaluates at x. An ξ- cut of A, written by A[ξ], defined as $\{t|\ A(t) \geq \xi\}$, for $0 \leq \xi \leq 1$ and A[ξ] is always closed and bounded. Represent an arbitrary fuzzy stochastic number by a pair of functions ($\underline{a}(\xi)$, $\bar{a}(\xi)$), [2], which satisfies the following arguments:

1. $\underline{a}(\xi)$ is a bounded below continuous non decreasing function over [0,1].
2. $\bar{a}(\xi)$ is a bounded above continuous non increasing function over [0,1].
3. $\underline{a}(\xi) \leq \bar{a}(\xi)$, $0 \leq \xi \leq 1$.

A crisp stochastic number ξ is simply represented by $\underline{a}(\xi) = \bar{a}(\xi) = \xi$, $0 \leq \xi \leq 1$. For arbitrary fuzzy stochastic numbers $e = (\underline{e}(\xi), \bar{e}(\xi))$, $u = (\underline{u}(\xi), \bar{u}(\xi))$ and real number k,[1].

1. $e = u$ if $\underline{e}(\xi) = \underline{u}(\xi)$ and $\bar{e}(\xi) = \bar{u}(\xi)$.
2. $e + u = (\underline{e}(\xi) + \underline{u}(\xi)\ ,\ \bar{e}(\xi) + \bar{u}(\xi))$.

3. $ke = \begin{cases} (k\underline{e}, k\bar{u}), & k \geq 0 \\ (k\bar{e}, k\underline{u}), & k < 0 \end{cases}$

Let F be the set of all upper semi continuous normal convex fuzzy stochastic numbers with bounded ξ- level sets, [1].

Definition 2.1: [8]

Let F: $(a,b) \to F''$ and $t_0 \in (a,b)$. We say that F is differentiable at t_0 if there are $F(t_0 + h) - F(t_0)$, $F(t_0) - F(t_0 - h)$, then we have right and left differentiable forms as follows:

1. There exist an element $F'(t_0) \in F''$ such that, for all $h > 0$ sufficiently near to 0, and the limits:

$$\lim_{h \to 0^+} \frac{F(t_0+h)-F(t_0)}{h} = \lim_{h \to 0^+} \frac{F(t_0)-F(t_0-h)}{h} = F'(t_0) \qquad \ldots(1)$$

2. There exist an element $F'(t_0) \in F''$ such that, for all $h < 0$ sufficiently near to 0, and the limits

$$\lim_{h \to 0^-} \frac{F(t_0+h)-F(t_0)}{h} = \lim_{h \to 0^-} \frac{F(t_0)-F(t_0-h)}{h} = F'(t_0) \qquad \ldots(2)$$

Theorem 2.1

Let $F: T \to F$ be a function and denote $[F(t)]^{\xi} = [f_{\xi}(t), g_{\xi}(t)]$, for each $\xi \in [0,1]$. Then:

1. If F is right differentiable, then f_{ξ} and g_{ξ} are differentiable functions and $[F'(t)]^{\xi} = [f'_{\xi}(t), g'_{\xi}(t)]$ …(3)

2. If F is left differentiable, then f_{ξ} and g_{ξ} are differentiable functions and $[F'(t)]^{\xi} = [f'_{\xi}(t), g'_{\xi}(t)]$, [2]. …(4)

Hint: All the fuzzy random functions denoted in this paper has the form remind in the above theorem.

3- Preliminaries

The mathematical theory of fuzzy probability with observations (experiments, trials), which can be repeated many times under similar conditions.

Definition (3.1): [2]

Let (Ω, P, ξ) be a fuzzy stochastic probability space and let T be an index set. **Fuzzy random function** is a real valued function $X(t,w,\xi)$ on $T\text{x}\Omega$ such that for each fixed t, $X(t,w,\xi)$ is a random variable. The function $X(t,w,\xi)$ can be denoted by $X(t,\xi)$ and a fuzzy random function can be considered as a collocation $\{X(t,\xi), t \in T, \xi \in [0,1]\}$ of a fuzzy random variables.

At the index set T contains only one element, then the fuzzy random function has a single random variable, so the distribution function of this fuzzy random function is

$$F^{\xi}{}_{X^{\xi}(t,\xi)}(x^{\xi}(t,\xi)) = P^{\xi}[X^{\xi}(t,\xi) \leq x^{\xi}(t,\xi)] \qquad \ldots(5)$$

Definition (3.2): [3]

A fuzzy random function $X^{\xi}(t,\xi)$ will be called **stationary** if all the finite dimensional joint distribution functions defining $X^{\xi}(t,\xi)$ remain the same if whole group of points $t_1, \ldots, t_n$ is shifted along the time axis i.e. if

$$F^{\xi}_{X^{\xi}(t_1+t,\xi),X^{\xi}(t_2+t,\xi)\cdots,X^{\xi}(t_n+t,\xi)}(x^{\xi}(t_1,\eta),\ldots,x^{\xi}(t_n,\xi))=F^{\xi}_{X^{\xi}(t_1,\xi),\cdots,X^{\xi}(t_n,\xi)}(x^{\xi}(t_1,\xi),\ldots,x^{\xi}(t_n,\xi)) \quad \ldots(6)$$

or

$$P^{\xi}\{X^{\xi}(t_1+t,\xi) \le x_1, \ldots, X^{\xi}(t_n+t,\xi)\le x_n\} = P^{\xi}\{X^{\xi}(t_1,\xi) \le x_1, \ldots, X^{\xi}(t_n,\xi) \le x_n\} \quad \ldots(7)$$

for any n, $t_1,\ldots, t_n$ and t. In particular this implies that for a stationary fuzzy random function, all the one-dimensional distribution functions have to be identical (i.e., $F^{\xi}(x^{\xi}(t,\xi))$ as given in equation (5) not depend on (t,ξ), all the two-dimensional joint distribution functions can only depend on the difference $(t_2 - t_1,\xi)$, and so on.

Definition (3.3):

A fuzzy random function $\{X^{\xi}(t,\xi),\ t \in T,\ \xi \in [0,1]\}$ is called **strictly stationary** if the whole family of its finite dimensional distribution is invariant under any translation in parameter h. Also, that is mean that for every finite sequence of time points $\{t_1, t_2\ldots,t_n,\xi\}$, the joint distribution function of n random variables $\{X^{\xi}(t_1+h,\xi),\ldots, X^{\xi}(t_n+h,\xi)\}$ should be independent of h. i.e.

$$F^{\xi}_{X^{\xi}(t_1,\xi),\cdots,X^{\xi}(t_n,\xi)}(x_{t_1},\ldots,x_{t_n},\xi) = F^{\xi}_{X^{\xi}(t_1+h+\xi),\cdots,X^{\xi}(t_n+h+\xi)}(x_{t_1},\ldots,x_{t_n},\xi)$$

for any n, $t_1,\ldots, t_n$ and h.

So if $\{X^{\xi}(t,\xi),\ t \in T,\ \xi \in [0,1]\}$ is strictly stationary fuzzy random functions, then

1. The distribution function of the fuzzy random variable is the same for every point in the index set T, [4].

2. The joint distribution function depends only on the distance between the elements in the index set T, [1].

3. If $E^{\xi}\{X^{\xi}(t,\xi)\} < \infty$, then $\text{var}^{\xi}[X^{\xi}(t,\xi)] < \infty$, [4].

4- Fuzzy Moments and Fuzzy Correlation Function

There are many problems in mathematical statistics in which it is difficult, or at least not feasible, to determine completely the fuzzy correlation function of a fuzzy random variable. In such cases it is often possible to describe the distribution of the fuzzy random variable in completely. Although usefully by moments and certain functions of fuzzy moments of the fuzzy random variable, [2].

Definition (4.1): [3]

Consider the finite dimensional distribution function of the function $X(t,\eta)$

$F^{\xi}((x_1, x_2,\ldots, x_n, t_1, t_2,\ldots, t_n),\xi)$

the **mixed moment** of order $(m_1,m_2,\ldots,m_n,\xi)$ is defined as

$$\mu^{\xi}_{m_1,\ldots,m_n}((t_1,\ldots,t_n),\xi) = E^{\xi}(X^{\xi m_1}(t_1,\xi)\ldots X^{\xi m_n}(t_n,\xi))$$
$$= \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} x_1^{\xi m_1}\ldots x_n^{\xi m_n}\partial^n F^{\xi}((x_1,\ldots,x_n;t_1,\ldots,t_n),\xi) \quad \ldots(8)$$

where $X^{\xi}(t_1,\xi), \ldots, X^{\xi}(t_n,\xi)$ represents the family of stationary fuzzy random functions.

When we have a one dimensional distribution functions, then the **fuzzy moments of this one dimensional distribution functions** defined as follow,

$$\mu_{m_1}^{\xi}(t_1,\xi) = \mathrm{E}^{\xi}(\mathrm{X}^{\xi m_1}(t_1,\xi)) = \int_{-\infty}^{\infty} x_1^{\xi m_1} d\,\mathrm{F}^{\xi}(x_1;t_1,\xi)$$

when we have the two dimensional distribution functions $\mathrm{F}^{\xi}(x_1,x_2;t_1,t_2,\xi)$, then the moments are

$$\mu_{m_1,m_2}^{\xi}(t_1,t_2,\xi) = \mathrm{E}^{\xi}(\mathrm{X}^{\xi m_1}(t_1,\xi)\cdot \mathrm{X}^{\xi m_2}(t_2,\xi)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^{\xi m_1} x_2^{\xi m_2} \partial^2\, \mathrm{F}^{\xi}((x_1,x_2;t_1,t_2),\xi)$$

and so on.

Remark (4.2): [3]

If the fuzzy random function $\mathrm{X}^{\xi}(t,\xi)$ is stationary, then the distribution function of $\mathrm{X}^{\xi}(t,\xi)$ does not depend on the index set T and because of that the fuzzy mean value (first moment) of the stationary fuzzy random function $\mathrm{X}^{\xi}(t,\xi)$ is a fuzzy constant number i.e.

$\mu_{m_1}^{\xi}(t,\xi) = \mu_1^{\xi}(t,\xi) = \mathrm{E}^{\xi}[\mathrm{X}^{\xi}(t,\xi)] = a$, a is fuzzy constant number.

Definition (4.3): [3]

The first fuzzy moment of the stationary fuzzy random function $\mathrm{X}^{\xi}(t,\xi)$ describe the coarsest properties of $\mathrm{X}^{\xi}(t,\xi)$, to know much more description of $\mathrm{X}^{\xi}(t,\xi)$ is given by its second moment, so

$$\mu_{m_1,m_2}^{\xi}(t,s,\xi) = \mathrm{E}^{\xi}[\mathrm{X}^{\xi m_1}(t,\xi)\,\mathrm{X}^{\xi m_2}(s,\xi)]$$
$$= \mathrm{B}^{\xi}(t,s,\xi) \qquad t,s \in \mathrm{T}, \xi \in [0,1]$$

the function $\mathrm{B}^{\xi}(t,s,\xi)$ is called the **fuzzy correlation function** of $\mathrm{X}^{\xi}(t,\xi)$ when $m_1 = m_2 = 1$ which is depending on the difference $t - s$, i.e.

$$\mu_{m_1,m_2}^{\xi}(t,s,\xi) = \mathrm{E}^{\xi}[\mathrm{X}^{\xi}(t,\xi)\mathrm{X}^{\xi}(s,\xi)] = \mathrm{B}^{\xi}(t-s,\xi) = \mathrm{B}^{\xi}(\tau,\xi)$$

The fuzzy correlation function of any stationary fuzzy random function can be represented in the form of an integral

$$\mathrm{B}^{\xi}(\tau,\xi) = \int_{-\infty}^{\infty} e^{i\lambda\tau} d\,\mathrm{F}^{\xi}(\lambda,\xi) \qquad \ldots(9)$$

where $\mathrm{F}^{\xi}(\lambda,\xi)$ is a real non-decreasing function continuous to the right, [5].

or, $\mathrm{B}^{\xi}(\tau,\xi) = \int_{-\infty}^{\infty} e^{i\lambda\tau} f^{\,\xi}(\lambda,\xi)\, d\lambda$, [5]. …(10)

formula (9) is the same of formula (10) and $\mathrm{F}^{\xi}(\lambda,\xi) = \int_{-\infty}^{\infty} f^{\,\xi}(\lambda,\xi)\, d\lambda$ and it is clear that $\mathrm{F}'^{\xi}(\lambda,\xi) = f^{\xi}(\lambda,\xi)$, where $f^{\xi}(\lambda,\xi)$ is the spectral density function of the fuzzy random function $\mathrm{X}^{\xi}(t,\xi)$, $f^{\xi}(\lambda,\xi) \geq 0$.

More generally, when $\mathrm{X}^{\xi}(t,\xi)$ is a complex strictly stationary fuzzy random function with $\mathrm{E}^{\xi}[|\mathrm{X}^{\xi}(t,\xi)|^2]$, $\mathrm{E}^{\xi}[\mathrm{X}^{\xi}(t,\xi)] = a$, a is fuzzy constant. i.e. $\mathrm{X}^{\xi}(t,\xi)$ have a finite fuzzy variance and constant fuzzy mean. Because the stationarity of $X^\xi(t,\xi)$, we consider that $E^\xi[X^\xi(t,\xi)] = 0$ and the fuzzy variance which gives more precise description of $X^\xi(t,\xi)$ can be defined for $t_1 \neq t_2$ as follows:

$$\mathrm{var}^\xi[X^\xi(t,\xi)] = E^\xi\{[X^\xi(t_1,\xi) - \overline{X}^\xi(t_1,\xi)]\ \overline{[X^\xi(t_2,\xi) - \overline{X}^\xi(t_2,\xi)]}\}$$

$$= E^\xi[X^\xi(t_1,\xi)X^\xi(t_2,\xi)] - 0 = B^\xi(t_1 - t_2,\xi) = B^\xi((t_1,t_2),\xi = B^\xi(\tau,\xi)$$

where $B^\xi(\tau,\xi) = B^\xi(t_1 - t_2,\xi)$ is the correlation function of $X^\xi(t,\xi)$, so by Schwartz inequality

$$|B^\xi(t_1 - t_2,\xi)|^2 \leq E^\xi|X^\xi(t_1,\xi) - X^\xi(t_2,\xi)|^2\ E^\xi|X^\xi(t_2,\xi) - \overline{X}^\xi(t_2,\xi)|^2$$

which means that the correlation function $B^\xi(t_1 - t_2,\xi)$ is finite for all t_1, t_2. Furthermore $B^\xi(t_1 - t_2,\xi)$ has the following two properties:

1. $B^\xi(t_1 - t_2,\xi) = \overline{B^\xi(t_2 - t_1,\xi)}$ or $B^\xi(t_1,t_2,\xi) = \overline{B^\xi(t_2,t_1,\xi)}$ …(11)

2. $B^\xi(0,\xi) = E^\xi[|X^\xi(t,\xi)|^2] \geq 0$ for any t

A fuzzy random function $\{X^\xi(t,\xi)\}$ is said to be **stationary in the wide sense** if the following conditions are hold, [6]:

1. $E^\xi[|X^\xi(t,\xi)|^2] < \infty$, for all $t \in T$, $\xi \in [0,1]$.

2. $E^\xi\{X^\xi(t,\xi)\} = a$, a is constant.

3. $E^\xi\{[X^\xi(t+\tau,\xi) - \overline{X}^\xi(t+\tau,\xi)]\ \overline{[X^\xi(t,\xi) - \overline{X}^\xi(t,\xi)]}\} = B^\xi(\tau,\xi)$

or

$$E^\xi[X^\xi(t+\tau,\xi)\ \overline{X^\xi(t,\xi)}\,] = B^\xi(\tau,\xi) \qquad \text{…(12)}$$

where $B^\xi(\tau,\xi)$ is the fuzzy correlation function of $X^\xi(t,\xi)$ and does not depend on t. Furthermore $B^\xi(\tau,\xi)$ has the following properties:

1. $B^\xi(0,\xi) = E^\xi[X^\xi(t+0,\xi)\ \overline{X^\xi(t,\xi)}\,]$

$= E^\xi[|X^\xi(t,\xi)|^2] > 0$

2. The fuzzy correlation function of $X^\xi(t,\xi)$ is an even function i.e. $B^\xi(\tau,\xi) = \overline{B^\xi(-\tau,\xi)}$

3. The fuzzy correlation function of the stationary random function is positive semidefinite.

Definition (4.3): [3], [7]

A fuzzy function $f^\xi(x,\xi)$ defined to be **semi definite** if it satisfies

$$\sum_{j=1}^{h}\sum_{k=1}^{n} a_j a_k f^\xi(t_j - t_k,\xi)$$

for any set of real numbers $a_1,a_2, \ldots, a_n$ and any $(t_1, t_2,\ldots, t_n)$ such that $t_j - t_k \in X$ for all $j, k = 1, 2,\ldots, n$.

5- Fuzzy Cross-Correlation Functions

For describing any system of two (or more) fuzzy random function, the fuzzy cross-correlation function of one fuzzy random function which describes completely behavior of the fuzzy random function.

Definition (5.1): [8]

Let $X^{\xi}(t,\xi)$, $Y^{\xi}(t,\xi)$ be two fuzzy random functions, then the **cross-correlation function** of $X^{\xi}(t,\xi)$, $Y^{\xi}(t,\xi)$ defined by the equality

$$B^{\xi}_{xy}(t_2,t_1,\xi) = E^{\xi}\{\overline{[X^{\xi}(t_1,\xi)-E^{\xi}[X^{\xi}(t_1,\xi)]}[Y^{\xi}(t_2,\xi)-E^{\xi}[Y^{\xi}(t_2,\xi)]\}.$$

5.1 Some Properties of the fuzzy Cross-Correlation Function:

1. $B^{\xi}_{xy}(t_1,t_2,\xi) = \overline{B^{\xi}_{xy}(t_2,t_1,\xi)}$

2. $\left|B^{\xi}_{xy}(t_1,t_2,\xi)\right| \le \sqrt{k^{\xi}_x(t_1,t_2,\xi)k^{\xi}_y(t_1,t_2,\xi)}$, where k_x, k_y are the variances of the random functions $X^{\xi}(t,\xi)$, $Y^{\xi}(t,\xi)$ respectively. which has the properties

1. $B^{\xi}_{xy}(\tau,\xi) = \overline{B^{\xi}_{yx}(-\tau,\xi)}$ complex

$B^{\xi}_{xy}(\tau,\xi) = B^{\xi}_{yx}(-\tau,\xi)$ real

2. $\left|B^{\xi}_{xy}(\tau,\xi)\right| \le \sqrt{B^{\xi}_x(0,\xi)B^{\xi}_y(0,\xi)}$ |.

5.2 Derivative of a Fuzzy Random Function [5]

A fuzzy random function $X^{\xi}(t,\xi)$ is differentiable or is mean square at a point t. If for any given sequence of numbers $h_1, \ldots, h_n, \ldots$ converges to zero, the sequence of random variables

$$\frac{X^{\xi}(t+h_j,\xi)-X^{\xi}(t,\xi)}{h_j} \qquad j=1,\ldots,n,\ldots$$

Converges in the mean to a unique random variable.

We call this unique random variable the **derivative of $X^{\xi}(t,\xi)$ at the point t**, and denoted by $X'^{\xi}(t,\xi)$.

The preceding definition is a general for all random functions $X^{\xi}(t,\xi)$ for a stationary random function $X^{\xi}(t,\xi)$. It is easy to show that this stationary random function is differentiable for any t, but under the condition that a correlation function $B^{\xi}(\tau)$ of $X^{\xi}(t,\xi)$ must has a continuous second derivative with respect to τ, and $X'^{\xi}(t,\xi)$ must be continuous stationary function with correlation function

$$B^{\xi}_{y}(\tau,\xi) = -B^{\xi}_{x''}(\tau,\xi) \qquad \ldots(13)$$

and (13) means that $B''^{\xi}(\tau,\xi)$ is a correlation function of $X'^{\xi}(t,\xi)$ and $B^{\xi}(\tau,\xi)$ always has a continuous second derivative, [2].

5.3 The fuzzy Spectral Density Function of a Derivatives Fuzzy Random Function, [5]:

Let $X^{\xi}(t,\xi)$ be a stationary random function, then

$$V^{\xi}(t,\xi) = X'^{\xi}(t,\xi) = \frac{d}{dt}X^{\xi}(t,\xi) \qquad \ldots(14)$$

here $V^{\xi}(t,\xi)$ is also a stationary random function and the correlation function of $V^{\xi}(t,\xi)$ can be determined by

$B^{\xi}(\tau,\xi) = -(B''^{\xi}(\tau,\xi))$ …(15)

Also, since

$$B_X^{\xi}(\tau,\xi) = \int_{-\infty}^{\infty} e^{i\omega\tau} f_X^{\xi}(\omega,\xi)\,d\omega$$

$$B_X'^{\xi}(\tau,\xi) = \int_{-\infty}^{\infty} i\,\omega\, e^{i\omega\tau} f_X^{\xi}(\omega,\xi)\,d\omega$$

$$B_X''^{\xi}(\tau,\xi) = -\int_{-\infty}^{\infty} e^{i\omega\tau} \omega^2 f_X^{\xi}(\omega,\xi)\,d\omega$$

By (13), since

$$B_v^{\xi}(\tau,\xi) = \int_{-\infty}^{\infty} e^{i\omega\tau} \omega^2 f_v^{\xi}(\omega,\xi)\,d\omega \qquad \text{…(16)}$$

But $V^{\xi}(t,\xi)$ is stationary

$$B_v^{\xi}(\tau,\xi) = \int_{-\infty}^{\infty} e^{i\omega\tau} f_v^{\xi}(\omega,\xi)\,d\omega \qquad \text{…(17)}$$

where $f_v^{\xi}(\omega,\xi)$ is a spectral density function of $V^{\xi}(t,\xi)$ and by comparing (16) and (17)

$$f_v^{\xi}(\omega,\xi) = \omega^2 f_X^{\xi}(\omega,\xi) \qquad \text{…(18)}$$

6- Spectral Representation of Fuzzy Random Function

Definition (6.1): [8]

Every stationary $X^{\xi}(t,\xi)$ can be assigned a function with orthogonal increments, such that for each fixed t that spectral representation

$$X^{\xi}(t,\xi) = \int_{-\infty}^{\infty} e^{i\lambda t} f^{\xi}(\lambda,\xi)\,d\lambda \qquad \text{…(19)}$$

6.1 The Spectral Density Function of Stationary Fuzzy Random Function: [3]

Given the following analytic expression of a fuzzy random function $\{X^{\xi}(t,\xi), -\infty < t < \infty\}$ by

$$X^{\xi}(t,\xi) = x\, f^{\xi}(t,\xi) \qquad \text{…(20)}$$

where $E^{\xi}(X^{\xi}(t,\xi)\} = 0$, x is a fuzzy random variable and $f(t,\xi)$ is a numerical function of t.

For the purpose that $X^{\xi}(t,\xi)$ is a stationary function, it must be that [3]

$f^{\xi}(t,\xi) = r\, e^{-i(\lambda t+\theta)}$

where $r, \lambda, \theta \in \mathbb{R}$, $\xi \in [0,1]$.

Thus, the fuzzy random function (20) is stationary if it has the form

$X^{\xi}(t,\xi) = x\, r\, e^{-i(\lambda t+\theta)} = x\, r\, e^{-i\theta}\, e^{-i\lambda t}$

So, by including the numerical factor $r\,e^{-i\theta}$ in the fuzzy random variable x, the fuzzy random function (20) is stationary iff it has the form

$X^{\xi}(t,\xi) = x\, e^{-i\lambda t}$

the correlation function of the stationary function $X^{\xi}(t,\xi)$ is

$$B_x^{\xi}(\tau,\xi) = E^{\xi}[x(t+\tau,\xi)\, \overline{X^{\xi}(t_1,\xi)}]\ ,\ \tau > 0,\ \xi \in [0,1]$$

$$= E^{\xi}[x\, e^{-i\lambda(t+\tau,\xi)}\, \overline{x\, e^{-i\lambda(t,\xi)}}\,] = E^{\xi}[\,|x\,\overline{x}\,|\, e^{-i\lambda\tau}\,]$$

$$= E^{\xi}[\,|x|^2\, e^{-i\lambda\tau}\,]$$

$$= E^{\xi}|x|^2\, e^{-i\lambda\tau}$$

So, $B_x^{\xi}(\tau,\xi) = \sigma^2 e^{-i\lambda\tau}$, $\tau > 0$, $\xi \in [0,1]$

or

$$B_x^{\xi}(\tau,\xi) = \sigma^2 e^{-i\lambda|\tau|},\ |\tau| < \infty,\ \xi \in [0,1] \qquad \ldots(21)$$

where $E^{\xi}[\,|x|^2] = \sigma^2$ is the mathematical expectation of the square of the amplitude.

The spectral density function is the Fourier transformation of $B^{\xi}(\tau)$. Hence, if the correlation function $B^{\xi}(\tau,\xi)$ is known, the spectral density function can be obtained by using the formula for the inversion of a Fourier transformation, [3].

$$\hat{B}^{\xi}(w,\xi) = f\,(w,\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iw\tau} B^{\xi}(\tau,\xi)\, d\tau \qquad \ldots(22)$$

6.2 The fuzzy stochastic Probability of the Solutions of a Single F.R.L.D.Eq with Stationary Fuzzy Random Function on the F.R.H.S:

Consider the following non-homogenous R.L.D.Eq with constant coefficients $a_1, a_2, \ldots, a_n$ where $a_1, a_2, \ldots, a_n$ is fuzzy numbers

$$\frac{d^{(n)} y^{\xi}(t,\xi)}{dt^n} + a_1 \frac{d^{(n-1)} y^{\xi}(t,\xi)}{dt^{n-1}} + \ldots + a_n y^{\xi}(t,\xi) = X^{\xi}(t,\xi) \qquad \ldots(23)$$

where $y^{\xi}(t,\xi)$ is fuzzy random function and $X^{\xi}(t,\xi)$ is a stationary fuzzy random function with given spectral density function

$$f_X^{\xi}(\lambda) = \frac{1}{\pi\lambda^2} \qquad ,\lambda > 0 \qquad \ldots(24)$$

It is known that, the complete solution $y^{\xi}(t,\xi)$ of equation (23) for initial condition $t = t_0$ is the sum of the complementary solution $y_c^{\xi}(t,\xi)$ of the corresponding homogenous equation and a particular integral $y_p^{\xi}(t,\xi)$ of the non-homogenous equation with zero initial condition $t = t_0$ [3], (i.e.)

$$y^{\xi}(t,\xi) = y_c^{\xi}(t,\xi) + y_p^{\xi}(t,\xi) \qquad \ldots(25)$$

To find $y_c^{\xi}(t,\xi)$, we consider the corresponding homogenous equation of (23) which is

$$\frac{d^{(n)}y^{\xi}(t,\xi)}{dt^{n}}+a_1\frac{d^{(n-1)}y^{\xi}(t,\xi)}{dt^{n-1}}+\ldots+a_n y^{\xi}(t,\xi)=0 \qquad \ldots(26)$$

or

$(D^{n} + a_1 D^{n-1} + \ldots + a_n)\, y^{\xi}(t,\xi) = 0$

or the auxiliary equation

$$m^{n}+a_1 m^{n-1}+\ldots+a_n=0 \qquad , y^{\xi}(t,\xi)\neq 0 \qquad \ldots(27)$$

So, we have n distinct values of roots which are m_j, $j = 1, 2, \ldots, n$ and according to these roots, we get

$$y_1^{\xi}(t,\xi)=c_1 e^{m_1 t}, y_2^{\xi}(t,\xi)=c_2 e^{m_2 t}, \ldots, y_n^{\xi}(t,\xi)=c_n e^{m_n t} \qquad \ldots(28)$$

and therefore, the complementary solution of (26) will be

$$y_c^{\xi}(t,\xi)=\sum_{j=1}^{n} c_j e^{m_j t} \qquad \ldots(29)$$

or

$$y_c^{\xi}(t,\xi)=\sum_{j=1}^{n} c_j (\cos m_j t+\sin m_j t) \qquad \ldots(30)$$

By considering that the given differential equation (23) is stable if all roots of equation (27) and having non negative real parts and for $t \to \infty$, the complementary solution $y_c^{\xi}(t,\eta)$ can be neglected. Therefore the solution of the given differential equation (23) can be considered only as a solution of the particular integral $y_p^{\xi}(t,\xi)$ which can be found by the formula (9),

$$y_p^{\xi}(t,\xi)=\int_{t_0}^{t} p^{\xi}(t-t_1,\xi)\,X^{\xi}(t_1,\xi)\,dt_1 \qquad \ldots(31)$$

To find completely $y_p^{\xi}(t,\xi)$, we first find the function $p^{\xi}(t-t_1,\xi)$ as follows:

By differentiating $y_p^{\xi}(t,\xi)$ with respect to t under the initial condition $\frac{dy_p^{\xi}(t,\xi)}{dt}=0$ for $t = t_0$, we obtain

$$\frac{dy_p^{\xi}(t,\xi)}{dt}=\int_{t_0}^{t}\frac{d}{dt}[p^{\xi}(t-t_1,\xi)X^{\xi}(t_1,\xi)]dt_1$$
$$=\int_{t_0}^{t}[p^{\xi}(t-t_1,\xi)X^{\xi}(t_1,\xi)\frac{dt}{dt_1}+X^{\xi}(t_1,\xi)\frac{d}{dt}p^{\xi}(t-t_1,\xi)]dt_1$$
$$=\int_{t_0}^{t}p^{\xi}(t-t_1,\xi)X^{\xi}(t_1,\xi)dt_1+\int_{t_0}^{t}X^{\xi}(t_1,\xi)\frac{d}{dt}p^{\xi}(t-t_1,\xi)dt_1$$

$$\frac{dy_p^{\xi}(t,\xi)}{dt} = p^{\xi}(t-t_1,\xi)\mathrm{X}^{\xi}(t_1,\xi)\frac{dt_1}{dt}\Big|_{t_0}^{t} + \int_{t_0}^{t}\mathrm{X}^{\xi}(t_1,)\xi\frac{d}{dt}p^{\xi}(t-t_1,\xi)dt_1$$

$$= p^{\xi}(t-t,\xi)\mathrm{X}^{\xi}(t,\xi)\frac{dt}{dt} - p^{\xi}(t-t_0,\xi)\mathrm{X}^{\xi}(t_0,\xi)\frac{dt_0}{dt} + \int_{t_0}^{t}\mathrm{X}^{\xi}(t_1,\xi)\frac{d}{dt}p^{\xi}(t-t_1,\xi)dt_1$$

or

$$\frac{dy_p^{\xi}(t,\xi)}{dt} = p^{\xi}(t-t,\xi)\mathrm{X}^{\xi}(t,\xi) + \int_{t_0}^{t}\mathrm{X}^{\xi}(t_1,\xi)\frac{d}{dt}p^{\xi}(t-t_1,\xi) \quad \ldots(32)$$

By hypothesis $\frac{dy_p^{\xi}(t,\xi)}{dt} = 0$ for $t = t_0$ and since the integral on the right hand-side of equation (32) vanish when $t = t_0$, moreover $p^{\xi}(t_0 - t_0,\xi)\mathrm{X}^{\xi}(t_0,\xi) = 0$. Also, since $\mathrm{X}^{\xi}(t,\xi)$ is an arbitrary random function for any $t = t_1 = 0$ and can not be equal to zero, the function $p^{\xi}(t - t_1,\xi) = 0$ …(33)

Hence, equation (32) can now be written as

$$\frac{dy_p^{\xi}(t,\xi)}{dt} = \int_{t_0}^{t}\mathrm{X}^{\xi}(t_1,\xi)\frac{d}{dt}p^{\xi}(t-t_1,\xi)dt_1 \quad \ldots(34)$$

By the same way, the successive derivatives of $y_p^{\xi}(t,\xi)$, $(n–1)$ times under the initial condition $\frac{d^{(j-1)}y_p^{\xi}(t,\xi)}{dt^{j-1}} = 0$ for $t = t_0$ (j = 3,4,…,n) give us

$$\frac{d^2y_p^{\xi}(t,\xi)}{dt^2} = \frac{d}{dt}p^{\xi}(t-t,\xi)\mathrm{X}^{\xi}(t,\xi) + \int_{t_0}^{t}\mathrm{X}^{\xi}(t_1,\xi)\frac{d^2}{dt^2}p^{\xi}(t-t_1,\xi)dt_1$$

$$\frac{d^3y_p^{\xi}(t,\xi)}{dt^3} = \frac{d^2}{dt^2}p^{\xi}(t-t,\xi)\mathrm{X}^{\xi}(t,\xi) + \int_{t_0}^{t}\mathrm{X}^{\xi}(t_1,\xi)\frac{d^3}{dt^3}p^{\xi}(t-t_1,\xi)dt_1$$

$$\vdots$$

$$\frac{d^{(n-1)}y_p^{\xi}(t,\xi)}{dt^{(n-1)}} = \frac{d^{(n-2)}}{dt^{(n-2)}}p^{\xi}(t-t,\xi)\mathrm{X}^{\xi}(t,\xi) + \int_{t_0}^{t}\mathrm{X}^{\xi}(t_1,\xi)\frac{d^{(n-1)}}{dt^{(n-1)}}p^{\xi}(t-t_1,\xi dt_1$$

…(35)

and by the same reasons of obtaining equations (33) and (34)

$$\frac{d}{dt}p^{\xi}(t-t,\xi) = \frac{d^2}{dt^2}p^{\xi}(t-t,\xi) = \ldots = \frac{d^{(n-2)}}{dt^{(n-2)}}p^{\xi}(t-t,\xi) = 0 \quad \ldots(36)$$

the system of equation (35), will be

$$\frac{d^2 y_p^{\xi}(t,\xi)}{dt^2} = \int_{t_0}^{t} X^{\xi}(t_1,\xi)\frac{d^2}{dt^2} p^{\xi}(t-t_1,\xi)dt_1$$

$$\frac{d^3 y_p^{\xi}(t,\xi)}{dt^3} = \int_{t_0}^{t} X^{\xi}(t_1,\xi)\frac{d^3}{dt^3} p^{\xi}(t-t_1,\xi)dt_1$$

$\vdots$

$$\frac{d^{(n-1)} X^{\xi}(t,\xi)}{dt^{(n-1)}} = \int_{t_0}^{t} X^{\xi}(t_1,\xi)\frac{d^{(n-1)}}{dt^{(n-1)}} p^{\xi}(t-t_1,\xi)dt_1$$

…(37)

Finally, by taking the *n*-th derivative of $y_p^{\xi}(t,\xi)$, or directly the derivative of $\frac{d^{(n-1)} y_p^{\xi}(t,\eta)}{dt^{n-1}}$, we get

$$\frac{d^{(n)} y_p^{\xi}(t,\xi)}{dt^n} = \int_{t_0}^{t} [X^{\xi}(t,\xi)\frac{d^{(n)}}{dt^n} p^{\xi}(t-t_1,\xi) + X^{\xi}(t_1,\xi)\frac{d^{(n-1)}}{dt^{n-1}} p^{\xi}(t-t_1,\xi)\frac{dt_1}{dt}]dt_1$$

$$= \int_{t_0}^{t} X^{\xi}(t_1,\xi)\frac{d^{(n)}}{dt^n} p^{\xi}(t-t_1,\xi)dt_1 + \int_{t_0}^{t} X^{\xi}(t_1,\xi)\frac{d^{(n-1)}}{dt^{n-1}} p^{\xi}(t-t_1,\xi)\frac{dt_1}{dt}dt_1$$

$$= \int_{t_0}^{t} X^{\xi}(t_1,\xi)\frac{d^{(n)}}{dt^n} p^{\xi}(t-t_1,\xi)dt_1 + X^{\xi}(t,\xi)\frac{d^{(n-1)}}{dt^{n-1}} p^{\xi}(t-t,\xi) - X^{\xi}(t_0,\xi)\frac{d^{(n-1)}}{dt^{n-1}} p^{\xi}(t-t_0,\xi)\frac{dt_0}{dt}$$

and by assuming that

$$\frac{d^{(n-1)}}{dt^{n-1}} p^{\xi}(t-t,\xi) = 1,\ \frac{d^{(n-1)}}{dt^{n-1}} p^{\xi}(t-t_0,\xi) = 0$$

the *n*-th derivative of $y_p^{\xi}(t,\xi)$ can be written as

$$\frac{d^{(n)} y_p^{\xi}(t,\xi)}{dt^n} = \int_{t_0}^{t} X^{\xi}(t_1,\xi)\frac{d^{(n)}}{dt^n} p^{\xi}(t-t_1,\xi)dt_1 + X^{\xi}(t,\xi) \quad \text{…(38)}$$

Now, by substituting the equations (31), (34), (37) and (38) into the given equation (23), we get

$$\int_{t_0}^{t} \left[\frac{d^{(n)}}{dt^n} + a_1\frac{d^{(n-1)}}{dt^{n-1}} + \ldots + a_n\right] X^{\xi}(t_1,\xi p^{\xi}(t-t_1,\xi) + X^{\xi}(t,\xi) = X^{\xi}(t,\xi)$$

$$\int_{t_0}^{t} \left[\frac{d^{(n)}}{dt^n} + a_1\frac{d^{(n-1)}}{dt^{n-1}} + \ldots + a_n\right] X^{\xi}(t_1,\xi) p^{\xi}(t-t_1,\xi) = 0$$

$$\int_{t_0}^{t}\left[\frac{d^{(n)}}{dt^{n}}+a_1\frac{d^{(n-1)}}{dt^{n-1}}+\ldots+a_n\right]p^{\xi}(t-t_1,\xi)=0\ ,\ \mathrm{X}^{\xi}(t_1,\xi)\neq 0 \qquad \ldots(39)$$

and by solving the differential equation (39) by the same way of solving the differential equation (26), we get

$$p^{\xi}(t-t_1,\xi)=\sum_{j=1}^{n}b_j e^{\lambda_j\,(t-t_1)} \qquad \ldots(40)$$

where b_j are constants and λ_j are the roots of the auxiliary equation (j = 1, 2,…, n)

$\lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0$

Hence, the particular integral $y_p^{\xi}(t,\xi)$ which represent the complete (general) solution of the given differential equation (23) is (see equation (31)).

$$y_p^{\xi}(t,\xi)=\int_{t_0}^{t}\mathrm{X}^{\xi}(t_1,\xi)\sum_{j=1}^{n}b_j e^{\lambda_j\,(t-t_1)}dt_1,\quad \lambda_j > 0,\quad j = 1,\ 2,\ldots,\ n.$$

…(41)

6.3 Fuzzy Correlation Function of the Solution or(of $y_p(t,\xi)$):

By putting $\tau = t - t_1$, for all $\tau > 0$ into equation (42), it becomes

$$y_p^{\xi}(t,\xi)=\int_{0}^{\infty}\mathrm{X}^{\xi}(t-\tau,\xi)\,p^{\xi}(\tau,\xi)d\tau \qquad \ldots(42)$$

and by solving this integral by parts, we get

$$y_p^{\xi}(t,\xi)=p^{\xi}(\tau,\xi)\int_{0}^{\infty}\mathrm{X}^{\xi}(t-\tau,\xi)d\tau-\int_{0}^{\infty}p'(\tau,\xi)\mathrm{X}^{\xi}(t-\tau,\xi)d\tau$$

but, by equation (33) $p^{\xi}(t - t_1,\xi) = p^{\xi}(\tau,\xi) = 0$, so

$$y_p^{\xi}(t,\xi)=c\int_{0}^{\infty}p'^{\xi}(\tau,\xi)\mathrm{X}^{\xi}(t-\tau,\xi)d\tau\ \ ,c=-1 \qquad \ldots(43)$$

Now, multiply $y_p^{\xi}(t,\xi)$ by $y_p^{\xi}(t + h,\xi)$, $h > 0$ and by [7] the fuzzy correlation function of this product can be found as follows:

$$\mathrm{B}^{\xi}_{y_p}(h,\xi)=\mathrm{E}^{\xi}\left[\overline{y_p^{\xi}(t,\xi)}\;y_p^{\xi}(t+h,\varsigma)\right]$$
$$=\mathrm{E}^{\xi}\left\{\left[\overline{c\int_{0}^{\infty}p'^{\xi}(\tau_1,\xi)\mathrm{X}^{\xi}(t-\tau_1,\xi)d\tau_1}\right]\left[c\int_{0}^{\infty}p'^{\xi}(\tau_2,\xi)\mathrm{X}^{\xi}(t+h-\tau_2,\xi)d\tau_2\right]\right\}$$

$$\mathrm{B}^{\xi}_{y_p}(h,\xi)=\int_0^{\infty}\int_0^{\infty}\left[\overline{p'^{\xi}(\tau_1,\xi)}p'^{\xi}(\tau_2,\xi)\right]\mathrm{E}^{\xi}\left[\overline{\mathrm{X}^{\xi}(t-\tau_1,\xi)}\mathrm{X}^{\xi}(t+h-\tau_2,\xi)\right]\overline{d\tau_1}d\tau_2,\bar{c}c=1 \qquad \ldots(44)$$

then, by (9),

$$\mathrm{E}^{\xi}\left[\overline{\mathrm{X}^{\xi}(t-\tau_1,\xi)}\mathrm{X}^{\xi}(t+h-\tau_2,\xi)\right]=\mathrm{B}^{\xi}_{\mathrm{X}}(h-\tau_2+\tau_1,\xi)=\int_{-\infty}^{\infty}e^{i\lambda(h-\tau_2+\tau_1)}f^{\xi}_{\mathrm{X}}(\lambda)d\lambda$$

where $f^{\xi}_{\mathrm{X}}(\lambda)$ is the spectral density function of the random function $\mathrm{X}^{\xi}(t,\xi)$.

So, the equation (44) can now be written after interchanging the order of integration as follows:

$$\mathrm{B}^{\xi}_{y_p}(h,\xi)=\int_{-\infty}^{\infty}e^{i\lambda h}\left[\int_0^{\infty}\overline{p'^{\xi}(\tau_1,\xi)}\,e^{i\lambda\tau_1}d\overline{\tau_1}\right]\times\left[\int_0^{\infty}p'^{\xi}(\tau_2,\xi)\,e^{-i\lambda\tau_2}d\tau_2\right]f^{\xi}_{\mathrm{X}}(\lambda)d\lambda \qquad \ldots(45)$$

and by using the integration by parts method to solve the two integrals inside curly bracket regarding that $p^{\xi}(\tau,\xi)=p^{\xi}(t-t_1,\xi)=0$.

$$\mathrm{B}^{\xi}_{y_p}(h,\xi)=\int_{-\infty}^{\infty}e^{i\lambda h}\left[(-i\lambda)\int_0^{\infty}\overline{p^{\xi}(\tau_1,\xi)}\,e^{i\lambda\tau_1}d\overline{\tau_1}\right]\times \left[(i\lambda)\int_0^{\infty}p^{\xi}(\tau_2,\xi)e^{-i\lambda\tau_2}d\tau_2\right]f^{\xi}_{\mathrm{X}}(\lambda)d\lambda$$

$$=\int_{-\infty}^{\infty}\lambda^2e^{i\lambda h}\left[\int_0^{\infty}\overline{p^{\xi}(\tau_1,\xi)}e^{i\lambda\tau_1}d\overline{\tau_1}\right]\left[\int_0^{\infty}p^{\xi}(\tau_2,\xi)e^{-i\lambda\tau_2}d\tau_2\right]f^{\xi}_{\mathrm{X}}(\lambda)d\lambda$$

and from $\tau_1=\tau_2=\tau$ and by substituting equation (24)

$$\mathrm{B}^{\xi}_{y_p}(h,\xi)=\frac{1}{\pi}\int_{-\infty}^{\infty}e^{i\lambda h}\left[\int_0^{\infty}\overline{p^{\xi}(\tau,\xi)}\,e^{i\lambda\tau}d\bar{\tau}\right]\left[\int_0^{\infty}p^{\xi}(\tau,\xi)\,e^{-i\lambda\tau}d\tau\right]d\lambda$$

or

$$\mathrm{B}^{\xi}_{y_p}(h,\xi)=\frac{1}{\pi}\int_{-\infty}^{\infty}e^{i\lambda h}\left|\int_0^{\infty}p^{\xi}(\tau,\xi)e^{i\lambda\tau}d\tau\right|^2d\lambda$$

where $p^{\xi}(\tau,\eta)$ is defined by equation (40) such that $\tau=t-t_1$,

$$\mathrm{B}^{\xi}_{y_p}(h,\xi)=\frac{1}{\pi}\int_{-\infty}^{\infty}e^{i\lambda h}\left|\int_0^{\infty}\sum_{j=1}^{n}b_j\,e^{\lambda_j\tau}e^{i\lambda\tau}d\tau\right|^2d\lambda$$

or

$$\mathrm{B}_{y_p}^{\xi}(h,\xi)=\frac{1}{\pi}\int_{-\infty}^{\infty}e^{i\lambda h}\left|\sum_{j=1}^{n}b_j\int_{0}^{\infty}e^{(\lambda_j+i\lambda)\tau}\,d\tau\right|^2 d\lambda$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}e^{i\lambda h}\left|\sum_{j=1}^{n}b_j\left(\frac{1}{\lambda_j+i\lambda}\right)\int_{0}^{\infty}e^{(\lambda_j+i\lambda)\tau}\,(\lambda_j+i\lambda)d\tau\right|^2 d\lambda$$

but since $\tau = t - t_1$ and when $t \to t_1$, then $(\lambda_j + i\lambda)\tau \to 0$, so that

$$\mathrm{B}_{y_p}^{\xi}(h,\xi)=\frac{1}{\pi}\int_{-\infty}^{\infty}e^{i\lambda h}\left|\sum_{j=1}^{n}\frac{b_j}{\lambda_j+i\lambda}\right|^2 d\lambda$$

$$\mathrm{B}_{y_p}^{\xi}(h,\xi)=\frac{1}{\pi}\int_{-\infty}^{\infty}e^{i\lambda h}\sum_{j=1}^{n}\frac{b_j^2}{\left|\lambda_j+i\lambda\right|^2}d\lambda$$

$$=\frac{1}{\pi}\sum_{j=1}^{n}b_j^2\int_{-\infty}^{\infty}\frac{e^{i\lambda h}}{\lambda_j^2+\lambda^2}$$

$$\mathrm{B}_{y_p}^{\xi}(h,\xi)=\frac{1}{\pi}\sum_{j=1}^{n}b_j^2\int_{-\infty}^{\infty}\frac{e^{i\lambda h}}{(\lambda-i\lambda_j)(\lambda+i\lambda_j)}d\lambda \qquad \ldots(46)$$

For finding completely, the fuzzy correlation function of $y_p^{\xi}(t,\xi)$, we let z = λ ⇒ dz = dλ and use the Cauchy formula (6) to evaluate the integral of equation (46), so,

$$\mathrm{B}_{y_p}^{\xi}(h,\xi)=\sum_{j=1}^{n}b_j^2\int_{-\infty}^{\infty}\frac{e^{izh}}{(z-i\lambda_j)(z+i\lambda_j)}dz$$

let $f(z,\xi)=\dfrac{e^{izh}}{(z-i\lambda_j)(z+i\lambda_j)}$, $j=1,2,\ldots,n$

which is analytic inside on a simple closed contour C except for isolated singular points (simple poles) at $z = \pm i\lambda_j$, and the residues of $f(z)$ at these two simple poles are respectively

$$R_1=\lim_{z\to -i\lambda_j}(z+i\lambda_j)f(z)=-\frac{e^{-\lambda_j h}}{2i\lambda_j}$$

$$R_2=\lim_{z\to i\lambda_j}(z-i\lambda_j)f(z)=\frac{e^{\lambda_j h}}{2i\lambda_j}$$

So that by Cauchy formula (6)

$$\int_{-\infty}^{\infty}f(z,\xi)dz=\int_{-\infty}^{\infty}\frac{e^{izh}}{(z-i\lambda_j)(z+i\lambda_j)}\;dz$$

$$=2\pi i\,(R_1+R_2)$$

$$\int_{-\infty}^{\infty} f(z,\xi)dz = \pi(\frac{e^{-\lambda_j h} - e^{\lambda_j h}}{\lambda_j}) \quad \ldots(47)$$

Therefore, by substituting equation (47) into equation (46), we get the fuzzy correlation function of the solution of the given differential equation (23), which is

$$\mathbf{B}_{y_p}^{\xi}(h,\xi) = \sum_{j=1}^{n} b_j^2 (\frac{e^{-\lambda_j h} - e^{\lambda_j h}}{\lambda_j}), \lambda_j > 0.$$

6.4 Fuzzy Spectral Density Function of the Solution or (of y_p^{ξ} (t,ξ)):

By the formula (9), equation (46) can be written as follows:

$$\int_{-\infty}^{\infty} e^{i\lambda h} f_{y_p}^{\xi}(\lambda) d\lambda = \sum_{j=1}^{n} b_j^2 \int_{-\infty}^{\infty} \frac{e^{i\lambda h}}{(\lambda - i\lambda_j)(\lambda + i\lambda_j)} d\lambda$$

$$f_{y_p}^{\xi}(\lambda) = \sum_{j=1}^{n} b_j^2 \left[\frac{1}{(\lambda - i\lambda_j)(\lambda + i\lambda_j)} \right]$$

$$f_{y_p}^{\xi}(\lambda) = \sum_{j=1}^{n} \frac{b_j^2}{\lambda^2 + \lambda_j^2}, \lambda > 0 \quad \ldots(48)$$

which represent the fuzzy spectral density function of the solution of the given differential equation (23).

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