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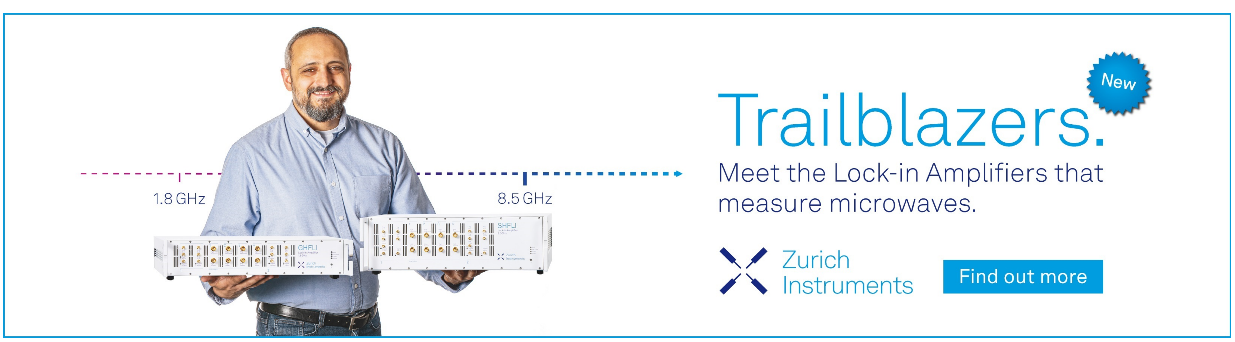

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Approximation of the Mean and Variance for the Median Estimator of the Shape Parameter for Pareto European Distribution

Adwea Naji Atewi [a], Bushra Swedan Naser [b], Safaa Jawad Ali [c] and Muataz Adnan Harhoosh [d]

Institute of Medical Technology Al-Mansur, Middle Technical University, Baghdad, Iraq.

[a] Corresponding author: adwea-naji@mtu.edu.iq
[b] safajawad1970@gmail.com
[c] bushrasewdan@gmail.com
[d] Muataz_adnan@mtu.edu.iq

Abstract. In this research, some of Pareto properties from the first class are studied beside some of its theories and Pareto distribution parameters were obtained using a median amount. The amount of shape parameter contains a very complex model and its properties are difficult to know. In this paper, we approximated the mean and the variance using Tyler series of second degree derivation to obtain a mathematical model.

INTRODUCTION

Pareto distribution of two parameters is widely used as a model in many applications such as population distribution over residential complexes, economy and global oil reserves. this distribution was named after the Italian economist professor Vilfredo Pareto (1848 – 1923) Pareto proposed this distribution for first time, when a comprehensive law was drawn up to deal with income distribution for a specific community and had the Pareto principle known as rule 80 – 20 in administration science [1-3].

He is also has the two economic theories of Pareto optimization Pareto preference. In this paper, estimation of Pareto distribution parameters of the first type was obtained using the median estimator.Then the mean approximation and variance of the median estimator of the Pareto distribution was extracted.

- **Definition:**

Let X be a random variable of a statistical community, defined as a Pareto dist. of type I if it has a probability density function pdf as follows [4]:

$$f(x;\alpha,c)=\begin{cases}\dfrac{\alpha c^{\alpha}}{x^{\alpha+1}} & x\geq c\ ,c>0\,,\alpha>0\\ 0 & \text{otherwise}\end{cases} \tag{1}$$

Where α is the shape parameter and c is the scale parameter of the Pareto distribution, and the maximum distribution value can be obtained when x = c

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$$f(c) = \frac{\alpha}{c} \qquad \alpha < c \tag{2}$$

Figure (1) represent different forms of distribution with different values for each α and c:

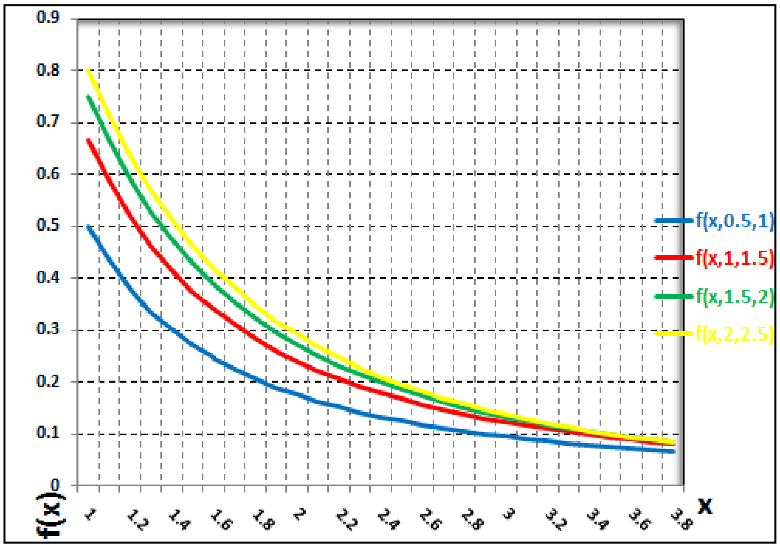

FIGURE 1. Pareto pdf's.

- **Cumulative Distribution[4]:**

$$F(x;\alpha,c) = 1 - \left(\frac{c}{x}\right)^{\alpha} \tag{3}$$

- **The kth moment about origin:**

$$E\left(x^k\right) = \int_c^{\infty} x^k f(x) dx \tag{4}$$

In the case of Pareto distribution, moment r can be found by the relation:

$$E\left(x^r\right) = \frac{\alpha c^r}{\alpha - r} \qquad \alpha \neq r \tag{5}$$

When r = 1, the first moment represents the mean:

$$E(x) = \frac{\alpha c}{\alpha - 1} \qquad \alpha \neq 1 \tag{6}$$

For k=2 then

$$E\left(x^2\right) = \frac{\alpha c^2}{\alpha - 2} \qquad \alpha \neq 2 \tag{7}$$

Thus the variance can be found by the relationship:

$$\mathrm{var}(x) = E\left(x^2\right) - \left(E(x)\right)^2 \tag{8}$$

By butting eq.(4) & eq.(5) in eq.(6), we get:

$$Var(x) = \frac{\alpha c^2}{(\alpha-1)^2(\alpha-2)} \qquad \alpha > 2 \tag{9}$$

- **Median [5]:**

The median of the dist. is value of x for a r.v X so that $F(x) = \mathrm{pr}(X \le x) = \frac{1}{2}$, ie, it represents the value of x, which is located in the center of the data after their order.

In the case of Pareto dist., the broker represents:

$$Md = c\sqrt[\alpha]{2} \tag{10}$$

- **Mode [5]:**

The mode of dist. is value of x for a r.v X so that the value of the p.d.f, f (x) maximizes.

For the Pareto dist., it is clear that the value of x that makes the p.d,f f (x) the greatest value is when x is the smallest value, that is, when x = c.

$$Mode = c \tag{11}$$

- **Median Method [6]:**

This method is used to measure the shape parameter $\hat{\alpha}$ by assuming the scale parameter c is Maximum Likelihood Estimator, MLE

This method is used to determine the estimator of the shape parameter $\hat{\alpha}$ assuming that the scale parameters estimator is the MLE that is:

$$\hat{c} = \min\{x_1, x_2, x_3 \ldots, x_n\} = x_{(1)} \tag{12}$$

From eq.(8) we get:

$$x_{md} = \hat{c}\sqrt[\alpha]{2}$$

$$\text{or } x_{md} = x_{(1)}\sqrt[\alpha]{2}$$

Taking the logarithm of both

$$\ln x_{md} = \ln x_{(1)} + \frac{1}{\alpha}\ln 2$$

$$\ln \frac{x_{md}}{x_{(1)}} = \frac{1}{\alpha}\ln 2$$

$$\hat{\alpha} = \frac{\ln 2}{\ln\left(\frac{x_{md}}{x_{(1)}}\right)} \tag{13}$$

SOME THEORIES CONCERNING THE DISTRIBUTION OF PARETO TYPE I

- **Theorem (1):**

"If the r.v. X ~par(α,c) then the r.v. Y=bX~Par(α,bc), b is constant" [7,8].

Proof:

$$Y = bX \Rightarrow X = y/b$$

$$dx/dy = \frac{1}{b} \Rightarrow |J| = \frac{1}{b}$$

$$g(y) = \alpha c^{\alpha} / (y/b)^{(\alpha+1)} \left(\frac{1}{b}\right)$$

$$= \alpha c^{\alpha} \left(b^{\alpha} / y^{(\alpha+1)}\right)$$

$$\Rightarrow g(y) = \begin{cases} \alpha (cb)^{\alpha} / y^{(\alpha+1)} & cb \le y < \infty \\ 0 & e.w \end{cases}$$

which is the p.d.f. of Pareto dist. with parameters α, tc.

- **Theorem (2):**

Let $x_1, x_2, x_3, \ldots, x_n$ be a r.s of size n from $par(\alpha, c)$ then:

1- The first order statistical $Y_1 = \min(X_1, X_2, \ldots, X_n) \sim \mathrm{par}(\mathrm{n}\alpha, \mathrm{c})$

2- The limiting distribution of $\mathrm{Y}_1 Deg(c)$.

Proof:

1- by using eqs (1), (2):

$$y_1 = \min\{x_1, x_2, \ldots.., x_n\}$$

$$P(y_1 \le y) = \begin{cases} 1 - P(y_1 > y) & , y < c \\ 0 & , y \ge c \end{cases}$$

$$\because P(x_i \le y) = 1 - P(x_i > y) \qquad , i = 1, 2, \ldots\ldots.., n$$

We have:

$$P(x_i \le y) = \int_c^y f(x_i, c, \alpha)\, dx = 1 - \left(\frac{c}{y}\right)^{\alpha}$$

Therefore:

$$p(Y_1 \geq y) = \prod_{i=1}^{n} p(x_i \geq y) = \left(\frac{c}{y}\right)^{n\alpha}$$

$$P(Y_1 < y) = 1 - \left(\frac{c}{y}\right)^{n\alpha}$$

$$h(y) = \begin{cases} \dfrac{d}{dy}\left[1 - \left(\dfrac{c}{y}\right)^{n\alpha}\right] & , c \leq y < \infty \\ 0 & e.w \end{cases}$$

$$h(y) = \begin{cases} \dfrac{n\alpha c^{n\alpha}}{Y^{n\alpha+1}} & , c \leq y < \infty \\ 0 & e.w \end{cases}$$

That is $Y_1 \sim \text{par}(\text{n}\alpha, \text{c})$

2- $\because Y_1 \sim \text{par}(\text{n}\alpha, \text{c}), \textit{then c.d.f of } Y_1 \textit{ is}:$

$$G(Y_1) = \begin{cases} 0 & Y_1 \leq c \\ 1 - \left[\dfrac{c}{Y_1}\right]^{n\alpha} & c < Y_1 < \infty \end{cases}$$

Then:

$$\lim_{n \to \infty} G(Y_1) = \begin{cases} 0 & Y_1 \leq c \\ 1 & Y_1 > c \end{cases}$$

“This mean that the limit dist. of Y_1 is Degenerate Dist. with Parameter c, the mean of r.v Y_1 is”:

$$E(Y_1) = \frac{n\alpha c}{n\alpha - 1} \tag{14}$$

and The variance of r.v Y_1 is:

$$Var(Y_1) = \frac{n\alpha c^2}{(n\alpha - 1)^2 (n\alpha - 2)} \tag{15}$$

Let $X_1, X_2, \ldots, X_n$ r.v of size n follow the Pareto dist., We will approximation the mean & the variance of the median estimator method by using the Tyler series of the quadratic variable x and y at point (μ_x, μ_y) up to 2^{nd} order which is given by[8]:

$$"E[g(X,Y)] = g(\mu_X,\mu_Y) + \frac{1}{2}Var[X]\frac{\partial^2}{\partial X^2}g(X,Y)\Big|_{\substack{\mu_X\\ \mu_Y}} + \frac{1}{2}Var[Y]\frac{\partial^2}{\partial Y^2}g(X,Y)\Big|_{\substack{\mu_X\\ \mu_Y}}$$

$$+Cov[X,Y]\left[\frac{\partial^2}{\partial X\partial Y}g(X,Y)\Big|_{\substack{\mu_X\\ \mu_Y}}\right]" \tag{16}$$

$$"Var[g(X,Y)] = Var[X]\left[\frac{\partial}{\partial X}g(X,Y)\Big|_{\substack{\mu_X\\ \mu_Y}}\right]^2 + Var[Y]\left[\frac{\partial}{\partial Y}g(X,Y)\Big|_{\substack{\mu_X\\ \mu_Y}}\right]^2 +$$

$$2Cov[X,Y]\left[\frac{\partial}{\partial X}g(X,Y)\Big|_{\substack{\mu_X\\ \mu_Y}}\right]\left[\frac{\partial}{\partial Y}g(X,Y)\Big|_{\substack{\mu_X\\ \mu_Y}}\right]" \tag{17}$$

By butting $X = \overline{X}$ and $Y = Y_1$ in eq.(1) and eq.(2), we get:

$$"E[g(\overline{X},Y_1)] = g(\mu_{\overline{X}},\mu_{Y_1}) + \frac{1}{2}Var[\overline{X}]\frac{\partial^2}{\partial \overline{X}^2}g(\overline{X},Y_1)\Big|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}} + \frac{1}{2}Var[Y_1]\frac{\partial^2}{\partial Y_1^2}g(\overline{X},Y_1)\Big|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}}$$

$$+Cov[\overline{X},Y_1]\left[\frac{\partial^2}{\partial \overline{X}\partial Y_1}g(\overline{X},Y_1)\Big|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}}\right]" \tag{18}$$

and

$$"Var[g(\overline{X},Y_1)] = Var[\overline{X}]\left[\frac{\partial}{\partial \overline{X}}g(\overline{X},Y_1)\Big|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}}\right]^2 + Var[Y_1]\left[\frac{\partial}{\partial Y_1}g(\overline{X},Y_1)\Big|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}}\right]^2 +$$

$$2Cov[\overline{X},Y_1]\left[\frac{\partial}{\partial \overline{X}}g(\overline{X},Y_1)\Big|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}}\right]\left[\frac{\partial}{\partial Y_1}g(\overline{X},Y_1)\Big|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}}\right]" \tag{19}$$

We have the MED estimators given by eq.(10) and eq.(11) is:

$$\hat{c} = \min\{x_1, x_2, \ldots, x_n\} = x_{(1)}$$

$$\hat{\alpha} = \frac{Ln2}{Ln\left(\frac{x_{med}}{x_{(1)}}\right)}$$

By the relationship between the mean, median and mode:

Mode = 3 median - 2 mean

And from eq.(9) we have: $Mode = c$ then we get:

$$x_{med} = \frac{2\overline{X} + c}{3}$$

$$\hat{\alpha} = \frac{\ln 2}{\ln\left(\frac{\frac{2\overline{X} + c}{3}}{y_1}\right)}$$

$$\Rightarrow \hat{\alpha} = \frac{\ln 2}{\ln\left(\frac{2\overline{X} + y_1}{3y_1}\right)}$$

From equation (4) & (7), we have:

$$\mu_{\overline{X}} = \mu = \frac{\alpha c}{\alpha - 1}$$

$$Var\left[\overline{X}\right] = \frac{\sigma^2}{n} = \frac{\alpha c^2}{n(\alpha - 1)^2(\alpha - 2)} \qquad \alpha > 2 \tag{20}$$

From equation (12) and (13), we get:

$$Y_1 = \frac{n\alpha c}{n\alpha - 1} \quad , \quad \overline{X} = \frac{\alpha c}{\alpha - 1}$$

Let $g\left(\overline{X}, Y_1\right) = \hat{\alpha} = \frac{\ln 2}{\ln\left(\frac{2\overline{X} + y_1}{3y_1}\right)}$

$$g\left(\overline{X}, Y_1\right) = \hat{\alpha} = \frac{\ln 2}{\ln\left(\frac{2\overline{X} + y_1}{3y_1}\right)}$$

$$g\left(\mu_{\overline{X}}, \mu_{Y_1}\right) = \frac{\ln 2}{\ln\left(\frac{\frac{2\alpha c}{\alpha - 1} + \frac{n\alpha c}{n\alpha - 1}}{\frac{3n\alpha c}{n\alpha - 1}}\right)}$$

$$g\left(\mu_{\overline{X}}, \mu_{Y_1}\right) = \frac{\ln 2}{\ln\left(\frac{\frac{2\alpha c(n\alpha - 1) + n\alpha c(\alpha - 1)}{(\alpha - 1)(n\alpha - 1)}}{\frac{3n\alpha c}{n\alpha - 1}}\right)}$$

$$g\left(\mu_{\overline{X}}, \mu_{Y_1}\right) = \frac{\ln 2}{\ln\left(\frac{2n\alpha^2 c - 2\alpha c + n\alpha^2 c - n\alpha c}{(\alpha - 1)(n\alpha - 1)} \times \frac{(n\alpha - 1)}{3n\alpha c}\right)}$$

$$g\left(\mu_{\overline{X}}, \mu_{Y_1}\right) = \frac{\ln 2}{\ln\left(\frac{3n\alpha^2 c - \alpha c(n + 2)}{3n\alpha c(\alpha - 1)}\right)}$$

$$g\left(\mu_{\overline{X}}, \mu_{Y_1}\right) = \frac{\ln 2}{\ln\left(\frac{\alpha c(3n\alpha - (n + 2))}{3n\alpha c(\alpha - 1)}\right)}$$

$$g\left(\mu_{\overline{X}}, \mu_{Y_1}\right) = \frac{\ln 2}{\ln\left(\frac{(3n\alpha - (n + 2))}{3n(\alpha - 1)}\right)} \qquad (21)$$

$$g\left(\overline{X}, Y_1\right) = \hat{\alpha} = \frac{\ln 2}{\ln\left(\frac{2\overline{X} + y_1}{3y_1}\right)}$$

and

$$\frac{\partial g\left(\overline{X},Y_1\right)}{\partial \overline{X}}=-\frac{\left(\frac{3y_1}{2\overline{X}+y_1}\cdot\frac{6y_1}{9{y_1}^2}\right)\ln 2}{\left(\ln\left(\frac{2\overline{X}+y_1}{3y_1}\right)\right)^2}$$

$$\frac{\partial g\left(\overline{X},Y_1\right)}{\partial \overline{X}}=-\frac{\frac{2\ln 2}{2\overline{X}+y_1}}{\left(\ln\left(\frac{2\overline{X}+y_1}{3y_1}\right)\right)^2}$$

$$\frac{\partial g\left(\overline{X},Y_1\right)}{\partial \overline{X}}=-\frac{2\ln 2}{\left(2\overline{X}+y_1\right)\left(\ln\left(\frac{2\overline{X}+y_1}{3y_1}\right)\right)^2}$$

$$\left.\frac{\partial g\left(\overline{X},Y_1\right)}{\partial \overline{X}}\right|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}}=-\frac{2\ln 2}{\left(\frac{3n\alpha^2c-\alpha c(n+2)}{(\alpha-1)(n\alpha-1)}\right)\left(\ln\frac{\left(\frac{\left(3n\alpha^2c-\alpha c(n+2)\right)}{(\alpha-1)(n\alpha-1)}\right)}{\frac{3n\alpha c}{n\alpha-1}}\right)^2}$$

$$\left.\frac{\partial g\left(\overline{X},Y_1\right)}{\partial \overline{X}}\right|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}}=-\frac{2\ln 2}{\left(\frac{n\alpha^2c+\alpha c(n-2)}{(\alpha-1)(n\alpha-1)}\right)\left(\ln\left(\frac{3n\alpha-(n+2)}{3n(\alpha-1)}\right)\right)^2} \tag{22}$$

$$\frac{\partial^2 g\left(\overline{X},Y_1\right)}{\partial \overline{X}^2}=\frac{2\ln 2\left[\left(2\overline{X}+y_1\right)2\ln\left(\frac{2\overline{X}+y_1}{3y_1}\right)\left(\frac{3y_1}{2\overline{X}+y}\right)\frac{6y_1}{9{y_1}^2}+2\left(\ln\left(\frac{2\overline{X}+y_1}{3y_1}\right)\right)^2\right]}{\left(2\overline{X}+y_1\right)^2\left(\ln\left(\frac{2\overline{X}+y_1}{3y_1}\right)\right)^4}$$

$$\frac{\partial^2 g\left(\overline{X},Y_1\right)}{\partial \overline{X}^2}=\frac{4\ln 2\ln\left(\frac{2\overline{X}+y_1}{3y_1}\right)\left[2+\ln\frac{2\overline{X}+y_1}{3y_1}\right]}{\left(2\overline{X}+y_1\right)^2\left(\ln\left(\frac{2\overline{X}+y_1}{3y_1}\right)\right)^4}$$

$$\frac{\partial^2 g(\overline{X},Y_1)}{\partial \overline{X}^2}=\frac{4\ln 2\left[\ln\left(\frac{2\overline{X}+y_1}{3y_1}\right)+2\right]}{(2\overline{X}+y_1)^2\left(\ln\left(\frac{2\overline{X}+y_1}{3y_1}\right)\right)^3}$$

$$\left.\frac{\partial^2 g(\overline{X},Y_1)}{\partial \overline{X}^2}\right|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}}=\frac{4\ln 2\left[2+\ln\left(\frac{\frac{2\alpha c}{\alpha-1}+\frac{n\alpha c}{n\alpha-1}}{\frac{3n\alpha c}{n\alpha-1}}\right)\right]}{\left(\frac{2\alpha c}{\alpha-1}+\frac{n\alpha c}{n\alpha-1}\right)^2\left(\ln\left(\frac{\frac{2\alpha c}{\alpha-1}+\frac{n\alpha c}{n\alpha-1}}{\frac{3n\alpha c}{n\alpha-1}}\right)\right)^3}$$

$$\left.\frac{\partial^2 g(\overline{X},Y_1)}{\partial \overline{X}^2}\right|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}}=\frac{4\ln 2\left[2+\ln\left(\frac{2\alpha c(n\alpha-1)+n\alpha c(\alpha-1)}{(\alpha-1)(n\alpha-1)}\cdot\frac{(n\alpha-1)}{3n\alpha c}\right)\right]}{\left(\frac{2\alpha c(n\alpha-1)+n\alpha c(\alpha-1)}{(\alpha-1)(n\alpha-1)}\right)^2\left(\ln\frac{2\alpha c(n\alpha-1)+n\alpha c(\alpha-1)}{(\alpha-1)(n\alpha-1)}\cdot\frac{(n\alpha-1)}{3n\alpha c}\right)^3}$$

$$\left.\frac{\partial^2 g(\overline{X},Y_1)}{\partial \overline{X}^2}\right|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}}=\frac{4\ln 2\left[2+\ln\left(\frac{(2n\alpha^2c-2\alpha c+n\alpha^2c-n\alpha c)}{(\alpha-1)(n\alpha-1)}\cdot\frac{(n\alpha-1)}{3n\alpha c}\right)\right]}{\left(\frac{2n\alpha^2c-2\alpha c+n\alpha^2c-n\alpha c}{(\alpha-1)(n\alpha-1)}\right)^2\left(\ln\frac{(2n\alpha^2c-2\alpha c+n\alpha^2c-n\alpha c)}{(\alpha-1)(n\alpha-1)}\cdot\frac{(n\alpha-1)}{3n\alpha c}\right)^3}$$

$$\left.\frac{\partial^2 g(\overline{X},Y_1)}{\partial \overline{X}^2}\right|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}}=\frac{4\ln 2\left[2+\ln\left(\frac{\alpha+n-2}{3n(\alpha-1)}\right)\right]}{\left(\frac{3n\alpha^2c-\alpha c(n+2)}{(\alpha-1)(n\alpha-1)}\right)^2\left(\ln\frac{3n\alpha-(n+2)}{3n(\alpha-1)}\right)^3}$$

$$\frac{\partial g(\overline{X},Y_1)}{\partial Y_1}=-\frac{(\ln 2)\frac{3y_1}{2\overline{X}+y_1}\left(\frac{3y_1-3(2\overline{X}+y_1)}{9{y_1}^2}\right)}{\left(\ln\left(\frac{2\overline{X}+y_1}{3y_1}\right)\right)^2}$$

(23)

$$\frac{\partial g(\overline{X},Y_1)}{\partial Y_1}=-\frac{\left(\frac{\ln 2(3y_1-3(2\overline{X}+y_1))}{3y_1(2\overline{X}+y_1)}\right)}{\left(\ln\left(\frac{2\overline{X}+y_1}{3y_1}\right)\right)^2}$$

$$\frac{\partial g(\bar{X},Y_1)}{\partial Y_1} = -\frac{\left(\frac{\ln 2(3y_1 - 6\bar{X} - 3y_1)}{3y_1(2\bar{X}+y_1)}\right)}{\left(\ln\left(\frac{2\bar{X}+y_1}{3y_1}\right)\right)^2}$$

$$\frac{\partial g(\bar{X},Y_1)}{\partial Y_1} = \frac{2\bar{X}\ln 2}{\left(2\bar{X}y_1 - y_1{}^2\right)\left(\ln\left(\frac{2\bar{X}+y_1}{3y_1}\right)\right)^2}$$

$$\left.\frac{\partial g(\bar{X},Y_1)}{\partial Y_1}\right|_{\substack{\mu_{\bar{X}}\\ \mu_{Y_1}}} = \frac{\frac{(2\ln 2)\alpha c}{\alpha-1}}{\left(\frac{2\alpha c}{\alpha-1}.\frac{n\alpha c}{n\alpha-1}+\frac{n^2\alpha^2c^2}{(n\alpha-1)^2}\right)\left[\ln\left(\frac{\alpha+n-2}{3n(\alpha-1)}\right)\right]^2}$$

$$\left.\frac{\partial g(\bar{X},Y_1)}{\partial Y_1}\right|_{\substack{\mu_{\bar{X}}\\ \mu_{Y_1}}} = \frac{\frac{(2\ln 2)\alpha c}{\alpha-1}}{\left(\frac{2n\alpha^2c^2}{(n\alpha-1)(\alpha-1)}+\frac{n^2\alpha^2c^2}{(n\alpha-1)^2}\right)\left[\ln\left(\ln\frac{3n\alpha-(n+2)}{3n(\alpha-1)}\right)^2\right]}$$

$$\left.\frac{\partial g(\bar{X},Y_1)}{\partial Y_1}\right|_{\substack{\mu_{\bar{X}}\\ \mu_{Y_1}}} = \frac{\frac{(2\ln 2)\alpha c}{\alpha-1}}{\left(\frac{2n\alpha^2c^2(n\alpha-1)}{(n\alpha-1)^2(\alpha-1)}+\frac{n^2\alpha^2c^2(\alpha-1)}{(n\alpha-1)^2(\alpha-1)}\right)\left[\ln\left(\ln\frac{3n\alpha-(n+2)}{3n(\alpha-1)}\right)\right]^2}$$

$$\left.\frac{\partial g(\bar{X},Y_1)}{\partial Y_1}\right|_{\substack{\mu_{\bar{X}}\\ \mu_{Y_1}}} = \frac{\frac{(2\ln 2)\alpha c}{\alpha-1}}{\left(\frac{2n^2\alpha^3c^2-2n\alpha^2c^2+n^2\alpha^3c^2-n^2\alpha^2c^2}{(n\alpha-1)^2(\alpha-1)}\right)\left[\ln\left(\ln\frac{3n\alpha-(n+2)}{3n(\alpha-1)}\right)\right]^2}$$

$$\left.\frac{\partial g(\bar{X},Y_1)}{\partial Y_1}\right|_{\substack{\mu_{\bar{X}}\\ \mu_{Y_1}}} = \frac{(2\ln 2)\alpha c}{\left(\frac{3n^2\alpha^3c^2-n\alpha^2c^2(n+2)}{(n\alpha-1)^2}\right)\left[\ln\left(\ln\frac{3n\alpha-(n+2)}{3n(\alpha-1)}\right)\right]^2}$$

$$\left.\frac{\partial g(\bar{X},Y_1)}{\partial Y_1}\right|_{\substack{\mu_{\bar{X}}\\ \mu_{Y_1}}} = \frac{2\ln 2}{\left(\frac{3n^2\alpha^2c-n\alpha c(n+2)}{(n\alpha-1)^2}\right)\left[\ln\left(\ln\frac{3n\alpha-(n+2)}{3n(\alpha-1)}\right)\right]^2} \qquad (24)$$

$$\frac{\partial^2 g(\overline{X},Y_1)}{\partial Y_1^2}=\frac{-2\overline{X}\ln 2\left[(2\overline{X}y_1+y_1^2)\frac{6y_1}{2\overline{X}+y_1}\left(\frac{3y_1-3(2\overline{X}+y_1)}{9y_1^2}\right)\left(\ln\left(\frac{2\overline{X}+y_1}{3y_1}\right)\right)+\left(\ln\left(\frac{2\overline{X}+y_1}{3y_1}\right)\right)^2(2\overline{X}+2y_1)\right]}{(2\overline{X}y_1+y_1^{\,2})^2\left(\ln\left(\frac{2\overline{X}+y_1}{3y_1}\right)\right)^4}$$

$$\frac{\partial^2 g(\overline{X},Y_1)}{\partial Y_1^2}=\frac{2\overline{X}\ln 2\left[(2\overline{X}y_1+y_1^2)\frac{2y_1}{2\overline{X}+y_1}\left(\frac{y_1-(2\overline{X}+y_1)}{y_1^2}\right)-\left(\ln\left(\frac{2\overline{X}+y_1}{3y_1}\right)\right)(2\overline{X}+2y_1)\right]}{(2\overline{X}y_1+y_1^{\,2})^2\left(\ln\left(\frac{2\overline{X}+y_1}{3y_1}\right)\right)^3}$$

$$\frac{\partial^2 g(\overline{X},Y_1)}{\partial Y_1^2}=\frac{2\overline{X}\ln 2\left[(4\overline{X})-\left(\ln\left(\frac{2\overline{X}+y_1}{3y_1}\right)\right)(2\overline{X}+2y_1)\right]}{(2\overline{X}y_1+y_1^{\,2})^2\left(\ln\left(\frac{2\overline{X}+y_1}{3y_1}\right)\right)^3}$$

$$\left.\frac{\partial^2 g(\overline{X},Y_1)}{\partial Y_1^2}\right|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}}=\frac{\frac{2(\ln 2)\alpha c}{\alpha-1}\left[\frac{4\alpha c}{\alpha-1}-\left(\ln\left(\frac{\frac{2\alpha c}{\alpha-1}+\frac{n\alpha c}{n\alpha-1}}{\frac{3n\alpha c}{n\alpha-1}}\right)\right)\left(\frac{2\alpha c}{\alpha-1}+\frac{2n\alpha c}{n\alpha-1}\right)\right]}{\left(\frac{2n\alpha^2c^2}{(\alpha-1)(n\alpha-1)}+\frac{n^2\alpha^2c^2}{(n\alpha-1)^2}\right)\left(\ln\left(\frac{\frac{2\alpha c}{\alpha-1}+\frac{n\alpha c}{n\alpha-1}}{\frac{3n\alpha c}{n\alpha-1}}\right)\right)^3}$$

$$\left.\frac{\partial^2 g(\overline{X},Y_1)}{\partial Y_1^2}\right|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}}=\frac{\frac{2(\ln 2)\alpha^2c^2}{\alpha-1}\left[\frac{4}{\alpha-1}-\left(\ln\left(\frac{3n\alpha-(n+2)}{3n(\alpha-1)}\right)\right)\left(\frac{2(n\alpha-1)}{(\alpha-1)(n\alpha-1)}+\frac{2n(\alpha-1)}{(n\alpha-1)(\alpha-1)}\right)\right]}{\alpha^2c^2\left(\frac{2n^2(n\alpha-1)+n^2(\alpha-1)}{(\alpha-1)(n\alpha-1)^2}\right)\left(\ln\left(\frac{3n\alpha-(n+2)}{3n(\alpha-1)}\right)\right)^3}$$

$$\left.\frac{\partial^2 g(\bar{X},Y_1)}{\partial Y_1^2}\right|_{\substack{\mu_{\bar{X}}\\ \mu_{Y_1}}} = \frac{\frac{2(\ln 2)\alpha^2 c^2}{\alpha-1}\left[\frac{4}{\alpha-1}-\left(\ln\left(\frac{3n\alpha-(n+2)}{3n(\alpha-1)}\right)\right)\left(\frac{2(n\alpha-1)}{(\alpha-1)(n\alpha-1)}+\frac{2n(\alpha-1)}{(n\alpha-1)(\alpha-1)}\right)\right]}{\alpha^2 c^2\left(\frac{2n^2(n\alpha-1)+n^2(\alpha-1)}{(\alpha-1)(n\alpha-1)^2}\right)\left(\ln\left(\frac{3n\alpha-(n+2)}{3n(\alpha-1)}\right)\right)^3}$$

$$\left.\frac{\partial^2 g(\bar{X},Y_1)}{\partial Y_1^2}\right|_{\substack{\mu_{\bar{X}}\\ \mu_{Y_1}}} = \frac{\frac{2(\ln 2)\alpha^2 c^2}{\alpha-1}\left[\frac{4}{\alpha-1}-\left(\ln\left(\frac{3n\alpha-(n+2)}{3n(\alpha-1)}\right)\right)\left(\frac{2(n\alpha-1)}{(\alpha-1)(n\alpha-1)}+\frac{2n(\alpha-1)}{(n\alpha-1)(\alpha-1)}\right)\right]}{\alpha^2 c^2\left(\frac{2n^2(n\alpha-1)+n^2(\alpha-1)}{(\alpha-1)(n\alpha-1)^2}\right)\left(\ln\left(\frac{3n\alpha-(n+2)}{3n(\alpha-1)}\right)\right)^3}$$

$$\left.\frac{\partial^2 g(\bar{X},Y_1)}{\partial Y_1^2}\right|_{\substack{\mu_{\bar{X}}\\ \mu_{Y_1}}} = \frac{\frac{2(\ln 2)}{\alpha-1}\left[\frac{4}{\alpha-1}-\left(\ln\left(\frac{3n\alpha-(n+2)}{3n(\alpha-1)}\right)\right)\left(\frac{2n\alpha-2}{(\alpha-1)(n\alpha-1)}+\frac{2n\alpha-2n}{(n\alpha-1)(\alpha-1)}\right)\right]}{\left(\frac{2n^3\alpha-2n^2+n^2\alpha-n^2}{(\alpha-1)(n\alpha-1)^2}\right)\left(\ln\left(\frac{3n\alpha-(n+2)}{3n(\alpha-1)}\right)\right)^3}$$

$$\left.\frac{\partial^2 g(\bar{X},Y_1)}{\partial Y_1^2}\right|_{\substack{\mu_{\bar{X}}\\ \mu_{Y_1}}} = \frac{\frac{2(\ln 2)}{\alpha-1}\left[\frac{4}{\alpha-1}-\left(\ln\left(\frac{\alpha+n-2}{3n(\alpha-1)}\right)\right)\left(\frac{2n-2}{(\alpha-1)(n\alpha-1)}\right)\right]}{\left(\frac{2n^2}{(\alpha-1)(n\alpha-1)}-\frac{n^2}{(n\alpha-1)^2}\right)\left(\ln\left(\frac{\alpha+n-2}{3n(\alpha-1)}\right)\right)^3} \tag{25}$$

$$\frac{\partial^2 g(\bar{X},Y_1)}{\partial \bar{X}\partial Y_1} = \frac{-2\ln 2\left[(2\bar{X}+y_1).2\left(\ln\frac{2\bar{X}+y_1}{3y_1}\right)\frac{3y_1-(2\bar{X}+y_1).3}{9y_1^2}+\left(\ln\left(\frac{2\bar{X}+y_1}{3y_1}\right)\right)^2\right]}{(2\bar{X}+y_1)^2\left(\ln\left(\frac{2\bar{X}+y_1}{3y_1}\right)\right)^4}$$

$$\frac{\partial^2 g(\bar{X},Y_1)}{\partial \bar{X}\partial Y_1} = \frac{2\ln 2\left[(2\bar{X}+y_1)\frac{4\bar{X}}{3y_1^2}-\left(\ln\left(\frac{2\bar{X}+y_1}{3y_1}\right)\right)\right]}{(2\bar{X}+y_1)^2\left(\ln\left(\frac{2\bar{X}+y_1}{3y_1}\right)\right)^3}$$

$$\left.\frac{\partial^2 g(\bar{X},Y_1)}{\partial \bar{X}\partial Y_1}\right|_{\substack{\mu_{\bar{X}}\\ \mu_{Y_1}}} = \frac{2\ln 2\left[\frac{3n\alpha^2 c-\alpha c(n+2)}{(\alpha-1)(n\alpha-1)}\cdot\frac{\frac{4\alpha c}{\alpha-1}}{\frac{6n\alpha c}{n\alpha-1}}-\ln\left(\frac{3n\alpha-(n+2)}{3n(\alpha-1)}\right)\right]}{\left(\frac{3n\alpha^2 c-\alpha c(n+2)}{(\alpha-1)(n\alpha-1)}\right)^2\left(\ln\left(\frac{3n\alpha-(n+2)}{3n(\alpha-1)}\right)\right)^3}$$

$$\left.\frac{\partial^2 g(\bar{X},Y_1)}{\partial\bar{X}\partial Y_1}\right|_{\substack{\mu_{\bar{X}}\\ \mu_{Y_1}}} = \frac{2\ln 2\left[\frac{4(3n\alpha^2c-\alpha c(n+2))}{6n(\alpha-1)^2}-\ln\left(\frac{3n\alpha-(n+2)}{3n(\alpha-1)}\right)\right]}{\left(\frac{3n\alpha^2c-\alpha c(n+2)}{(\alpha-1)(n\alpha-1)}\right)^2\left(\ln\left(\frac{3n\alpha-(n+2)}{3n(\alpha-1)}\right)\right)^3} \qquad (26)$$

In general $\bar{X}$ converge stochastically to μ_x and according to theorem (1.6), Y_1 converge stochastically to c. So $\bar{X}Y_1$ converge stochastically to $\mu_x c$, [23], therefore

$$\mu_{\bar{X}} = \mu = \frac{\alpha c}{\alpha-1}$$

$$E\left[\overline{XY_1}\right] = \mu_{\bar{X}} c = \frac{\alpha c^2}{\alpha-1}$$

$$Cov[\bar{X},Y_1] \cong E[\overline{XY_1}] - E[\bar{X}]E[Y_1] = \frac{\alpha c^2}{\alpha-1} - \frac{\alpha c}{(\alpha-1)}\frac{n\alpha c}{(n\alpha-1)}$$

$$Cov[\bar{X},Y_1] \cong \frac{-\alpha c^2}{(\alpha-1)(n\alpha-1)} \qquad (25)$$

Butting eq. (19), (18), (21), (13), (23), (24), (25) in eq. (16)

$$E\left[g(\bar{X},Y_1)\right] = \frac{\ln 2}{\ln\left(\frac{(3n\alpha-(n+2))}{3n(\alpha-1)}\right)} + \frac{1}{2}\frac{\alpha c^2}{n(\alpha-1)^2(\alpha-2)}\left(\frac{4\ln 2\left[2+\ln\left(\frac{\alpha+n-2}{3n(\alpha-1)}\right)\right]}{\left(\frac{3n\alpha^2c-\alpha c(n+2)}{(\alpha-1)(n\alpha-1)}\right)^2\left(\ln\frac{3n\alpha-(n+2)}{3n(\alpha-1)}\right)^3}\right)$$

$$+\frac{1}{2}\left(\frac{n\alpha c^2}{(n\alpha-1)^2(n\alpha-2)}\right)\frac{\frac{2(\ln 2)}{\alpha-1}\left[\frac{4}{\alpha-1}-\left(\ln\left(\frac{\alpha+n-2}{3n(\alpha-1)}\right)\right)\left(\frac{2n-2}{(\alpha-1)(n\alpha-1)}\right)\right]}{\left(\frac{2n^2}{(\alpha-1)(n\alpha-1)}-\frac{n^2}{(n\alpha-1)^2}\right)\left(\ln\left(\frac{\alpha+n-2}{3n(\alpha-1)}\right)\right)^3}$$

$$+\left(\frac{-\alpha c^2}{(\alpha-1)(n\alpha-1)}\right)\frac{2\ln 2\left[\frac{4(3n\alpha^2c-\alpha c(n+2))}{6n(\alpha-1)^2}-\ln\left(\frac{3n\alpha-(n+2)}{3n(\alpha-1)}\right)\right]}{\left(\frac{3n\alpha^2c-\alpha c(n+2)}{(\alpha-1)(n\alpha-1)}\right)^2\left(\ln\left(\frac{3n\alpha-(n+2)}{3n(\alpha-1)}\right)\right)^3}$$

Butting eq. (18), (20), (13), (22), (26) in eq. (17)

$$Var\left[g\left(\bar{X},Y_1\right)\right]=\frac{\alpha c^2}{n(\alpha-1)^2(\alpha-2)}\left[-\frac{2\ln 2}{\left(\frac{n\alpha^2c+\alpha c(n-2)}{(\alpha-1)(n\alpha-1)}\right)\left(\ln\left(\frac{3n\alpha-(n+2)}{3n(\alpha-1)}\right)\right)^2}\right]^2$$

$$+\frac{n\alpha c^2}{(n\alpha-1)^2(n\alpha-2)}\left[\frac{2\ln 2}{\left(\frac{3n^2\alpha^2c-n\alpha c(n+2)}{(n\alpha-1)^2}\right)\left[\ln\left(\ln\frac{3n\alpha-(n+2)}{3n(\alpha-1)}\right)\right]^2}\right]^2+2\left(\frac{-\alpha c^2}{(\alpha-1)(n\alpha-1)}\right)$$

$$\left(-\frac{2\ln 2}{\left(\frac{n\alpha^2c+\alpha c(n-2)}{(\alpha-1)(n\alpha-1)}\right)\left(\ln\left(\frac{3n\alpha-(n+2)}{3n(\alpha-1)}\right)\right)^2}\right)\left(\frac{2\ln 2}{\left(\frac{3n^2\alpha^2c-n\alpha c(n+2)}{(n\alpha-1)^2}\right)\left[\ln\left(\ln\frac{3n\alpha-(n+2)}{3n(\alpha-1)}\right)\right]^2}\right)$$

$$Var\left[g\left(\bar{X},Y_1\right)\right]=\alpha c^2\ln 2\;\frac{1}{n(\alpha-1)^2(\alpha-2)}\left[-\frac{2\ln 2}{\left(\frac{n\alpha^2c+\alpha c(n-2)}{(\alpha-1)(n\alpha-1)}\right)\left(\ln\left(\frac{3n\alpha-(n+2)}{3n(\alpha-1)}\right)\right)^2}\right]^2$$

$$+\frac{2}{\left(\frac{3n^2\alpha^2c-n\alpha c(n+2)}{(n\alpha-1)^3}\right)\left[\ln\left(\ln\frac{3n\alpha-(n+2)}{3n(\alpha-1)}\right)\right]^2}$$

$$\left[\frac{n}{(n\alpha-1)(n\alpha-2)}+\frac{2}{(\alpha-1)\left(\frac{n\alpha^2c+\alpha c(n-2)}{(\alpha-1)(n\alpha-1)}\right)\left(\ln\left(\frac{3n\alpha-(n+2)}{3n(\alpha-1)}\right)\right)^2}\right]$$

CONCLUSIONS AND RECOMMENDATIONS

We can conclude the following:

1. We can expand using the approximation method to find the mean and the variance of other capabilities for Pareto distribution or other distributions.

2. The presence of the function $g\left(\overline{X}, Y_1\right) = \hat{\alpha} = \dfrac{\ln 2}{\ln\left(\dfrac{2\overline{X} + y_1}{3y_1}\right)}$ is difficult in the process of finding the mean approximation and variance due to the difficulty of separating $\overline{X}$ from y_1 and thus the difficulty of simplifying the equation.

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