Results in Control and Optimization 4 (2021) 100046

Contents lists available at ScienceDirect

Results in Control and Optimization

journal homepage: www.elsevier.com/locate/rico

E-B-invexity in E-differentiable mathematical programming

Najeeb Abdulaleem *

Department of Mathematics, Hadhramout University, P.O. BOX : (50511-50512), Al-Mahrah, Yemen
Faculty of Mathematics and Computer Science, University of Łódź, Banacha 22, 90–238 Łódź Poland

ARTICLE INFO

MSC:
90C26
90C30
26B25

Keywords:
E-differentiable function
E-B-invex functions
E-optimality conditions

ABSTRACT

In this paper, a new concept of generalized convexity is introduced for (not necessarily) differentiable optimization problem with E-differentiable functions. Namely, for an E-differentiable function, the concept of E-B-invexity is defined. The E-differentiable E-B-invexity notion unify the concepts of convexity, invexity, E-convexity, E-invexity and B-invexity. Further, the sufficiency of the so-called E-Karush-Kuhn–Tucker optimality conditions are established for the considered E-differentiable optimization problem with both inequality and equality constraints under E-B-invexity hypotheses. Moreover, the example of a nonsmooth programming problem with E-differentiable functions is constructed to illustrate the aforesaid results.

1. Introduction

One of the most widely applied hypotheses in optimization theory is convexity. The convex functions appear in many pastures of economics, management science, engineering, finance, etc. Several definitions extending the concept of convexity of a function have been introduced, with the purpose of weakening the assumptions to establish some results concerning the sufficiency of Karush–Kuhn–Tucker conditions of a mathematical programming problem. A significant generalization of convex functions is invex functions, introduced by Hanson [1]. Over the years, many generalizations of this concept have been introduced in the literature (see, for example, [2–15], and others). Bector and Singh [16] introduced the B-vex functions as generalization of convex functions, namely B-vex functions satisfy many of the basic properties of convex functions. Bector et al. [6] defined the concept of B-invexity. Under the assumption of B-invexity imposed on the functions involving a mathematical programming problem, both the sufficiency of the Karush–Kuhn–Tucker necessary optimality conditions and Wolfe duality results hold.

Youness [15] discussed the generalize convex sets and convex functions which placed on a one-to-one and onto operator $E : R^n \to R^n$ in the same region in which functions defined and known as the E-convex sets and E-convex functions. Moreover, the results established by Youness [15] were improved by Yang [17]. The beginning outcomes of Youness [15] inspired the great study of succeeding work, which substantially developed the part of E-convexity in optimization theory (see, for example, [11,18–30], and others). Megahed et al. [29] presented the concept of an E-differentiable convex function which transforms a (not necessarily) differentiable convex function to a differentiable function also based on the effect of an operator $E : R^n \to R^n$. Recently, Abdulaleem [5] introduced a new concept of generalized convexity as a generalization of the notion of E-differentiable E-convexity. Namely, Abdulaleem defined the concept of E-differentiable E-invexity in the case of (not necessarily) differentiable vector optimization problems with E-differentiable functions.

In this paper, firstly, we characterize the class of E-differentiable E-B-invex functions. Namely, we show that every E-stationary point of any E-differentiable E-B-invex function is its global E-minimizer. Further, the concept of a so-called E-differentiable E-B-invex function for E-differentiable optimization problems is introduced. Moreover, the sufficient optimality conditions are

* Correspondence to: Department of Mathematics, Hadhramout University, P.O. BOX : (50511-50512), Al-Mahrah, Yemen.
E-mail address: nabbas985@gmail.com.

https://doi.org/10.1016/j.rico.2021.100046
Received 29 March 2021; Received in revised form 3 June 2021; Accepted 9 August 2021
Available online 14 August 2021

established for the considered E-differentiable optimization problem under E-B-invexity. This result is illustrated by the example of a nonconvex E-differentiable optimization problem in which the involved functions are E-B-invex.

2. Definition of the class of E-B-invex functions and preliminaries

We now give the definition of an E-differentiable function introduced by Megahed et al. [29].

Definition 1 (*[29]*)**.** Let $E : R^n \to R^n$ and $f : R^n \to R$ be a (not necessarily) differentiable function at a given point $u \in R^n$. It is said that f is an E-differentiable function at u if and only if $f \circ E$ is a differentiable function at u (in the usual sense), that is,

$$(f \circ E)(x) = (f \circ E)(u) + \nabla (f \circ E)(u)(x-u) + \theta(u, x-u) \|x-u\|, \tag{1}$$

where $\theta(u, x-u) \to 0$ as $x \to u$.

Now, we define a new concept of generalized convexity for an E-differentiable function.

Definition 2. Let $E : R^n \to R^n$ and $f : R^n \to R$ be an E-differentiable function on R^n. It is said that f is E-B-invex with respect to η and b at $u \in R^n$ on R^n if, there exist functions $\eta : R^n \times R^n \to R^n$ and $b : R^n \times R^n \to R_+$ such that, for all $x \in R^n$,

$$b(E(x), E(u)) [f(E(x)) - f(E(u))] \geq \nabla (f \circ E)(u) \eta(E(x), E(u)). \tag{2}$$

If inequality (2) holds for any $u \in R^n$, then f is E-B-invex with respect to η and b on R^n.

Remark 3. Note that Definition 2 generalizes and extends several generalized convexity notions, previously introduced in the literature. Indeed, there are the following special cases:

(a) If $b(x, u) = 1$, then the definition of an E-B-invex function reduces to the definition of an E-invex function introduced by Abdulaleem [5].
(b) If $E(x) \equiv x$ (E is an identity map), then the definition of an E-B-invex function reduces to the definition of a B-invex function introduced by Bector et al. [6].
(c) If $E(x) \equiv x$ (E is an identity map) and $b(x, u) = 1$, then the definition of an E-B-invex function reduces to the definition of an invex function introduced by Hanson [1].
(d) If $\eta : R^n \times R^n \to R^n$ is defined by $\eta(x, u) = x - u$ and $b(x, u) = 1$, then the definition of an E-B-invex function reduces to the definition of an E-differentiable E-convex function introduced by Megahed et al. [29].
(e) If f is differentiable, $E(x) = x$ (E is an identity map), $\eta : R^n \times R^n \to R^n$ is defined by $\eta(x, u) = x - u$ and $b(x, u) = 1$, then the definition of an E-B-invex function reduces to the definition of a differentiable convex function.
(f) If f is differentiable, $\eta : R^n \times R^n \to R^n$ is defined by $\eta(x, u) = x - u$ and $b(x, u) = 1$, then we obtain the definition of a differentiable E-convex function introduced by Youness [15].

Definition 4. Let $E : R^n \to R^n$ and $f : R^n \to R$ be an E-differentiable function on R^n. It is said that f is strictly E-B-invex with respect to η and b at $u \in R^n$ on R^n if, there exist functions $\eta : R^n \times R^n \to R^n$ and $b : R^n \times R^n \to R_+$ such that, for all $x \in R^n$,

$$b(E(x), E(u)) [f(E(x)) - f(E(u))] > \nabla (f \circ E)(u) \eta(E(x), E(u)). \tag{3}$$

If inequality (3) holds for any $u \in R^n$ ($E(x) \neq E(u)$), then f is strictly E-B-invex with respect to η and b on R^n.

Now, we present an example of a nondifferentiable E-B-invex function, which is not a B-invex function or an E-invex function.

Example 5. Let $E : R \to R$. A function $f : R \to R$ defined by $f(x) = e^{\sqrt[3]{x}}$ is E-B-invex with respect to $E(x) = x^3$, η and b defined by

$$b(E(x), E(u)) = \begin{cases} 1 & \text{if } x \geq u, \\ \frac{e^x}{e^u - e^x} & \text{if } x < u. \end{cases}$$

$$\eta(E(x), E(u)) = \begin{cases} \frac{e^x - e^u}{e^u} & \text{if } x \geq u, \\ -1 & \text{if } x < u. \end{cases}$$

But f is not E-invex with respect to η defined above as can be seen by taking $x = 1$, $u = 2$, since the inequalities

$$f(E(x)) - f(E(u)) < \nabla(f \circ E)(u)\eta(E(x), E(u))$$

hold. Hence, by the definition of an E-invex function [5], it follows that f is not E-invex with respect to η given above. Also, f is not a B-invex function with respect to η and b defined above (see [6]).

Now, we present an example of a nondifferentiable E-B-invex function, which is not an invex function or an E-invex function.

Example 6. A function $f : M \subseteq (0, \frac{\pi}{2}) \rightarrow R$ defined by $f(x) = \sqrt[3]{x} + \sin(\sqrt[3]{x})$ is E-B-invex with respect to $E : M \rightarrow M$, $\eta : M \times M \rightarrow R$ and $b : M \times M \rightarrow R_+$ defined by $E(x) = x^3$, $\eta(x,u) = \frac{2\sin(\sqrt[3]{x}) - \sin(\sqrt[3]{u})}{\cos(\sqrt[3]{u})}$ and $b(x,u) = 2$, respectively. But f is not invex with respect to η defined above, since the inequality

$$f(x) - f(u) < \nabla f(u)\eta(x,u)$$

holds. Hence, by the definition of an invex function [1], it follows that f is not invex with respect to η given above. Also, f is not E-invex with respect to η defined above, since the inequality

$$f(E(x)) - f(E(u)) < \nabla(f \circ E)(u)\eta(E(x), E(u))$$

holds. Hence, by the definition of an E-invex function [5], it follows that f is not E-invex with respect to η given above. Further, f is not B-invex function with respect to η and b defined above (see [6]).

Now we introduce necessary and sufficient conditions for an E-B-invex function.

Theorem 7. *Let $E : R^n \rightarrow R^n$. $f : R^n \rightarrow R$ is an E-differentiable E-B-invex function with respect to η and b on R^n, where $\eta : R^n \times R^n \rightarrow R^n$, $b : R^n \times R^n \rightarrow R_+$ and if and only if its every E-stationary point is a global E-minimum of f.*

Proof. "⇒" Let $E : R^n \rightarrow R^n$. Clearly, if f is an E-differentiable E-B-invex function with respect to η and b on R^n satisfying

$$b(E(x), E(u)) \neq 0 \quad if \quad x \neq u,$$

$$b(E(x), E(u)) = 0 \quad if \quad x = u, \tag{4}$$

and $E(\overline{x})$ its E-stationary point, then $\nabla f(E(\overline{x})) = 0$ implies $f(E(\overline{x})) \leq f(E(x))$, $\forall x \in R^n$.
"⇐"
If $\nabla f(E(\overline{x})) = 0$, take $\eta(E(x), E(\overline{x})) = 0$ and (4).
If $\nabla f(E(\overline{x})) \neq 0$, take

$$\eta(E(x), E(\overline{x})) = \frac{f(E(x)) - f(E(\overline{x}))}{\nabla f(E(\overline{x}))\nabla f(E(\overline{x}))}\nabla f(E(\overline{x}))$$

and b is any nonnegative real-valued function defined on $R^n \times R^n$ satisfying (4). ■

Example 8. Let $E : R \rightarrow R$, $f : R \rightarrow R$ be an E-differentiable function on R defined by

$$f(x) = \sqrt[3]{x^2}, \; E(x) = x^{\frac{9}{2}}.$$

Note that f is not an E-differentiable E-B-invex function. Since the differentiable function $f(E(x)) = x^3$ has an E-stationary point at $E(x) = 0$, but it is not a global E-minimum.

3. Optimality conditions for E-differentiable optimization problem

In the paper, we consider the following (not necessarily differentiable) optimization problem with both inequality and equality constraints:

$$\begin{aligned} & f(x) \rightarrow \min \\ \text{subject to } & g_i(x) \leq 0, \; i \in I = \{1, \dots, k\}, \\ & h_j(x) = 0, \; j \in J = \{1, \dots, s\}, \end{aligned} \tag{P}$$

where $f : R^n \rightarrow R$ and $g_i : R^n \rightarrow R$, $i \in I$, $h_j : R^n \rightarrow R$, $j \in J$, are E-differentiable functions on R^n. We will write $g := (g_1, \dots, g_k) : R^n \rightarrow R^k$ and $h := (h_1, \dots, h_s) : R^n \rightarrow R^s$ for convenience. Let

$$\Omega := \left\{x \in R^n : g_i(x) \leq 0, \; i \in I, \; h_j(x) = 0, \; j \in J\right\}$$

be the set of all feasible solutions of (P). Further, $I(\overline{x})$ is the set of all active inequality constraints at point $\overline{x} \in D$, i.e. $I(\overline{x}) = \left\{i \in I : g_i(\overline{x}) = 0\right\}$.

Definition 9. A point $\overline{x}$ is said to be an E-optimal solution of (P) if there exists no x such that

$$f(x) < f(\overline{x}).$$

Now, for the considered optimization problem (P), we define its associated differentiable E-optimization problem (P_E) as follows:

$$\begin{aligned} & f(E(x)) \rightarrow \min \\ \text{subject to } & g_i(E(x)) \leq 0, \; i \in I = \{1, \dots, k\}, \\ & h_j(E(x)) = 0, \; j \in J = \{1, \dots, s\}. \end{aligned} \tag{P_E}$$

We call the problem (P_E) an E-optimization problem. Let

$$\Omega_E := \{x \in R^n : g_i(E(x)) \le 0,\ i \in I,\ h_j(E(x)) = 0,\ j \in J\}$$

be the set of all feasible solutions of (P_E).

Lemma 10 (*[18]*). *Let $E : R^n \to R^n$ be a one-to-one and onto operator. Then $E\left(\Omega_E\right) = \Omega$.*

Definition 11. A point $E(\overline{x}) \in \Omega$ is said to be an E-optimal solution of (P) if there exists no $E(x) \in \Omega$ such that

$$f(E(x)) < f(E(\overline{x})).$$

Definition 12. Let $E : R^n \to R^n$ be a one-to-one and onto operator. If $\overline{x}$ is an optimal solution of (P_E). Then, there exists $E\left(\overline{x}\right)$ is an E-optimal solution of (P).

Theorem 13 (*[5]*). *(E-Karush-Kuhn–Tucker Necessary Optimality Conditions). Let the objective function f, the constraint functions g_i, $i \in I$, and h_j, $j \in J$, be E -differentiable at $\overline{x} \in \Omega_E$. Further, let $E\left(\overline{x}\right)$ be an E-optimal solution of the considered E-differentiable optimization problem (P) and the E-Guignard constraint qualification [5] be satisfied at $\overline{x}$. Then, there exist Lagrange multipliers $\overline{\zeta} \in R^k$ and $\overline{\mu} \in R^s$ such that*

$$\nabla f(E\left(\overline{x}\right)) + \sum_{i=1}^{k} \overline{\zeta}_i \nabla g_i(E\left(\overline{x}\right)) + \sum_{j=1}^{s} \overline{\mu}_j \nabla h_j(E\left(\overline{x}\right)) = 0, \tag{5}$$

$$\overline{\zeta}_i g_i(E\left(\overline{x}\right)) = 0,\ i \in I, \tag{6}$$

$$\overline{\zeta}_i \in R_+,\ \ i \in J. \tag{7}$$

Definition 14. It is said that $\left(E\left(\overline{x}\right), \overline{\zeta}, \overline{\mu}\right) \in \Omega \times R^k \times R^s$ is an E-Karush-Kuhn–Tucker point (E-KKT-point, in short) for the considered optimization problem (P) if the E-Karush-Kuhn–Tucker necessary optimality conditions (5)–(7) are satisfied at $E(\overline{x})$ with Lagrange multiplier $\overline{\zeta}$, $\overline{\mu}$.

Now, we prove the sufficiency of the E-Karush-Kuhn–Tucker (E-KKT, in short) necessary optimality conditions for the considered E-differentiable optimization problem (P) under E-B-invexity hypotheses.

Theorem 15. *Let $\left(\overline{x}, \overline{\zeta}, \overline{\mu}\right) \in \Omega_E \times R^k \times R^s$ be a Karush–Kuhn–Tucker point (KKT-point, in short) of the E-optimization problem (P_E). Let $J_E^+\left(E\left(\overline{x}\right)\right) = \{j \in J : \overline{\mu}_j > 0\}$ and $J_E^-\left(E\left(\overline{x}\right)\right) = \{j \in J : \overline{\mu}_j < 0\}$. Furthermore, assume the following hypotheses are fulfilled:*

(a) the objective function f, is an E-B_f-invex function with respect to η and b_f at $\overline{x}$ on Ω_E,
(b) each inequality constraint g_i, $i \in I_E\left(\overline{x}\right)$, is an E-B_{g_i}-invex function with respect to η and b_{g_i} at $\overline{x}$ on Ω_E,
(c) each equality constraint h_j, $j \in J^+\left(E\left(\overline{x}\right)\right)$, is an E-B_{h_j}-invex function with respect to η and b_{h_j} at $\overline{x}$ on Ω_E,
(d) each function $-h_j$, $j \in J^-\left(E\left(\overline{x}\right)\right)$, is an E-B_{h_j}-invex function with respect to η and b_{h_j} at $\overline{x}$ on Ω_E.

Then $\overline{x}$ is an optimal solution of the problem (P_E) and, thus, $E\left(\overline{x}\right)$ is an E-optimal solution of the problem (P).

Proof. By assumption, $\left(\overline{x}, \overline{\zeta}, \overline{\mu}\right) \in \Omega_E \times R^k \times R^s$ is a Karush–Kuhn–Tucker point of the differentiable optimization problem (P_E). Then, by Definition 14, the Karush–Kuhn–Tucker necessary optimality conditions (5)–(7) are satisfied at $\overline{x}$ with Lagrange multipliers $\overline{\zeta} \in R^k$ and $\overline{\mu} \in R^s$. We proceed by contradiction. Suppose, contrary to the result, that $\overline{x}$ is not an optimal solution of the problem (P_E). Hence, by Definition 11, there exists another $x^* \in \Omega_E$ such that

$$f(E\left(x^*\right)) < f\left(E\left(\overline{x}\right)\right). \tag{8}$$

Using hypotheses (a)–(d), by Definition 2, the following inequalities

$$b_f\left(E\left(x^*\right), E\left(\overline{x}\right)\right)\left[f\left(E\left(x^*\right)\right) - f\left(E\left(\overline{x}\right)\right)\right] \ge \nabla f\left(E\left(\overline{x}\right)\right)\eta\left(E\left(x^*\right), E\left(\overline{x}\right)\right), \tag{9}$$

$$b_{g_i}\left(E\left(x^*\right), E\left(\overline{x}\right)\right)\left[g_i(E\left(x^*\right)) - g_i(E\left(\overline{x}\right))\right] \ge \nabla g_i\left(E\left(\overline{x}\right)\right)\eta\left(E\left(x^*\right), E\left(\overline{x}\right)\right), i \in I\left(E\left(\overline{x}\right)\right), \tag{10}$$

$$b_{h_j}\left(E\left(x^*\right), E\left(\overline{x}\right)\right)\left[h_j(E\left(x^*\right)) - h_j(E\left(\overline{x}\right))\right] \ge \nabla h_j\left(E\left(\overline{x}\right)\right)\eta\left(E\left(x^*\right), E\left(\overline{x}\right)\right),\ \ j \in J^+\left(E\left(\overline{x}\right)\right), \tag{11}$$

$$b_{h_j}\left(E\left(x^*\right), E\left(\overline{x}\right)\right)\left[-h_j(E\left(x^*\right)) + h_j(E\left(\overline{x}\right))\right] \ge -\nabla h_j\left(E\left(\overline{x}\right)\right)\eta\left(E\left(x^*\right), E\left(\overline{x}\right)\right),\ \ j \in J^-\left(E\left(\overline{x}\right)\right) \tag{12}$$

hold, respectively. Where $b_f\left(E(x^*), E\left(\overline{x}\right)\right) > 0$, for all $x^* \in \Omega_E$. Combining (8) and (9), we get

$$\nabla(f \circ E)\left(\overline{x}\right)\eta\left(E\left(x^*\right), E\left(\overline{x}\right)\right) < 0. \tag{13}$$

Multiplying inequalities (10)–(12) by the corresponding Lagrange multipliers, respectively, we obtain

$$b_{g_i}\left(E\left(x^*\right),E\left(\overline{x}\right)\right)\left[\overline{\zeta}_i g_i(E\left(x^*\right))-\overline{\zeta}_i g_i(E\left(\overline{x}\right))\right]\geq\overline{\zeta}_i\nabla g_i\left(E\left(\overline{x}\right)\right)\eta\left(E\left(x^*\right),E\left(\overline{x}\right)\right),\ \ i\in I\left(E\left(\overline{x}\right)\right), \tag{14}$$

$$b_{h_j}\left(E\left(x^*\right),E\left(\overline{x}\right)\right)\left[\overline{\mu}_j h_j(E\left(x^*\right))-\overline{\mu}_j h_j(E\left(\overline{x}\right))\right]\geq\overline{\mu}_j\nabla h_j\left(E\left(\overline{x}\right)\right)\eta\left(E\left(x^*\right),E\left(\overline{x}\right)\right),\ \ j\in J^+\left(E\left(\overline{x}\right)\right), \tag{15}$$

$$b_{h_j}\left(E\left(x^*\right),E\left(\overline{x}\right)\right)\left[\overline{\mu}_j h_j(E\left(x^*\right))-\overline{\mu}_j h_j(E\left(\overline{x}\right))\right]\geq\overline{\mu}_j\nabla h_j\left(E\left(\overline{x}\right)\right)\eta\left(E\left(x^*\right),E\left(\overline{x}\right)\right),\ \ j\in J^-\left(E\left(\overline{x}\right)\right). \tag{16}$$

Using the E-Karush-Kuhn–Tucker necessary optimality condition (6) together with $x^*\in\Omega_E$ and $\overline{x}\in\Omega_E$, where $b_{g_i}\left(E\left(x^*\right),E\left(\overline{x}\right)\right)>0,\ i\in I\left(E\left(\overline{x}\right)\right)$ and $b_{h_j}\left(E\left(x^*\right),E\left(\overline{x}\right)\right)>0,\ j\in J\left(E\left(\overline{x}\right)\right)$ we get, respectively,

$$\overline{\zeta}_i\nabla g_i\left(E\left(\overline{x}\right)\right)\eta\left(E\left(x^*\right),E\left(\overline{x}\right)\right)\leq 0,\ i\in I\left(E\left(\overline{x}\right)\right), \tag{17}$$

$$\overline{\mu}_j\nabla h_j\left(E\left(\overline{x}\right)\right)\eta\left(E\left(x^*\right),E\left(\overline{x}\right)\right)\leq 0,\ j\in J^+\left(E\left(\overline{x}\right)\right), \tag{18}$$

$$\overline{\mu}_j\nabla h_j\left(E\left(\overline{x}\right)\right)\eta\left(E\left(x^*\right),E\left(\overline{x}\right)\right)\leq 0,\ \ j\in J^-\left(E\left(\overline{x}\right)\right)\ . \tag{19}$$

Combining (13) and (17)–(19), we obtain that the following inequality

$$\left[\nabla\left(f\circ E\right)\left(\overline{x}\right)+\sum_{i=1}^{k}\overline{\zeta}_i\nabla g_i\left(E\left(\overline{x}\right)\right)+\sum_{j=1}^{s}\overline{\mu}_j\nabla h_j\left(E\left(\overline{x}\right)\right)\right]\eta\left(E\left(x^*\right),E\left(\overline{x}\right)\right)<0$$

holds, which is a contradiction to the E-Karush-Kuhn–Tucker necessary optimality condition (5). By assumption, $E:R^n\to R^n$ is a one-to-one and onto operator. Since $\overline{x}$ is an optimal solution of the problem (P_E), by Definition 12, $E\left(\overline{x}\right)$ is an E-optimal solution of the problem (P). Thus, the proof of this theorem is completed. ∎

Theorem 16. *Let $\left(\overline{x},\overline{\zeta},\overline{\mu}\right)\in\Omega_E\times R^k\times R^s$ be a Karush–Kuhn–Tucker point (KKT-point, in short) of the E-optimization problem (P_E). Furthermore, assume that the following hypotheses are fulfilled:*

(a) the objective function f, is a strictly E-B_f-invex function with respect to η and b_f at $\overline{x}$ on Ω_E,
(b) each inequality constraint g_i, $i\in I_E\left(\overline{x}\right)$, is an E-B_{g_i}-invex function with respect to η and b_{g_i} at $\overline{x}$ on Ω_E,
(c) each equality constraint h_j, $j\in J^+\left(E\left(\overline{x}\right)\right)$, is an E-B_{h_j}-invex function with respect to η and b_{h_j} at $\overline{x}$ on Ω_E,
(d) each function $-h_j$, $j\in J^-\left(E\left(\overline{x}\right)\right)$, is an E-B_{h_j}-invex function with respect to η and b_{h_j} at $\overline{x}$ on Ω_E.

Then $\overline{x}$ is an optimal solution of the problem (P_E) and, thus, $E\left(\overline{x}\right)$ is an E-optimal solution of the problem (P).

We now present an example of an E-differentiable optimization problem in which the functions involved are E-B-invex, in order to illustrate the sufficient optimality conditions defined in the paper.

Example 17. Consider the following nondifferentiable optimization problem

$$\begin{aligned} &f(x)=e^{\sqrt[3]{8x}}-1\to\min\\ &\text{s.t. }\ g(x)=1-e^{\sqrt[3]{x}}\leq 0. \end{aligned} \tag{P1}$$

Note that the feasible solution set of the considered optimization problem (P1) is $\Omega=\{x\in R:1-e^{\sqrt[3]{x_1}}\leq 0\}$. Further, note that the functions constituting problem (P1) are nondifferentiable at $\overline{x}=0$. Let $E:R\to R$ be a one-to-one and onto mapping defined as follows $E\left(x\right)=x^3$. Now, for the problem (P1), we define its associated constrained E-optimization problem (P_E1) as follows

$$\begin{aligned} &f(E(x))=e^{2x}-1\to\min\\ &\text{s.t. }\ g(E(x))=1-e^{x}\leq 0. \end{aligned} \tag{P_E1}$$

Note that the feasible solution set of the problem (P_E1) is $\Omega_E=\{x\in R:1-e^x\leq 0\}$ and $\overline{x}=0$ is a feasible solution of the problem (P_E1). Let η and b be defined by $\eta(x,u)=e^{\sqrt[3]{x}}+e^{\sqrt[3]{u}}$, $b_f(x,u)=\frac{e^{\sqrt[3]{u}}}{e^{\sqrt[3]{x}+1}}$ and $b_g(x,u)=1$. Further, note that all functions constituting the considered optimization problem (P1) are E-differentiable E-B-invex at $\overline{x}=0$. Then, by Definition 2, it can be shown that the objective function f is E-B_f-invex with respect to η, b_f at $\overline{x}$ on Ω_E, the constraint function g is E-B_g-invex with respect to η, b_g at $\overline{x}$ on Ω_E. Thus, all hypotheses of Theorem 15 are fulfilled and, therefore, we conclude that $\overline{x}=0$ is an optimal solution of the E-optimization problem (P_E1) and, thus, $E(\overline{x})$ is an E-optimal solution of the considered multiobjective programming problem (P1). Note, moreover, that the sufficient optimality conditions under E-differentiable E-convexity (see, for example, [18]) are not applicable since the functions constituting problem (P_E1) are not E-convex at $\overline{x}$ on Ω_E. Also, the sufficient optimality conditions under invexity (see [1]) are not applicable in the considered case since the functions constituting problem (P1) are not invex with respect to η defined above on Ω. Moreover, the sufficient optimality conditions under B-invexity (see [6]) are not applicable in the considered case since the functions constituting problem (P1) are not B-invex with respect to η and b defined above on Ω.

4. Concluding remarks

In this paper, a new class of nondifferentiable optimization problems has been defined. Namely, the concept of E-B-invexity in which all involved functions are E-differentiable has been introduced. The so-called E-Karush-Kuhn–Tucker sufficient optimality conditions have been established for the considered E-differentiable optimization problem under E-B-invexity. These results have been illustrated in the paper by suitable examples.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] Hanson MA. On sufficiency of the Kuhn–Tucker conditions. J Math Anal Appl 1981;80:545–50.
[2] Antczak T. (p, r)-Invex sets and functions. J Math Anal Appl 2001;263:355–79.
[3] Antczak T. A class of B-(p, r)-invex functions and mathematical programming. J Math Anal Appl 2003;286:187–206.
[4] Antczak T. B-(p, r)-Pre-invex functions. Folia Math Acta Lodz 2004;11:3–15.
[5] Abdulaleem N. E-Invexity and generalized E-invexity in E-differentiable multiobjective programming. In: ITM web of conferences, Vol. 24. 2019, p. 01002.
[6] Bector CR, Suneja SK, Lalitha CS. Generalized B-vex functions and generalized B-vex programming. J Optim Theory Appl 1993;76:561–76.
[7] Hanson MA, Mond B. Further generalizations of convexity in mathematical programming. J Inf Optim Sci 1982;3:25–32.
[8] Hanson MA, Mond B. Necessary and sufficient conditions in constrained optimization. Math Program 1987;37:51–8.
[9] Jeyakumar V, Mond B. On generalised convex mathematical programming. ANZIAM J 1992;34:43–53.
[10] Jeyakumar V. Strong and weak invexity in mathematical programming. Math Methods Oper Res 1985;55:109–25.
[11] Jiao H, Liu S. Semilocal E-preinvexity and its applications in nonlinear multiple objective fractional programming. J Inequal Appl 2011;2011:1–13.
[12] Mond B, Weir T. Generalized concavity and duality. In: Schaible S, Ziemba WT, editors. Generalized concavity in optimization and economics. New York: Acadimic Press; 1981, p. 263–76.
[13] Pini R, Singh C. A survey of recent [1985-1995] advances in generalized convexity with applications to duality theory and optimality conditions. Optimization 1997;39:311–60.
[14] Rueda NG, Hanson MA. Optimality criteria in mathematical programming involving generalized invexity. J Math Anal Appl 1988;130:375–85.
[15] Youness EA. E-Convex sets, E-convex functions, and E-convex programming. J Optim Theory Appl 1999;102:439–50.
[16] Bector CR, Singh C. B-Vex functions. J Optim Theory Appl 1991;71:237–53.
[17] Yang XM. On E-convex sets, E-convex functions and E-convex programming. J Optim Theory Appl 2001;109:699–704.
[18] Antczak T, Abdulaleem N. E-Optimality conditions and wolfe E-duality for E-differentiable vector optimization problems with inequality and equality constraints. J Nonlinear Sci Appl 2019;12:745–64.
[19] Antczak T, Abdulaleem N. Optimality and duality results for E-differentiable multiobjective fractional programming problems under E-convexity. J Inequal Appl 2019;2019:292.
[20] Abdulaleem N. E-Duality results for E-differentiable E-invex multiobjective programming problems. J Phys Conf Ser 2019;1294:032027.
[21] Abdulaleem N. Wolfe E-duality for E-differentiable E-invex vector optimization problems with inequality and equality constraints. In: International conference on applied mathematics & computational science (ICAMCS.NET). Budapest, Hungary: IEEE; 2018, p. 156–63.
[22] Abdulaleem N. E-Optimality conditions for E-differentiable E-invex multiobjective programming problems. WSEAS Trans Math 2019;18:14–27.
[23] Duca DI, Duca E, Lupsa L, Blaga R. E-Convex functions. Bull Appl Comput Math 2000;43:93–103.
[24] Duca DI, Lupsa L. On the E-epigraph of an E-convex function. J Optim Theory Appl 2006;120:341–6.
[25] Fulga C, Preda V. Nonlinear programming with E-preinvex and local E-preinvex functions. European J Oper Res 2009;192:737–43.
[26] Kumari B, Jayswal A. Some properties of geodesic E-preinvex function and geodesic semi E-preinvex function on Riemannian manifolds. Opsearch 2018;55(3–4):807–22.
[27] Syau YR, Lee ES. Some properties of E-convex functions. Appl Math Lett 2005;18:1074–80.
[28] Syau YR, Jia L, Lee ES. Generalizations of E-convex and B-vex functions. Comput Math Appl 2009;58:711–6.
[29] Megahed AA, Gomma HG, Youness EA, El-Banna AZ. Optimality conditions of E-convex programming for an E-differentiable function. J Inequal Appl 2013;2013:246.
[30] Mishra SK, Mohapatra RN, Youness EA. Some properties of semi E-b-vex functions. Appl Math Comput 2011;217:5525–30.