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COEFFICIENT ESTIMATES OF CLASSES OF q-STARLIKE AND q-CONVEX FUNCTIONS

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ABSTRACT. In this paper, we introduce new classes of q-starlike and q-convex functions involving the q-derivative operator defined on the open unit disk. Furthermore, we obtain estimates on the second and third coefficients of these classes.

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1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. If f and g are analytic functions in $\mathbb{U}$, we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in $\mathbb{U}$, with $w(0) = 0, |w(z)| < 1$, for all $z \in \mathbb{U}$, such that $f(z) = g(w(z))$ for all $z \in \mathbb{U}$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subseteq g(\mathbb{U})$.

For a function $f \in \mathcal{A}$ given by (1) and $0 < q < 1$, the q-derivative of a function f is defined by ([2, 14]).

$$D_q(f(z)) = \frac{f(qz) - f(z)}{(q-1)z}, \quad q \neq 1, z \neq 0, \tag{2}$$

$D_q(f(0)) = f'(0)$. From (2), we deduce that

$$D_q(f(z)) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1},$$

where

$$[k]_q = \frac{1-q^k}{1-q}.$$

As $q \to 1, [k]_q \to k$. For a function $h(z) = z^k$, we observe that

$$\begin{aligned} D_q(h(z)) &= D_q(z^k) = \frac{1-q^k}{1-q} z^{k-1} = [k]_q z^{k-1} \\ \lim_{q\to 1}(D_q(h(z))) &= \lim_{q\to 1}\left([k]_q z^{k-1}\right) = kz^{k-1} = h'(z), \end{aligned}$$

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where h' is the ordinary derivative.
Making use of the q-derivative, we introduce the subclasses $\mathcal{S}_q^*(\alpha)$ and $\mathcal{C}_q(\alpha)$ of the class $\mathcal{A}$ for $0 \leq \alpha < 1$ which are defined by

$$\begin{aligned} \mathcal{S}_q^*(\alpha) &= \left\{ f \in \mathcal{A} : Re\left(\frac{zD_q(f(z))}{f(z)}\right) > \alpha, z \in \mathbb{U} \right\}, \\ \mathcal{C}_q(\alpha) &= \left\{ f \in \mathcal{A} : Re\left(1 + \frac{zqD_q(D_q(f(z)))}{D_q f(z)}\right) > \alpha, z \in \mathbb{U} \right\}. \end{aligned}$$

We note that

$$f \in \mathcal{C}_q(\alpha) \Leftrightarrow f \in \mathcal{S}_q^*(\alpha),$$

and

$$\begin{aligned} \lim_{q\to 1} \mathcal{S}_q^*(\alpha) &= \left\{ f \in \mathcal{A} : \lim_{q\to 1} Re\left(\frac{zD_q(f(z))}{f(z)}\right) > \alpha, z \in \mathbb{U} \right\} = \mathcal{S}^*(\alpha), \\ \lim_{q\to 1} \mathcal{C}_q(\alpha) &= \left\{ f \in \mathcal{A} : \lim_{q\to 1} Re\left(1 + \frac{zqD_q(D_q(f(z)))}{D_q f(z)}\right) > \alpha, z \in \mathbb{U} \right\} = \mathcal{C}(\alpha), \end{aligned}$$

where $\mathcal{S}^*(\alpha)$, $\mathcal{C}(\alpha)$ are, respectively, the classes of starlike of order α and convex of order α in $\mathbb{U}$ ([20]).

Now using the q-derivative of a function $f \in \mathcal{A}$ and the principle of subordination, we introduce the following classes of q-starlike and q-convex analytic functions.

Definition 1.1. *Let P be the class of all functions φ which are analytic and univalent in $\mathbb{U}$ and for which φ is convex with $\varphi(0) = 1$ and $Re(\phi(z)) > 0$ for $z \in \mathbb{U}$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_q^*(\varphi)$ if it satisfies the following subordination condition:*

(3)
$$\frac{z\, D_q f(z)}{f(z)} \prec \varphi(z), (\varphi \in P).$$

Definition 1.2. *A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{C}_q(\varphi)$ if it satisfies the following subordination condition:*

(4)
$$1 + \frac{zq\, D_q(D_q(f(z)))}{D_q(f(z))} \prec \varphi(z), (\varphi \in P).$$

We note that:

i: $\lim_{q\to 1} \mathcal{S}_q^*(\varphi) = \mathcal{S}^*(\varphi)$ and $\lim_{q\to 1} \mathcal{C}_q(\varphi) = \mathcal{C}(\varphi)$ (see [15])

ii: $\lim_{q\to 1} \mathcal{S}_q^*(\frac{1+z}{1-z}) = \mathcal{S}^*$ and $\lim_{q\to 1} \mathcal{C}_q(\frac{1+z}{1-z}) = \mathcal{C}$ (see [16, 18])

In order to establish our main results, we need the following lemma:

Lemma 1.1. [17] *If $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is a function with positive real part in $\mathbb{U}$ and μ a complex number, then*

$$|c_2 - \mu c_1^2| \leq 2 \max\{1; |2\mu - 1|\}.$$

The result is sharp for functions given by

$$p(z) = \frac{1+z}{1-z} \quad \textit{and} \quad p(z) = \frac{1+z^2}{1-z^2}.$$

2. Main Results

We assume throughout this paper that the function $\varphi \in P$, and $0 < q < 1$.

Theorem 2.1. *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots \in P$. If f given by (1) in the class $\mathcal{S}_q^*(\varphi)$ and μ is a complex number, then*

$$|a_3 - \mu a_2^2| \le \frac{B_1}{q[2]_q} \max\left\{1, \left|\frac{B_2}{B_1} + \frac{(1-[2]_q\mu}{q} B_1\right|\right\}. \tag{5}$$

The result is sharp.

Proof. If $f \in \mathcal{S}_q^*(\varphi)$, then there is a Schwarz function w in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ in $\mathbb{U}$ such that

$$\frac{z\, D_q(f(z))}{f(z)} = \varphi(w(z)). \tag{6}$$

Define the function $p(z)$ by

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + p_1 z + p_2 z^2 + \dots. \tag{7}$$

Since w is a Schwarz function, we see that $Re\,(p(z)) > 0$ and $p(0) = 1$. Define

$$g(z) = \frac{z\, D_q(f(z))}{f(z)} = 1 + d_1 z + d_2 z^2 + \dots. \tag{8}$$

In view of (6), (7) and (8), we have

$$g(z) = \varphi\left(\frac{p(z) - 1}{p(z) + 1}\right). \tag{9}$$

Since

$$\frac{p(z) - 1}{p(z) + 1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \left(p_3 + \frac{p_1^3}{4} - p_1 p_2\right) z^3 + \dots\right].$$

Therefore, we have

$$\varphi\left(\frac{p(z) - 1}{p(z) + 1}\right) = 1 + \frac{1}{2} B_1 p_1 z + \left[\frac{1}{2} B_1 \left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4} B_2 p_1^2\right] z^2 + \dots,$$

from the last equation and (8), we obtain

$$d_1 = \frac{1}{2} B_1 p_1,$$

and

$$d_2 = \frac{1}{2} B_1 \left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4} B_2 p_1^2.$$

A computation shows that

$$\frac{z\, D_q(f(z))}{f(z)} = 1 + q a_2 z + \{q[2]_q a_3 - q a_2^2)\} z^2 + \cdots.$$

Then, from (8), we find that

$$d_1 = q a_2 \quad \text{and} \quad d_2 = q[2]_q a_3 - q a_2^2,$$

or equivalently we have

$$a_2 = \frac{B_1 p_1}{2q},$$

and

$$a_3 = \frac{B_1}{2q[2]_q}(p_2 - \frac{p_1^2}{2}) + \frac{B_2 p_1^2}{4q[2]_q} + \frac{B_1^2 p_1^2}{4q^2[2]_q}.$$

Therefore

$$a_3 - \mu a_2^2 = \frac{B_1}{2q[2]_q}\{p_2 - \nu p_1^2\},$$

where

$$\nu = \frac{1}{2}\Big[1 - \frac{B_2}{B_1} - \frac{B_1}{q} + \frac{[2]_q \mu}{q} B_1\Big].$$

Our result now is followed by an application of Lemma 1.1. This completes the proof of Theorem 2.1.

Similarly, we can prove the following theorem for the class $\mathcal{C}_q(\varphi)$.

Theorem 2.2. *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + ... \in P$. If f given by (1) in the class $\mathcal{C}_q(\varphi)$ and μ is a complex number, then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{[2]_q[3]_q} \max\left\{1, \left|\frac{B_2}{B_1} + \frac{[2]_q - [3]_q \mu}{q[2]_q} B_1\right|\right\}.$$

The result is sharp.

Taking $q \to 1$ in Theorems 2.1 and 2.2 , we obtain the following results for functions belonging to $\mathcal{S}_q^*(\varphi)$ and $\mathcal{C}_q(\varphi)$.

Corollary 2.1. *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + ...$ with $B_1 \neq 0$. If f given by (1) in the class $\mathcal{S}^*(\varphi)$ and μ is a complex number, then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{2} \max\left\{1, \left|\frac{B_2}{B_1} + (1 - 2\mu) B_1\right|\right\}. \tag{10}$$

The result is sharp.

Corollary 2.2. *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + ...$ with $B_1 \neq 0$. If f given by (1) is in the class $\mathcal{C}(\varphi)$ and μ is a complex number, then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{6} \max\left\{1, \left|\frac{B_2}{B_1} + \left(1 - \frac{3}{2}\mu\right) B_1\right|\right\}.$$

The result is sharp.

3. Remarks

It is known that quantum calculus (q-calculus) is an ordinary classical calculus without the notion of limits. It defines as h-calculus and q-calculus. Here h stands for Planck's, while q stands for quantum. The application of q-calculus was perhaps initiated by Jackson ([1], [2]). In fact, he was the first to develop q-integral and q-derivative in a systematic way. Later, in the 80's, geometrical interpretation of q-analysis is recognized through studies on quantum groups. It also suggests a relation between integrable systems and q-analysis. In ([4, 5, 6]), the q-analogue of Baskakov Durrmeyer operator has been proposed, which is based on q-analogue of beta function. Other important q-generalization of complex operators are the q-Picard and q-Gauss-Weierstrass singular integral operators discussed in [7],[8] and [9]. The authors studied approximation and geometric properties of these q-operators in some subclasses of analytic functions in a compact disk. Very recently, other q-analogues of differential operators have been introduced in [11] see also ([12, 13, 19]). These q-operators are defined by using convolution of normalized analytic functions and q-hypergeometric functions, where several interesting results are obtained. Some of them are the ones in the above results. Some other work related to q-Differentiation, q-hypergeometric can also be looked up in [3, 10].

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