On generalization of ωb-open sets In Topological Spaces

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للنشر والتوزيع

On generalization of ωb-open sets In Topological Spaces

By

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INKY

بِسْمِ ٱللَّهِ ٱلرَّحْمَٰنِ ٱلرَّحِيمِ

﴿ قَالُواْ سُبْحَٰنَكَ لَا عِلْمَ لَنَآ إِلَّا مَا عَلَّمْتَنَآۖ

إِنَّكَ أَنتَ ٱلْعَلِيمُ ٱلْحَكِيمُ ﴾

صَدَقَ ٱللَّهُ ٱلْعَلِيُّ ٱلْعَظِيمُ

سورة البقرة (٣٢)

الإهداء

إلى أولئك الرجال

إلى من أقرضوا الله قرضاً حسناً

إلى من جاهدوا حتى تكون كلمة الله هي العليا

إلى شهداء العراق أهدي بحثي هذا ، ، لاسيما أخي الشهيد السيد أسامة عباس ناصر الصافي

زين العابدين

Introduction

This thesis introduces some concepts in general topology which are the concepts of separation axioms, connected,compactspace and lindelof space by usingω-open set.In the 1982[18] Hdeib introd-

uced the concept of ω-open sets in topological spaces .In 1996 [4] Andrjivic gave a new type of generalized open set in topological space called b-

open sets .Finally [22] 2008, Noiri,Al-Omari and Noorani introduced the concepts ofω-open and the complement ofω-set ω-

.In [23]. and [27]

concept of separation axioms In [15].M.C.

Gemignani studied the concept of connected space In[7] Burbaki studied the concept of compact space In[9],[19] studied the concept of countably S-closed and S-closed.In [13].R.Engleking studied the conce-

Pt of lindelof spaces,In[11] E.Ekici S- lindelof

we introduced the definition of the concept of -connected space,the definition of the concept of-compact space, countably -compact and the definition of the concept of -lindelof space which turns out to be equivalent to -lindelof andlindelof

space ,where they studied continuity by using these sets. This thesis consists of three chapters ,Chapter one is divided into two sections, Section

One deals with the basic definition have been,recall and results Section two we mentioned definition of -continuous function and prove some properties about it.Chapter two consists of three sections, Section one, we the defined of -closed and -

open functions and proves some results Section two

,we introduced fundamental. concepts of separation axioms and generalized by -open set also we prove some relations among them .In section three

,we explain the concept of -connected space and give some Generalization about it .Chapter three is divided into two. sections ,Section one,we the concept

of -compact space and give some important generalizations on this concept,In this section also

introduces a newconcept namely nearly compact space and we prove some results about it.In section two, we recall definition, proposition and theorems of -lindelof space, and also we introduce the concept nearly -lindelof space, moreover,we prove some results about it .

Chapter one

On Basic Definitions and Results

Introduction

This chapter consists of two sections In section one we mentioned some of the basic definitions which are needed in this thesis We introduce a new class of set called ωb*-open set Section two,we introduce the definition of ωb-continuous function and prove some properties about it,we discusses composition of ωb-continuous function and restriction function by ωb-open set .

1.1 On ωb-open Set

The section introduces a new class of sets called ωb-open set and give examples, remarks and propositions about this class .

Definition (1.1.1): [18]

A subset A is said to be ω-open set if for each $x \in A$ there exists an open set U_x such that $x \in U_x$ and U_x-A is countable The complement of ω-open set is called ω-closed. The family of ω-open sets denoted by ωO(X) .

Definition (1.1.2): [4]

Let X be topological space A is called b-open set in X, iff $A \subseteq \overline{\mathring{A}} \cup \overset{\circ}{\overline{A}}$ the complement of b-open set is called b-closed and it is easy to see that A is b-closed set iff $\overline{\mathring{A}} \cap \overset{\circ}{\overline{A}} \subseteq A$ the family of all b-open sub sets of aspace is denoted by BO(X) .

Proposition (1.1.3): **[3]**

Let $A \subseteq X$ then the following statements are equivalent: -

1- A is b-closed .

2- $\overline{\mathring{A}} \cap \overline{A}^{\circ} \subseteq A$.

Remarks (1.1.4):

It is clear every open set is b-open and the converse is not true in general

Let $X = \{1, 2, 3\}$, $\tau=\{X,\emptyset,\{2\}\}$,$BO(X) = \{X, \emptyset, \{2\}, \{1,2\}, \{2, 3\}$,then $\{2,3\}$ is b-open set but not open set .

Definition (1.1.5): **[22]**

A subsetA of a space X is said to be ωb-open,if for everyx $\in$ A,there exists a b-oopen subset $U_x \subseteq X$ containing x such thatU_x-A iscountable the comple- ment of an ωb-open subset is said to be ωb-closed, the family of all ωb-open sub sets of a space is denoted by ωBO(X).

We conclude from the above definition every ω-open set is ωb-open .

Lemma (1.1.6): [22]

For a subset of a topological space,both ω-openness and b-opennes imply ωb-opennes

Theorem (1.1.7): [22]

Let X be a space and $C \subseteq X$, If C is ωb-closed then $C \subseteq K \cup B$ for some b-closed subset K and a countable subset B.

Lemma (1.1.8): [22]

Asubset A of a space X is ωb-open if and only if for every $x \in A$,there exists a b-open subset U containing x and a countable subset C such that $U-C \subseteq A$

Remark (1.1.9): [22]

In any topological spaces

1- Any open set is ω-open

2- Any b-open is ωb-open

3- Any open set is ωb-open

In general the convers of above Remark (1.1.9) is not true in general as shown in the following

Examples (1.1.10):

1-Let $X = \{1, 2, 3\}$,$\tau = \{X, \emptyset, \{1\}, \{2\}, \{1,2\}, \{1,3\}\}$ then,$\{3\}$ is ω-open(since X is countable set) and is not open .

2- Let $X = N$,$\tau = \{A \subseteq X : A^c \text{ is finite}\} \cup \{\emptyset\}$, $BO(X)$ $= \{G: G \subseteq X, G \text{ is infinite and } G^c \text{ infinite}\} \cup$ $\{G: G \subseteq X, G^c \text{ is finite}\} \cup \{\emptyset\}$then $\{1\}$ is not b-open

thus, $\overline{\{1\}}^{\circ} \cup \overline{\{1\}^{\circ}} = \emptyset$

$\Longrightarrow \{1\} \nsubseteq \emptyset$ hence $\{1\}$ is not b-open,let $B = \{1\}$

Since $1 \in U$ =N-$\{2\}$ thus U is b-open set contain 1therefore,N-B is countable .

3- Let $X = \{a, b, c\}$,$\tau = \{X, \emptyset, \{b\}\}$,then $\{a, b\}$ is ωb-open (since X is a countable set) and it is not open.

Proposition (1.1.11): [4]

Let $\{A_\lambda\}_{\lambda\in\Lambda}$be acollection of b-open in a topological space X then,$\cup_{\lambda\in\Lambda} A_\lambda$ is b-open

Proposition (1.1.12): [22]

The union of any family of ωb-open sets is ωb-open.

Prposition (1.1.13):

1- The intersection of two ωb-open sets is not always ωb-open [**22**].

2- The intersection of b-open sets and open is b-open [**4**]

3-The intersection of ωb-open sets and ω-open is ωb-open [**22**]

4- The intersection of an ωb-open set and open set is ωb-open. [**22**]

5- The intersection of two ω-open sets is ω-open **[17]**

6- The intersection of ω-open sets and open is ω-open . [**21**]

7- The intersection of ωb-open set and b-open set is not ωb-open [**22**]

8- The intersection of two b-open sets is not always b-open **[3]**

Remark(1.1.14):

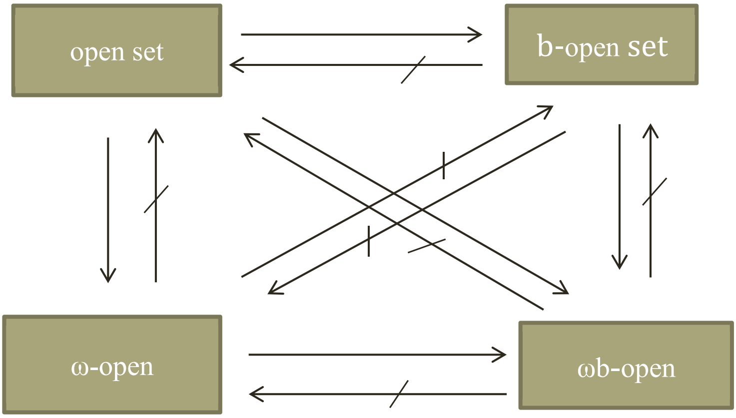
The conspest of b-open and ω-open are independent as the following example shows

Examples (1.1.15):

1- Let $X = \{1, 2, 3\}$, $\tau = \{X, \emptyset, \{2\}\}$, $BO(X) = \{X, \emptyset, \{2\}, \{1,2\}, \{2,3\}\}$ then, $\{3\}$ is ω-open (since X is a countable set) and it is not b-open.

2- Let X = R with the usual topology τ, Let A= Q be the set of all rational numbers Then A is b-open but it is not ω-open .

The following diagramshows the relation betweentypes of open set



<u>Definition (1.1.16):</u>

A subset A of a space X is said to be ωb^*-open,if for every $x \in A$,there exists a b-open subset $U_x \subseteq X$ containing x such that U_x-A is finite the complement of an ωb^*-open subset is said to be ωb^*-closed.

Remark (1.1.17):

Every closed set is ωb-closed set but the converse is not true as the following .

Example (1.1.18):

Let $X = \{a, b, c\}$,$\tau = \{X, \emptyset, \{a\}\}$ then ωb-closed set $= \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$ since X is countable set then $\{c\}$ is ωb-open thus, $\{a, b\}$is ωb-closed set but $\{a, b\}$ is not closed.

Definition (1.1.19):.[8]

Let X be a topological space and $A \subseteq X$, the b-closure of A is defined as the intersection of all b-closed sets in X,containing A and is denoted by $\overline{A}^{b}$ it is clear that $\overline{A}^{b}$ is b-closed set for any subset A of X and $A \subseteq \overline{A}^{b}$.

Definition (1.1.20):

Let X be a space and $A \subseteq X$,the intersection of all ωb-closed sets of X containing

A is called ωb-closure of A defined by $\overline{A}^{\omega b} = \cap$ $\{B: B \text{ is } \omega b\text{-closed in } X \text{ and } A \subseteq B\}$

Remark (1.1.21):

$\overline{A}^{\omega b}$ is the smallest ωb-closed set containing A

Proposition (1.1.22): [3]

Let X be a topological space and $A \subseteq X$ then, $x \in \overline{A}^{b}$ iff for each b-open set in X, contained point x, we have $U \cap A \neq \emptyset$.

Proposition (1.1.23):

Let X be a space and $A \subseteq B$ then.

1. $\overline{A}^{\omega b}$ is an ωb-closed set.
2. A is ωb-closed if and only if $A = \overline{A}^{\omega b}$.
3. $\overline{A}^{\omega b} = \overline{\overline{A}^{\omega b}}^{\omega b}$.
4. If $A \subseteq B$ then, $\overline{A}^{\omega b} \subseteq \overline{B}^{\omega b}$.
5. $\overline{A}^{\omega b} \subseteq \overline{A}$.

6. $\overline{A}^{\omega b} \subseteq \overline{A}^{b}$

Proof

1. By definition of ωb-closed set.

2. Let A be ωb-closed in X, since $A \subseteq \overline{A}^{\omega b}$ and $\overline{A}^{\omega b}$ is smallest ωb-closed set containing A then, $\overline{A}^{\omega b} \subseteq A$,thus $A = \overline{A}^{\omega b}$.

conversely:

Let $A = \overline{A}^{\omega b}$ Since $\overline{A}^{\omega b}$ is ωb-closed set therefore A is ωb-closed set

3.the prove complete From (1) and (2) .

4.Let $A \subseteq B$ since $B \subseteq \overline{B}^{\omega b}$ then $A \subseteq \overline{B}^{\omega b}$ but $\overline{A}^{\omega b}$ is smallest ωb-closed set containing A then, $\overline{A}^{\omega b} \subseteq \overline{B}^{\omega b}$.

5.Let $x \in \overline{A}^{\omega b}$ then,for all ωb-open set U such that $x \in U$,thus $U \cap A \neq \emptyset$ for all open set U hence $x \in U$ we have $U \cap A \neq \emptyset$ therefore $\overline{A}^{\omega b} \subseteq \overline{A}$.

6. Clear

Proposition (1.1.24):

Let X be a topological space and $A \subseteq X$ then $x \in \overline{A}^{\omega b}$ iff for each ωb-open U set in X contained point x we have $U \cap A \neq \emptyset$.

Proof.

Assume that$x \in \overline{A}^{\omega b}$ and let Ube ωb-open in X,such that $x \in U$ and suppose $U \cap A \neq \emptyset$ then $A \subseteq U^c$ since U ωb-open set in Xand $x \in U$,thusU^c ωb-closed set in X and $x \notin U^c$ and($\overline{A}^{\omega b}$ is the smallest ωb-closed set containing A)hence $\overline{A}^{\omega b} \subset U^c$ is contradiction

Therefore ,$x \notin \overline{A}^{\omega b}$

Conversely:

Suppose for each U is ωb-open set in X,such that $x \in U$ and $U \cap A \neq \emptyset$ to prove $x \in \overline{A}^{\omega b}$,let $x \notin \overline{A}^{\omega b}$ then, $x \in \left(\overline{A}^{\omega b}\right)^c$ Since that $\overline{A}^{\omega b}$ ωb-closed in X

$\left(\overline{A}^{\omega b}\right)^c$ is ωb-open in.X,and by the hypothes is we get

$\left(\overline{A}^{\omega b}\right)^c \cap A \neq \emptyset$ but $\left(\overline{A}^{\omega b}\right)^c \cap \overline{A}^{\omega b} = \emptyset$ then ,

$\left(\overline{A}^{\omega b}\right)^c \cap A = \emptyset$ this is contradiction since for every ωb-open set U in X,$U \cap A \neq \emptyset$.

Definition (1.1.25): [4]

Let X be topological space and $A \subseteq X$,the union of all b-open sets of X,contained A is called "b-Interior of A" denotedby $A^{\circ b}$, $A^{\circ b} = \cup \{B: B \text{ is b-open in X} \text{and } B \subseteq A\}$

Definition (1.1.26):

Let X be a space and $A \subseteq X$,the union of all ωb-open sets of X containing A is_called ωb-Interior of A denoted by $A^{\circ\omega b}$ or ωb-In(A) $A^{\circ\omega b} = \cup \{B: B \text{ is } \omega b\text{-open in X and } B \subseteq A\}$.

Remark (1.1.27):

$A^{\circ\omega b}$ is the largest ωb-open set containing A.

<u>Proposition (1.1.28):</u> [3]

Let X be a space and $A \subseteq X$ then,$x \in A^{\circ b}$ iff there exists b-open set G containing x such that $x \in G \subseteq A$.

<u>Proposition (1.1.29):</u>

Let X be a space and $A \subseteq X$ then,$x \in A^{\circ\omega b}$ if and only if there exists ωb-open set Gcontaining x such that $x \in G \subseteq A$.

Proof

Let $x \in A^{\circ\omega b}$ then $x \in \cup\, G$ such that G is ωb-open set and $x \in G \subseteq A$.

Conversely

Let there exists G ωb-open set such that $x \in G \subseteq A$

then $x \in\cup\, G, G \subseteq A$ and G ωb-open set then $x \in A^{\circ\omega b}$.

<u>Proposition (1.1.30):</u>

Let X be topological space and $A \subseteq B \subseteq X$ then

(i) $A^{\circ\omega b}$ is ωb-open set.

(ii) A is ωb-open if and only if $A = A^{\circ\omega b}$.

(iii) $A^{\circ} \subseteq A^{\circ\omega b}$.

(iv) $A^{\circ\omega b} = \left(A^{\circ\omega b}\right)^{\circ\omega b}$.

(v) if $A \subseteq B$, then $A^{\circ\omega b} \subseteq B^{\circ\omega b}$.

(vi) $A^{\circ b} \subseteq A^{\circ\omega b}$

proof

(i) and (ii) From def (1.1.26) .

(iii) Le $x \in A^{\circ\omega b}$ then there exists U open set such that $x \in U \subseteq A$ thus,$x \in A^{\circ\omega b}$

(iv) To prove this special From (i) and (ii)

(v) Let $x \in A^{\circ\omega b}$ then there exists V ωb-open set such that $x \in V \subseteq A$ by propostion (1.1.29) thus $A^{\circ\omega b} \subseteq B^{\circ\omega b}$

(vi) To prove this we use Proposition (1.1.28) and Proposition (1.1.29)

Proposition (1.1.31):

Let X be aspace and $A \subseteq X$, then

1) $(\overline{A}^{\omega b})^{c} = (A^{c})^{\circ\omega b}$.

2) $(A^{\circ\omega b})^c = \overline{(A^c)}^{\omega b}$.

Proof

1) since $A \subseteq \overline{A}^{\omega b}$ then, $(\overline{A}^{\omega b})^c \subseteq A^c$ and $\overline{A}^{\omega b}$ ωb-closed set in X thus, $(\overline{A}^{\omega b})^c$ is ωb-open set in X but $(A^c)^{\circ\omega b}$ is ωb-open set in X and $(A^c)^{\circ\omega b}$ A^c by using Propositionition(1.1.27)then $(\overline{A}^{\omega b})^c \subseteq (A^c)^{\circ\omega b}$…(1) now let $x \in (A^c)^{\circ\omega b}$ then there exists ωb-open set U in Xsuch that $x \in U \subseteq A^c$ to prove $x \in (\overline{A}^{\omega b})^c$,let $x \notin (\overline{A}^{\omega b})^c$,thus $x \in \overline{A}^{\omega b}$since $x \in U$ and U ωb-open set in X,therefore $U \cap A \neq \emptyset$ this is contradiction with $U \subseteq A^c$ so $x \in (\overline{A}^{\omega b})^c$ hence $(A^c)^{\circ\omega b} \subseteq (\overline{A}^{\omega b})^c$ …(2) from (1),(2) we get $(A^c)^{\circ\omega b} = (\overline{A}^{\omega b})^c$.

2) by using (1), $(\overline{A^c}^{\omega b})^c = A^{\circ\omega b}$ then,$(\overline{A^c}^{\omega b}) = (A^{\circ\omega b})^c$.

Definition (1.1.32): [14]

Let X be a space and $x \in X, A \subseteq X$ the point x is called b-limit point of A,if every b-open set containingx contains a point of A distinct from x we call the set of all b-limit point of A the b-derived set of A and denoted by $\acute{A}^{b}$ therefore,$x \in \acute{A}^{b}$ if and only if for every b-open set V in X such that $x \in V$ such that $(V \cap A) - \{x\} \neq \emptyset$.

Definition (1.1.33):

Let X be a space and $x \in X$, $A \subseteq X$ the point x is called ωb-limit point of A,if every ωb-open set containing x contains a point of A distinct from x,we call the set of all ωb-limitpoint of A the ωb-derived set of A and denoted by $\acute{A}^{\omega b}$ therefore, $x \in \acute{A}^{\omega b}$ if and only if for every ωb-open set V in X such that $x \in V$ such that $(V \cap A) - \{x\} \neq \emptyset$.

Proposition (1.1.34):

Let X be a space and $A \subseteq B \subseteq X$ then

1) $\overline{A}^{\omega b} = A \cup \acute{A}^{\omega b}$.

2) A ωb-closed set if and only if $\acute{A}^{\omega b} \subseteq A$.

3) $\acute{A}^{\omega b} \subseteq \acute{B}^{\omega b}$.

4) $\acute{A}^{\omega b} \subseteq \acute{A}^{b}$.

Proof.

1) $x \in \acute{A}^{\omega b}, x \notin \overline{A}^{\omega b}$ there exists ωb-open U,thus $x \in U$ such that $U \cap A = \emptyset, (U \cap A) - \{x\} \neq \emptyset$,then $x \notin \acute{A}^{\omega b}$ is contradiction,thus $x \in A^{-\omega b}$ hence $\acute{A}^{\omega b}\ \overline{A}^{\omega b}$, therefore $\acute{A}^{\omega b} \cup A \subseteq \overline{A}^{\omega b}$.

Conversely:

Let $x \in \overline{A}^{\omega b}$ then,either $x \in A$ or $x \notin A$, if $x \in A$,thus $x \in A \cup \acute{A}^{\omega b}$, if $x \notin A$ Since $x \in \overline{A}^{\omega b}$ for all U ωb-open set contains x such that $U \cap A \neq \emptyset$ since $x \notin A$, then $(U \cap A) - \{x\} \neq \emptyset$, $x \in \acute{A}^{\omega b}$ then,$x \in A \cup \acute{A}^{\omega b}$, therefore $\overline{A}^{\omega b} \subseteq A \cup \acute{A}^{\omega b}$.

2) Let A be an ωb-closed set, to prove $\acute{A}^{\omega b} \subseteq A$,let $x \notin A$ then $x \in A^c$,since A is ωb-closed set,then A^c is ωb-open set and $A \cap A^c = \emptyset$,thus $(A \cap A^c) - \{x\} = \emptyset$ hence $x \notin \acute{A}^{\omega b}$,thus $\acute{A}^{\omega b} \subseteq A$.

Conversely:

Let $\acute{A}^{\omega b} \subseteq A$, to prove A ωb-closed set,Since $\overline{A}^{\omega b} = A \cup \acute{A}^{\omega b}$ then,$\overline{A}^{\omega b} = A$ thus, A is ωb-closed set .

3) Let $x \in \acute{A}^{\omega b}$ then for all U ωb-open set contain x such that $(U \cap A) - \{x\} \neq \emptyset$, since $A \subseteq B$ then $(U \cap B) - \{x\} \neq \emptyset$ thus $x \in \acute{B}^{\omega b}$ therefore $\acute{A}^{\omega b} \subseteq \acute{B}^{\omega b}$.

4) Let $x \in \acute{A}^{\omega b}$ then,for every U ωb-open set contains x,hence $(U \cap A)$-$\{x\} \neq \emptyset$ thus,for every U b-open set contains x such that $(U \cap A)$-$\{x\} \neq \emptyset$ then,$x \in \acute{A}^{b}$ then $\acute{A}^{\omega b} \subseteq \acute{A}^{b}$.

Definition (1.1.35): [7]

Let X be a space and $B \subseteq X$,A neighborhood of B is any subset of X,which contains an.open set containing

B the neighborhood of a subset $\{x\}$ is also called neighborhood of the point x.

Definition (1.1.36):

Let X be a space and $B \subseteq X$,an ωb-neighborhood of B is any subset of X,which contains an ωb-open set containing B,the ωb-neighborhood of a subset $\{x\}$ is also called ωb-neighborhood of the point x.

Definition (1.1.37):

Let A be a subset of a space X,for each $x \in X$,then x is said to be ωb-boundary point of A,if each ωb-open $U_x$of x,we have $U_x \cap A \neq \emptyset$ and $U_x \cap A^c \neq \emptyset$,the set of all ωb-Boundary point of A is denoted by $b_{\omega b}(A)$.

Proposition (1.1.38):

Let X be a space and $A \subseteq X$ then:

1- $b_{\omega b}(A) = \overline{A}^{\omega b} \cap \overline{A^c}^{\omega b}$.

2- $A^{\circ \omega b} = A - b_{\omega b}(A)$.

3- $\overline{A}^{\omega b} = A \cup b_{\omega b}(A)$.

Proof

1- Let $x \in b_{\omega b}(A)$ if and only if for each ωb-open U in X,such that $x \in U, U_x \cap A \neq \emptyset$ and $U_x \cap A^c \neq \emptyset$ by Definition (1.1.37) $\Leftrightarrow x \in \overline{A}^{\omega b}$ and $x \in \overline{A^c}^{\omega b}$ by Proposition (1.1.24) $\Leftrightarrow x \in \overline{A}^{\omega b} \cap \overline{A^c}^{\omega b}$.

2- Let $x \in A^{\circ\omega b}$ then,$x \in A$ to prove $x \notin b_{\omega b}(A)$

Suppose $x \in b_{\omega b}(A)$ then for each ωb-open of U_x we have $U_x \cap A \neq \emptyset$ and $U_x \cap A^c \neq \emptyset$ since that $x \in A^{\circ\omega b}$ then,there exists ωb-open set V such that $x \in V \subseteq A$ by Proposition (1.1.29) Since $A \cap A^c = \emptyset$ and $V \subseteq A$ then,$V \cap A^c = \emptyset$ which is contradiction hence $x \notin b_{\omega b}(A)$ therefore, $A^{\circ\omega b}$

$\subseteq A - b_{\omega b}(A)$

Conversely:

Let $x \in A - b_{\omega b}(A)$ to prove $x \in A^{\circ\omega b}$ since $x \in A - b_{\omega b}(A)$ then, $x \in A$ and $x \notin b_{\omega b}(A)$ then,there exist ωb-open of x such that $V_x \cap A = \emptyset$ or $V_x \cap A^c = \emptyset$ Since $x \in V_x$ and $x \in A$ then,$V_x \cap A \neq \emptyset$ hence $V_x \cap A^c = \emptyset$, $V_x \subset A$ then,$x \in V \subseteq A$,therefore $x \in A^{\circ\omega b}$ by proposition (1.1.29)

3- Assume that $x \in \overline{A}^{\omega b}$ to prove $x \in A \cup b_{\omega b}(A)$ Suppose $x \notin A \cup b_{\omega b}(A)$,then $x \notin A$ and $x \notin b_{\omega b}(A)$,Since that $x \notin b_{\omega b}(A)$ then,there exists ωb-open U_x of x,thus $A \cap U_x = \emptyset$ or $A^c \cap U_x\emptyset$,Since that $x \notin A$ hence $x \in A^c$ and $A^c \cap A \neq \emptyset$ hence $A \cap U_x = \emptyset$, therefore $x \notin \overline{A}^{\omega b}$ which is a contradiction .

Conversely:

Let $x \in A \cup b_{\omega b}(A)$ to prove $x \in \overline{A}^{\omega b}$ since $x \in A \cup b_{\omega b}(A)$ thus,$x \in A$or $x \in b_{\omega b}(A)$if $x \in A$ then,$x \in \overline{A}^{\omega b}$,if $x \in b_{\omega b}(A)$ then,$x \in A^c \cap A^{c-\omega b}$ hence $x \in \overline{A}^{\omega b}$ therefore, $A \cup b_{\omega b}(A) \subseteq \overline{A}^{\omega b}$.

Definition (1.1.39):

Let Y be subspace of.space X,A subset B of space Y is said to be ωb-open set in Y,if for every $x \in B$,there exists a b-open subset U_x in Y contain x such that $U_x - B$ is a countable.

Proposition (1.1.40): [24]

Let X be a topological space and $Y \subseteq X$,if G is a b-open set in X and Y is an open set in X then,$G \cap Y$ is b-open set in Y.

Proposition (1.1.41): [14]

Let X be a topological space,let Y be an open subset of X and A is b-open set in Y, then A is b-open in X .

Proposition (1.1.42):

Let X be a topological space and $Y \subseteq X$,if G is ωb-open set in X and Y is an open set in X then,$G \cap Y$ is ωb-open set in Y.

Proof

Let $x \in G \cap Y$ then $x \in G$ since G is b-open set in X there exists a b-open set U_x in X contains x such that $U_x - G$ is countable by using Proposition (1.1.40) then,$U_x \cap Y$ is b-open set in Y,since $(U_x \cap Y) - (G \cap Y) \subseteq (U_x - G) \cap Y$,since $U_x - G$ is countable,then $(U_x - G) \cap Y$ is countable therefore, $(U_x \cap Y) - (G \cap Y)$ is countable hence $(G \cap Y)$ is ωb-open set in Y.

Corollary (1.1.43):

Let X be topological space Y be non-empty open in X,if B is an ωb-closed set in X , then $B \cap Y$ is ωb-closed set in Y.

Proof

Since B is an ωb-closed set in X,so B^c is an ωb-open set in X by using Proposition (1.1.42) $B^c \cap Y$ is an ωb-open set in Y then,$Y - (B^c \cap Y)$ is an ωb-closed set in Yand $Y - (B^c \cap Y) = Y \cap (B^c \cap Y)^c = Y \cap (B \cup Y^c)$ $= (Y \cap Y^c) \cup (B \cap Y) = B \cap Y$ isan ωb-closed set in Y.

<u>Proposition (1.1.44)</u>:

Let X be a topological space,let Y be an open subset ofXandA is ωb-open set inY,thenA is ωb-open set

Proof.

Let A be ωb-open set in Y,then there exists a b-open set U_x in Y, contains x such that $U_x - A$ is countable by using Proposition (1.1.41) thus U_x is b-open in X hence A is ωb-open in X .

<u>Corollary (1.1.45):</u>

Let X be space and Y be an open subset of X,if A is ωb-closed set in Y then,A is ωb-
closed set in X .

Proof

Since A is ωb-closed set in Y,then A^c is ωb-open set in Y,Since Y is open by using Proposition(1.1.44) thus,A^c is ωb-open set in X therefore,A is ωb-closed set in X.

Remark (1.1.46):

It is clear,if Y is open in X and $A \subseteq Y$ then, A is ωb-open (ωb-closed) inX, iff A is A is ωb-open(ωb-closed) in Y.

Definition (1.1.47): [5]

Let X be a topological space and $A \subseteq X$,A is called regular open set in X,if $A = \overline{A}^{\circ}$,The complement of regular open set is called regular closed and it is easy to see that A is regular closed if $A = \overline{A^{\circ}}$.

Definition (1.1.48): [3]

Let X be topological space and $A \subseteq X$,A is called b-regular open set in X,iff $A = \overline{A}^{b\,\circ b}$ the complement of b-regular open set is called b-regular closed and it is easy to see that A is b-regular closed set if $A = \overline{A^{\circ b}}^{b}$.

Definition (1.1.49):

Let X be topological space and $A \subseteq X$,A is called regular-ωb-open set in X,if$A = \overline{A}^{\omega b\,\circ\omega b}$ the complement of regular- ωb-open set is called regular-ωb-closed and it is easy to see that A is regular-ωb-closed set if $A = \overline{A^{\circ\omega b}}^{\omega b}$.

Proposition (1.1.50):

For any subset A of a topological space X,if A is an ωb-open set then, $\overline{A}^{\omega b\,\circ\omega b}$ is regular-ωb-open

Proof

Since $A^{\circ\omega b} \subseteq \overline{A}^{\omega b\,\circ\omega b}$ and since A is an ωb-open set then,$A = A^{\circ\omega b}$ hence $A \subseteq \overline{A}^{\omega b\,\circ\omega b}$

So that $\overline{A}^{\omega b\,\circ\omega b} \subseteq \overline{\overline{A}^{\omega b\,\circ\omega b}}^{\omega b\,\circ\omega b}$ since $\overline{A}^{\omega b\,\circ\omega b} \subseteq \overline{A}^{\omega b}$

then, $\overline{\overline{A}^{\omega b\,\circ\omega b}}^{\omega b} \subseteq \overline{\overline{A}^{\omega b}}^{\omega b} = \overline{A}^{\omega b}$ thus ,$\overline{\overline{A}^{\omega b\,\circ\omega b}}^{\omega b\,\circ\omega b}$

$\subseteq \overline{A}^{\omega b^{\circ \omega b}}$ hence $\overline{A}^{\omega b^{\circ \omega b}} = \overline{\overline{A}^{\omega b^{\circ \omega b}}}^{\omega b^{\circ \omega b}}$ $\therefore A^{-\omega b^{o\omega b}}$ is ωb-regular open set

Definition (1.1.51): [2]

A subset A is said to be ω-regular open if for each $x \in A$,there exists an regularopen set U_x containing x such that U_x-A iscountable,the complement of ω-regular open set is called ω-regular closed set.

Definition (1.1.52):

A subset A is said to be ωb-regular open if for each $x \in A$,there exists an b-regular open set G_x containing x such that G_x-A is countable the complement of ωb-regularopen set is called ωb-regular closed set.

Lemma (1.1.53):

A subset A of a topological space X is ωb-regular open iff for every $x \in A$ there exists an b-regular open set V_x containing x and acountable subset B such that $V_x - B \subseteq A$.

Proof .

Let A be ωb-regular open and $x \in A$ then,there exists b-regular open subset V_x containing x such that $V_x - A$ is countable,let $B = V_x - A = V_x \cap (X - A)$ Then ,$V_x - B \subseteq A$, .

conversely:

Let $x \in A$ then there exists an b-regular open subset V_x containing x and acountable subset B thus $V_x - B \subseteq A$ thus $V_x - A \subseteq B$ and $V_x - A$ is countable.set.

Proposition (1.1.54)

Let X be a topological space and $A \subseteq X$,if A is ωb-regular closed,then $A \subseteq K \cup B$.

for some b-regular closed subset K and countable subset B

Proof

If A is ωb-regular closed then $X - A$,is ωb-regular open and hence for every $x \in X - A$,there exists a b-regular open set U containing x and a countable set B thus,$U -$

$B \subseteq X - A$ thus,$A \subseteq X - (U - B) = X - \big(U \cap (X - B)\big) = (X - U) \cup B$,let $K = X - U$,then K is ωb-regular closed therefore $A \subseteq K \cup B$.

Definitin (1.1.55):

A subset A is said to be ωb*-regular open if for each $x \in A$,there exists an b-regular open set V_x containing x such that $V_x - A$ is finite set .

Lemma (1.1.56)

Let (X, τ) be topological space and $A \subseteq X$,A is ωb*-regular open if and only if for every $x \in A$,there exist an b-regular open set V_x containing x and a finite subset B such that $V_x - B \subseteq A$.

Proof

the same prove of lemma (1.1.53)

<u>Proposition (1.1.57):</u>

Let X be a space and A ⊆ X,if A is ωb*-regular closed then A ⊆ K ∪ B for some b-regularclosed subset K and a finite subset B.

Proof

the same prove of Proposition (1.1.54) .

1.2.Certain Types of ωb-continuous Functions

In this section,we reviewed the definition of ωb-continuous,remarks and propositions about this subject further more ,we mentioned some properties of ωb-irresolute function and it is relation with ωb-continuous function .

Definition (1.2.1): [7]

Let $f: X \to Y$ be a function of a space X into a space Y then,f is called A continuous. function if $f^{-1}(A)$ is an open set in X,for every open set A in Y.

Theorem (1.2.2): [25]

Let $f: X \to Y$ be a function of a space X into a space Y then:

i) f is a continuous function.

ii) $f^{-1}(A)$ is a closed set in X,for every closed set A in Y.

iii) $f(\overline{A}) \subseteq \overline{f(A)}$ for every set A of X.

iv) $\overline{f^{-1}(A)} \subseteq f^{-1}(\overline{A})$ for every set A of Y.

v) $f^{-1}(A^\circ) \subseteq (f^{-1}(A))^\circ$ for every set A of Y.

Now we review ageneral definition of the pervious concepts and prove some results.

Definition (1.2.3) [12]

Let $f: X \to Y$ be a function of a space X into a space Y then, f is called an b-
continuous function if $\mathrm{f}^{-1}(\mathrm{A})$ is an b-open set in X,for every open set A inY .

Definition (1.2.4): [22]

Let $f: X \to Y$ be a function of a space X into a space Y then,f is called an ωb-continuous function if $f^{-1}(A)$ is an ωb-open set in X, for every open set A in Y.

Proposition (1.2.5):

i- Every continuous is b-continuous.

ii- Every b-continuous is ωb-continuous.

iii- Every continuous is ωb-continuous.

Proof

i- Let $f: X \longrightarrow Y$ be continuous function and A open set in Ythen, $f^{-1}(A)$ is open set in X thus,$f^{-1}(A)$ is b-continuous set in X,therefore f is b-continuous.

ii- Let $f: X \longrightarrow Y$ b-open function and A open set in Y then,$f^{-1}(A)$ b-open set in X,thus $f^{-1}(A)$ ωb-open set in X,therefore f is ωb-continuous.

iii- Let $f: X \longrightarrow Y$ continuous function, and A open set in Y then $f^{-1}(A)$ open set in X,Thus $f^{-1}(A)$ ωb-open set in X therefore f is ωb-continuous .But the convers of i and ii, iii is not true in general for.

Examples (1.2.6)

i- Let $X = \{1,2,3\}$,$\tau = \{\emptyset, X, \{1,2\}\}$ be topology on X and $Y = \{a, b, c\}$,$\tau_y = \{\emptyset, Y,\{a\}, \{a, b\}\}$ topology on Y and f is function from X into Y, $f: X \longrightarrow Y$ be defined by

$f(1) = a$,$f(2) = c$,$f(3) = b$,the inverse images of $\{a\}$and $\{a, b\}$are $\{1\}$ and $\{1,3\}$ Respectively which is b-continuous but is not continuous.

ii-Let $X = \{1,2,3\}$,$\tau_x = \{\emptyset, X, \{1\}\}$,$Y = \{a, b, c\}$ then $\tau_y = \{\emptyset, Y, \{b\}\}$ and $f(1) = f(3) = a$, $f(2) = b$,thus f is ωb-continuous but $f^{-1}(\{b\}) = \{2\}$ is not b-open in X therefore f is not b-continuous.

iii- Let $X = \{1,2\}$ and $Y = \{a, b\}$, τ be indiscrete topology on X and $\tau' = \{\emptyset, Y, \{a\}\}$ be topology on Y,let $f: X \longrightarrow Y$ be function defined by $f(1) = a$, $f(2) = b$,thus f is ωb-continuous is not continuous.

<u>Remarks (1.2.7):</u>

Let $f: X \longrightarrow Y$ be a function of a space X into a space Y then:

i. The constant function is an ωb-continuous function.

ii. If(X, τ) is discrete then,f is an ωb-continuous function.

iii. If X finite set and τ any topology on X then, f ωb-continuous.

iv. If (Y, τ^*) indiscrete topology on Y then,f ωb-continuous.

Proposition (1.2.8):

Let $f: X \to Y$ be a function of a space X into a space Y then the following

statements are equivalent :

1- f is ωb-continuous function.

2- $f^{-1}(A^\circ) \subseteq (f^{-1}(A))^{\circ\omega b}$ for every set A of Y.

3- $f^{-1}(A)$ ωb-closed set inX for everyclosed set A in Y.

4- $f(A^{-\omega b}) \subseteq \overline{f(A)}$ for every set A of X.

5- $\overline{f^{-1}(A)}^{\omega b} \subseteq \mathrm{f}^{-1}(\bar{A})$ for every set A of Y.

Proof

$(1 \longrightarrow 2)$

Let A ⊆ Y since A°open set in Y,then $f^{-1}(A^\circ)$ ωb-open set in X thus, $f^{-1}(A)^\circ = (f^{-1}(A)^\circ)^{\circ\omega b} \subseteq$

$(f^{-1}(A))^{\circ\omega b}$ ∴ $f^{-1}(A)^\circ \subseteq (f^{-1}(A))^{\circ\omega b}$.

$(2 \longrightarrow 3)$.

Let A be a closed subset of Y then A^c is an open set in Y thus $A^c = (A^c)^\circ$,thus $f^{-1}(A^c)^\circ \subseteq \left(f^{-1}(A^c)\right)^{\circ\omega b}$ and hence $(f^{-1}(A))^c \subseteq ((f^{-1}(A))^c)^{\circ\omega b}$ and therefore $(f^{-1}(A))^c = ((f^{-1}(A))^c)^{\circ\omega b}$ hence $(f^{-1}(A))^c$ is an ωb-open set in X and $f^{-1}(A)$ is ωb-closed set in X .

$(3 \longrightarrow 4)$

Let $A \subseteq Y$,then $\overline{f(A)}$ is closed set in Y thus by (3) we have $f^{-1}\left(\overline{f(A)}\right)$ is ωb-closed set in X,containing A,thus $A^{-\omega b} \subseteq f^{-1}\left(\overline{f(A)}\right)$,hence $f(A^{-\omega b}) \subseteq \overline{f(A)}$

$(4 \longrightarrow 5)$

Let $A \subseteq Y$ then by (4) we have $f\overline{\left(f^{-1}(A)\right)}^{\omega b} \subseteq \overline{f(f^{-1}(A))}$ thus $\overline{\left(f^{-1}(A)\right)}^{\omega b} \subseteq f^{-1}(\overline{A})$.

$(5 \longrightarrow 1)$

Let V open set in Y then $V^c = \overline{V^c}$ by hypothesis $\overline{\left(f^{-1}(V^c)\right)}^{\omega b} \subseteq f^{-1}(\overline{V^c})$ hence

$\overline{\left(f^{-1}(V^c)\right)}^{\omega b} \subseteq f^{-1}(V^c)$,therefore $f^{-1}(V)$ is an ωb-open set in X thus f ωb-continuous function.

Now, we introduce another type of continuous function .

Definition (1.2.9):

Let $f: X \to Y$ be a function of a topological space (X, τ) into a topological space (Y, τ') then,f is called an ωb-irresolute (ωb́-continuous) function if $f^{-1}(A)$ is an ωb-open set in X,for every ωb-open set A in Y.

Definition (1.2.10):

Let $f: X \to Y$ be a function of a topological space (X, τ) into a topological space space (Y, τ'),then f is called an b-irresolute function if $f^{-1}(A)$ is an b-open set in X, for every b-open set A in Y.

Remark (1.2.11):

1-The constant function is ωb́-continuous function.

2- Let X and Y are finite sets and f: X → Y be a function of a space X into a space Y,thenf is an ωb́-continuous.

Proposition (1.2.12):

Every ωb́-continuous function is ωb-continuous function.

Proof

Let $f:(X,\tau) \longrightarrow (Y,\grave{\tau})$ be ωb́-continuous function and A open set in Y;then A is ωb-open since f is ωb́-continuous,thus $f^{-1}(A)$ is ωb-open inX,therefore f is ωb-continuous

Remark (1.2.13):

But the converse not true in general as the following shows

Example (1.2.14):

Let U be usual topology on R and τ be indiscrete topology on Y = {2,3},let $f: R \rightarrow$Y be a function defined.by $f(x) = \begin{cases} 2 \ if \ \ x \in Q \\ 3 \ \ if \ \ x \in Q^c \end{cases}$,

then f is ωb-continuous but is not ωb́-continuous since $f^{-1}(\{2\}) = Q$ is not ωb-open in R .

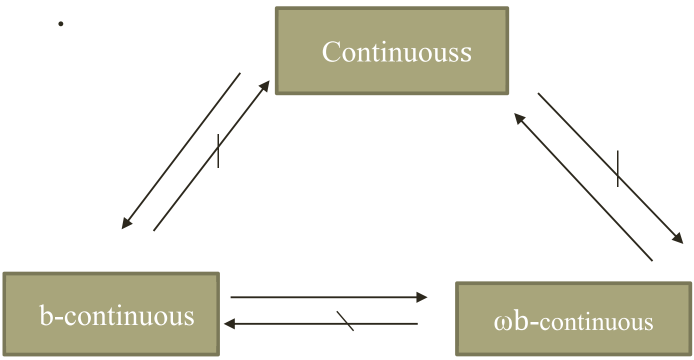
Remark (1.2.15):

Every continuous is ωb́-continuous function but the converse not true in general

Example (1.2.16):

Let f: x ⟶ y and Let $X = \{a, b, c\}$,$\tau = \{\emptyset, X, \{\mathrm{c}\}\}$ is topology on X and Y $= \{1,2\}$, $\acute{\tau} = \{\emptyset, Y, \{1\}\}$ is topology on Y,such that $f(a) = f(c) = 2$, $f(b) = 1$,it is clear that f is ωb́-continuous but is not continuous.

The following diagram shows the relations among the different types of continuous function .



Proposition (1.2.17)

Let $f: X \rightarrow Y$ be a function of a topological space (X, τ) into a topological space(Y, τ') then f is an ωb́-continuous function iff the inverse image of every ωb-closed .in Y is an ωb-closed set in X .

Proof

Let A be ωb-closed set in Y then,A^c ωb-open in Y

,Since f ωb́-continuous then $f^{-1}(A^c)$ is ωb-open in X by definition(1.2.9) since $f^{-1}(A^c) = \left(f^{-1}(A)\right)^c$ thus is. ωb-open set in X therefore,$f^{-1}(A)$ ωb-closed set in X for all A ωb-closed set in Y.

Proposition (1.2.18)

Let $f: X \to Y$ be.b-irresolute,one-to-one function from aspace into a space Y Then f is an ωb-irresolute .

Proof

Let A be an ωb-open subset of Y and $x \in f^{-1}(A)$ then $f(x) \in A$ and there exists an.b-open set V containing f(x) such that V-A countable set since f is b-irresolute ,thus$f^{-1}(V)$ is b-open in xcontaining x and since f is one-to-one ,hence $f^{-1}(V - A)$ is a countable set f^{-1}(V-A)is acountable set but $f^{-1}(V) - f^{-1}(A) = f^{-1}(V - A)$.hence $f^{-1}(V) - f^{-1}(A)$ is a countable set therefore $f^{-1}(A)$is an ωb-open set in X.

Proposition (1.2.19):

Let $f: X \to Y$ be a function of a topological space (X, τ) into a topological space (Y, τ') then the following statements are equivalent:

(1) f is an ωb́-continuous function.

(2) $f\left(\overline{A}^{\omega b}\right) \subseteq \overline{f(A)}^{\omega b}$ for every set $A \subseteq X$.

(3) $\overline{f^{-1}(B)}^{\omega b} \subseteq f^{-1}(B^{-\omega b})$ for every set $B \subseteq Y$.

Proof.

(1)→(2)

Let $A \subseteq X$,then $f(A) \subseteq Y$,$\overline{f(A)}^{\omega b}$ is ωb-closed set in Y, since f is an ωb́-continuous,Thus $f^{-1}\left(\overline{f(A)}^{\omega b}\right)$ is ωb-closed set in X by proposition (1.2.17) since $f(A) \subseteq \overline{f(A)}^{\omega b}$,Hence $f^{-1}(f(A)) \subseteq f^{-1}\left(\overline{f(A)}^{\omega b}\right)$, $A \subseteq f^{-1}(f(A))$ then $A \subseteq f^{-1}\left(\overline{f(A)}^{\omega b}\right)$,since $f^{-1}\left(\overline{f(A)}^{\omega b}\right)$is ωb-closedthus $\overline{A}^{\omega b} \subseteq$

$f^{-1}\left(\overline{f(A)}^{\omega b}\right)$,hence,$f\left(\overline{A}^{\omega b}\right) \subseteq f\left(f^{-1}\left(\overline{f(A)}^{\omega b}\right)\right)$

$\subseteq \overline{f(A)}^{\omega b}$ therefore $f\left(\overline{A}^{\omega b}\right) \subseteq \overline{f(A)}^{\omega b}$.

(2)→(3)

Let $\mathrm{f}\left(\overline{A}^{\omega b}\right) \subseteq \overline{f(A)}^{\omega b}$ $\forall$ A ⊆ X and B ⊆ Y then

$\mathrm{f}^{-1}(\mathrm{B}) \subseteq \mathrm{X}$, $\mathrm{f}\left(\overline{f^{-1}(B)}^{\omega b}\right) \subseteq \overline{f\left(f^{-1}(B)\right)}^{\omega b}$ since

$\mathrm{f}\left(\mathrm{f}^{-1}(\mathrm{B})\right) \subseteq \mathrm{B}$,hence $\overline{f\left(f^{-1}(B)\right)}^{\omega b} \subseteq \overline{B}^{\omega b}$,

$f^{-1}\left(\overline{f\left(f^{-1}(B)\right)}^{\omega b}\right) \subseteq \mathrm{f}^{-1}\left(\mathrm{B}^{-\omega \mathrm{b}}\right)$

therefore $\overline{f^{-1}(B)}^{\omega b} \subseteq f^{-1}\left(B^{-\omega b}\right)$

(3)→(1)

Let B ωb-closed set in Y then $\mathrm{B} = \overline{\mathrm{B}}^{\omega \mathrm{b}}$ since $\overline{\mathrm{f}(\mathrm{B})}^{\omega \mathrm{b}} \subseteq$

$\mathrm{f}^{-1}\left(\mathrm{B}^{-\omega \mathrm{b}}\right)$ then $\overline{\mathrm{f}(\mathrm{B})}^{\omega \mathrm{b}} \subseteq$

$\mathrm{f}^{-1}(\mathrm{B})$ since $f^{-1}(B) \subseteq \overline{f(B)}^{\omega b}$ thus $f^{-1}(B) =$

$\overline{f^{-1}(B)}^{\omega b}$ hence $f^{-1}(B)$ is ωb-closed set in X therefore f is an,ωb́-continuous function

Remark (1.2.20):

A composition of two ωb-continuous function not necessary be an ωb-continuous
function as the following shows

Example (1.2.21):

Let $X = R$, $Y = \{1,2\}$, $Z = \{a, b\}$,τ be the indiscre topology on Y,σ be the discreteTopology on Z and Let U be the usual topology on R, If $f: R \to Y$ is function defined

By $f(x) = \begin{cases} 1, & \text{if } x \in Q \\ 2, & \text{if } x \in Q^C \end{cases}$ and $g: Y \to Z$ is a function defined by $g(1) = a$,
$g(2) = b$ then f, g are ωb-continuous function but $g \circ f$ is not an ωb-continuous since $(g \circ f)^{-1}(\{a\})$ is not ωb-open set in X .

Proposition (1.2.22):

Every an ωb-irresolute function is an ωb-continuous function.

Proof

same prove (1.2.12)

Proposition (1.2.23):

Let X, Yand Z are spaces and $f: X \to$ Y be ωb-continuous if $g: Y \to Z$ is continuous, then $g \circ f: X \to Z$ is ωb-continuous .

Proof

Let B open set in Z,since g is continuous,then $g^{-1}(B)$ open in Y,Since f is ωb-contiuous, thus $f^{-1}\left(g^{-1}(B)\right)$ is ωb-continuous in X, hence $(g \circ f)^{-1}(B)$ is ωb-open in X therefore, $g \circ f: X \longrightarrow Z$ ωb-continuous

Proposition (1.2.24):

Let X, Y and Z be spaces and $f: X \to Y, g: Y \to Z$ be functions then,if f is an ωb́-

continuous and g is an ωb-continuous thus, $g \circ f : X \rightarrow Z$ is ωb-continuous

Proof

Let B be an open set in Z,then $\mathrm{g}^{-1}(\mathrm{B})$ is an ωb-open set in Y,since f is an ωb́-continuous then, $f^{-1}\left(g^{-1}(B)\right) = (g \circ f)^{-1}(B)$ is an ωb-open set in X, hence g ∘ f is ωb-continuous.

Proposition(1.2.25):

Let f: X → Yand g: Y → Z are be ωb́-continuous then $g \circ f : X \rightarrow Z$ is ωb́-continuous

Proof.

Let M be ωb-open set in Z,$(g \circ f)^{-1}(M) = f^{-1}\left(g^{-1}(M)\right)$ since g is ωb́-continuous and M ωb-open in Z then $\mathrm{g}^{-1}(\mathrm{M})$ is ωb-open in Y,Since f ωb́-continuous,thus$\left(\mathrm{g}^{-1}(\mathrm{M})\right)$ is ωb-open in X but $(g \circ f)^{-1}(M)$=$f^{-1}\left(g^{-1}(M)\right)$ thus,$(\mathrm{g} \circ \mathrm{f})^{-1}(\mathrm{M})$ is ωb-open in X,therefore g ∘ f is ωb́-continuous $g = f/Z$

Now, we study restriction of ωb-continuous (ωb́-continuous).function.

Definition (1.2. 26): [25]

Let be function $f: X \to Y$and the function $g: Z \to Y$ be defined ∀ x ∈ Z, $g(x) = f(x)$ is said restriction f on Z such that f$|Z: Z \longrightarrow Y$ and g =f$|g$

Proposition (1.2.27):

Let f: X ⟶ Y be a function and A be a nonempty open set in X :

(1) If f ωb-continuous, then f |A: A ⟶ Y is ωb-continuous.

(2) If f ωb́-continuous, thus f |A: A ⟶ Y is ωb́-continuous.

Proof.

(1) Let V be any open set in Y,Since f is ωb-continuous then,$f^{-1}(V)$ is ωb-open in X thus $f^{-1}(V) \cap A$ is ωb-

open in A such that A $(f|_A)^{-1}(V) = f^{-1}(V) \cap A$ is ωb-open therefore is ωb-continuous.

(2) Let B be an ωb-open inset in Y Since f is ωb́-continuous then $f^{-1}(B)$ is ωb-open set in X, $f^{-1}(B) \cap A$ is ωb-open set in A but $\left(f|_A(B)\right)^{-1} = f^{-1}(B) \cap A$,then $\left(f|_A(B)\right)^{-1}$ is ωbopen set in A hence $f|_A$ is ωb́-continuous.

Chapter two

on ωb-open function, ωb-separation Axioms and ωb-connected space

Introduction

In This chapter is divided into three sections,section one introduced the definition of ωb-open and ωb-closed functions and some proposition, remarks, theorems of about it. In section two we gave a different concept of the separation axiom by using ωb-open set and we introduced proposition, remarks,theorems of about it,also we introduce the definition ωb- R_1 space

space,ωb-R_2) space and we study the relation between ωb-separation axiom and ωb-R_i spaces,i=1,2 In section three,we introduce the fundamentual concept of connected space and generalized by ωb-open sets and we prove some result about it.

2.1 ωb-closed and ωb-open Functions

In this section,we defined of ωb-closed and ωb-open functions and some propositions and remark about that subject .

Definition (2.1.1): [7]

Let $f: X \to Y$ be a function of a space X into a space Y then , f is called an open function if f(A) is an open set in Y for every open set A in X.

Theorem (2.1.2): [9]

Let $f: X \longrightarrow Y$ be a function of space X into space Y then, thefollowing statement
are equivalent :

1. f is open function .
2. $f(A^{\circ}) \subseteq (f(A))^{\circ}$ for every subset A of X .
3. $(f^{-1}(A))^{\circ} \subseteq f^{-1}(A^{\circ})$ for every subset A of Y.
4. $f^{-1}(\overline{A}) \subseteq \overline{f^{-1}(A)}$ for every subset A of Y.

Definition (2.1.3): [3]

A function $f: X \longrightarrow Y$ is said to be b-open for every open subset A of X,if f(A) is an b-open set in Y.

Definition (2.1.4):

A function $f: X \longrightarrow Y$ is said to be ωb-open for every open subset A of X,if $f(A)$ is an ωb-open set in Y.

Example (2.1.5):

Let $X = \{a, b, c\}$ and $Y = \{1,2\}, \tau = \{X, \emptyset, \{a\}\}$ $, \tau' = \{\emptyset, Y, \{1\}\}$, then $f: (X, \tau) \rightarrow (Y, \tau')$ $\ni f(a) = f(b) = 1, f(c) = 2$ such that b-open set in X is $X, \emptyset, \{a\}$, $f(\emptyset) = \emptyset$, $\emptyset$ is ωb-open set in Y, $f(X) = \{f(a), f(b), f(c)\} = \{1,2\} = Y$ is ωb-open set in Y thus, $f(\{a\}) = \{f(a)\} = \{1\}$ is ωb-open set in Y, therefore f is ωb-open function

Proposition (2.1.6):

A function $f: X \to Y$ is ωb-open,iff $f(A^{\circ}) \subseteq (f(A))^{\circ\omega b}$ for all $A \subseteq X$

Proof

Suppose that $f: X \to Y$ is an ωb-open function,

let $A \subseteq X$ since A° open in X,then $f(A^{\circ})$ is an ωb-open in Y since $A^{\circ} \subseteq A$ thus,$f(A^{\circ}) \subseteq f(A)$ hence$(f(A^{\circ}))^{\circ\omega b} \subseteq (f(A))^{\circ\omega b}$ but $(f(A^{\circ}))^{\circ\omega b} = f(A^{\circ})$ therefore $f(A^{\circ}) \subseteq (f(A))^{\circ\omega b}$.

Conversely:

Let A be open in X then $A^{\circ} = A$ since $f(A^{\circ}) \subseteq (f(A))^{\circ\omega b}$ thus, $f(A) \subseteq (f(A))^{\circ\omega b}$ such that $f(A) = (f(A))^{\circ\omega b}$ hence f(A) is ωb-open in Y therefore $f: X \to Y$ is an ωb-open function .

<u>Remark (2.1.7):</u>

Every open function is an ωb-open function, but the converse is not true in general as the following example show.

<u>Example (2.1.8):</u>

Let $X = \{1,2,3\}$, $\tau = \{\emptyset, X, \{3\}\}$ be a topology on X, then $Y = \{a, b\}$ and τ' be indiscrete topology onY, let$f: X \to Y$ beafunction define by $f(1) = f(2) = a$ $f(3) = b$ then f is an $\omega b -$ open function but is not a an open function.

<u>Proposition (2.1.9):</u>

If $f: X \longrightarrow Y$ is open function and $g: Y \longrightarrow Z$ is ωb-open function then $g \circ f$ is ωb-open function.

Proof

Let A open set in X,Since f is an open function,then f(A)is open set in Y,Since g is ωb-open function thus,g$\big($f(A)$\big)$ is an ωb-open set in Z therefore $g \circ f: X \longrightarrow Z$ is $\omega\mathrm{b} - \text{open function.}$ ωb-open function.

We introduce and study ωb-closed function also some properties about them

<u>Definition (2.1.10):</u> [7]

Let $f: X \rightarrow Y$ be a function of a space X into a space Y,then f is called an closed function if f(A) is an closed set in Y,for every closed set A in X.

<u>Definition (2.1.11):</u> [3]

A function $f: X \longrightarrow Y$ is said to be b-closed if $f(A)$ is an b-closed set in Y,for every closed subset A of X .

<u>Definition (2.1.12):</u>

A function $f: X \longrightarrow Y$ is said to be ωb-closed,if $f(A)$ is an ωb-closed set in Y, for every closed subset A of X .

Remark (2.1.13):

The constant function is an ωb-closed .

Proposition (2.1.14):

A function $f: X \to Y$ is an ωb-closed iff $\overline{\mathrm{f(A)}}^{\omega \mathrm{b}} \subseteq \mathrm{f}(\overline{\mathrm{A}})$ for all $\mathrm{A} \subseteq \mathrm{X}$.

Proof

Suppose that $f: X \to Y$ is an ωb-closed function,let $\mathrm{A} \subseteq \mathrm{X}$,since $\overline{\mathrm{A}}$ is closed set in X, then $\mathrm{f}(\overline{\mathrm{A}})$ is ωb-closed set in Y since $\mathrm{A} \subseteq \overline{\mathrm{A}}$ thus,$\mathrm{f(A)} \subseteq \mathrm{f}(\overline{\mathrm{A}})$ hence $\overline{f(A)}^{\omega b} \subseteq \overline{f(\overline{A})}^{\omega b}$ but $\overline{f(\overline{A})}^{\omega b} = f(\overline{A})$ therefor $\overline{f(A)}^{\omega b} \subseteq \mathrm{f}(\overline{\mathrm{A}})$.

Conversely:

Let F be a closed set of X then,$\mathrm{F} = \overline{\mathrm{F}}$ by hypothesis $\overline{f(F)}^{\omega b} \subseteq f(\overline{F})$ hence $\overline{\mathrm{f(F)}}^{\omega \mathrm{b}} \subseteq f(F)$ thus $f(F)$ is an ωb-closed set inY therefore $f: X \rightarrow Y$ is an ωb-closed function .

Proposition (2.1.15):

Let $f: (X, \tau) \rightarrow (Y, \tau')$ be a function and $f(\overline{A}) = \overline{f(A)}^{\omega b}$ for each set A of X then f is ωb-closed and continuous function .

Proof

By proposition (2.1.14) f is an ωb-closed function, now to prove that f is continuous, let F ⊆ X then,

$f(\overline{F}) = \overline{f(A)}^{\omega b}$ by proposition (1.1.23)(5) thus, $\overline{\mathrm{f(F)}}^{\omega \mathrm{b}} \subseteq \overline{\mathrm{f(F)}}$ hence $f(\overline{F}) \subseteq \overline{f(F)}$ by Throrem (1.2.2)(3) therefore f is continuous function.

Proposition (2.1.16):

Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be function.then if f is aclosed and g is an ωb-closed,thus g ∘ f is a ωb-closed function.

Proof

It is clear.

Remark (2.1.17):

Every closed function is an ωb-closed function, but the converse is not true in general as the following

Example (2.1.18):

Let $X = \{1,2,3\}, \tau = \{\emptyset, X, \{3\}\}$ be a topology on X then,$Y = \{4,5\}$ and τ' be Indiscrete topology On Y , let $f: X \to Y$ be a function defined by $f(1) = f(2) = 4$, $f(3) = 5$ thus f is an ωb-closed function,but is not a an closed function .

Proposition (2.1.19):

Let $f: X \to Y$ be a ωb-closed function then the restriction of f to a closed subset F of X is an ωb-closed of F into Y.

Proof

Since F is a closed subset in X then the inclusion function $i/_F : F \to X$ is a closed function Since $f : X \to Y$ is an ωb-closed function thus by proposition (2.1.16) $f \circ i/_F : F \to Y$ is an ωb-closed function, but $f \circ i/_F = f/_F$ is an ωb-closed function.

Definition (2.1.20):

Let X and Y are topologica space then a function $f : X \to Y$ is called an

ωb-homeomorphism if:

(1) f is bijective .

(2) f is an ωb-continuous .

(3) f is an ωb-closed (ωb-open).

It is clear that every homeomorphism is an ωb-homeomorphism

Now we introduce the definition of ωb́-closed (ωb́-open) function and some . propositions about it .

Definition (2.1.21):

Let $f: X \longrightarrow Y$ be a function of space X into space Y then:f is called ωb́-closed function if f(A)is ωb-closed set inY ,for every ωb-closed A in X, f is called ωb́-openfunction if f(A) is ωb-open set in Y,for every ωb-open A in X.

Remark (2.1.22):

The constant function is an ωb́-closed function .

Proof

Clear.

Proposition (2.1.23):

A function $f: (X, \tau) \rightarrow (Y, \tau')$ is ωb́-closed if $\overline{\mathrm{f(A)}}^{\omega b} \subseteq \mathrm{f}\left(\overline{\mathrm{A}}^{\omega b}\right)$ for all A ⊆ X.

Proof

Suppose that $f: X \rightarrow Y$is an ωb́-closed function and

$A \subseteq X$,since $\overline{A}^{\omega b}$ is ωb-closed set in X, then $f\left(\overline{A}^{\omega b}\right)$ is ωb-closed set in Y ... $(*)$,since $A \subseteq \overline{A}^{\omega b}$ thus $f(A) \subseteq f\left(\overline{A}^{\omega b}\right)$,hence $\overline{f(A)}^{\omega b} \subseteq \overline{f(\overline{A})}^{\omega b}$ since $f\left(\overline{A}^{\omega b}\right) = \overline{f\left(\overline{A}^{\omega b}\right)}^{\omega b}$ by $(*)$ therefore $\overline{f(A)}^{\omega b} \subseteq f\left(\overline{A}^{\omega b}\right)$

Conversely:

Let A be a ωb-closed set of X then $A = \overline{A}^{\omega b}$ hypoth is $\overline{f(A)}^{\omega b} \subseteq f\left(\overline{A}^{\omega b}\right)$ hence $\overline{f(A)}^{\omega b} \subseteq f(A)$ thus f(A) is an ωb-closed set in Y ,therefore $f: X \to Y$ is an ωb́- Closed function.

Proposition (2.1.24):

Let Y and Z be space and $f: X \to Y, g: Y \to Z$ be function then:

(1) If f and g are ωb́-closed function then,$g \circ f$ is ωb́-closed function

(2) If g ∘ f is ωb́-closed function f is ωb́-continuous and onto then g is ωb́-closed function

(3) If g ∘ f is ωb́-closed function,g is ωb́-continuous and one-to-one then f is ωb́- Closed function.

Proof

(1) Let F be a ωb-closed set in X then f(F) is an ωb-closed set in Y thus $g\big(f(F)\big)$ is an ωb-closed set in Z but $(g \circ f)(F) = g\big(f(F)\big)$ hence g ∘ f is ωb́-closed function.

(2) Let F be a ωb-closed set in Yby proposition (1.2.17) $f^{-1}(F)$ is ωb-closed set inX, Thus $g \circ f\big(f^{-1}(F)\big)$ is ωb-closed set in Z,since f is onto then $g \circ f\big(f^{-1}(F)\big) = g(F)$hence g(F) is ωb-closed set in Z therefore g is ωb́-closed.

(3) Let F be a ωb-closed set in X,then $g \circ f(F)$ is ωb-closed set in Z then by Proposition (1.2.17) $g^{-1}(g \circ f(F))$is ωb-closed set in Y, since g is one -to-one thus $g^{-1}(g \circ f(F)) = f(F)$ is ωb-closed set in Y therefore f is ωb́ -closed.

Proposition (2.1.25):

LetX, Y, Z be space and $f: X \to Y$, g: Y → Z be function then:

(1) If f and g ar ωb́-open function,then $g \circ f$ is ωb́-open function .

(2) If $g \circ f$ is ωb́-open function f is ωb́-continuous and onto then g is ωb́-open .

(3) If $g \circ f$ is ωb́-open function g is ωb́-continuous and one-to-one then,f is ωb́-open

Proof

Similar to proof proposition (2.1.24).

Proposition(2.1.26):

A function $f: (X, \tau) \to (Y, \tau')$ is ωb́-open function if and only if $f(A^{\circ\omega b}) \subseteq (f(A))^{\circ\omega b}$ for all A ⊆ X

Proof

Suppose $f: X \to Y$ is an ωb́-open.function,let A ⊆ X since $A^{\circ\omega b}$ is ωb-open in X,Then $f(A^{\circ\omega b})$ is ωb-open in Y hence $f(A^{\circ\omega b}) = (f(A))^{\circ\omega b} \subseteq (f(A))^{\circ\omega b}$

Conversely:

Let A is an ωb-open in X since , $f(A^{\circ\omega b}) \subseteq (f(A))^{\circ\omega b}$ then $f(A) \subseteq (f(A))^{\circ\omega b}$ thus $f(A) = (f(A))^{\circ\omega b}$ hence $f(A)$ is ωb-open in Y therefore $f: X \to Y$ is an ωb́-open function.

Definition (2.1.27):

Let X and Y be space then a function f: X → Y is called an ωb́-homeomorphism if:

(1) f is bijective .

(2) f is an ωb́-continuous.

(3) f is an ωb́-closed (ωb́-open).

Proposition (2.1.28):

Let $f:(X,\tau)\longrightarrow(Y,\acute{\tau})$ be bijective function,then the following statements are equivalent:

i. f is ωb́-homeomorphism .

ii. f is ωb́-continuous and ωb́-closed .

iii. $f\left(\overline{A}^{\omega b}\right)=\overline{f(A)}^{\omega b}$ $\forall\ A\subseteq X$.

Proof

i ⟶ ii

By definition of ωb́-homeomorphism

ii ⟶ iii

Since f is ωb́-continuous then $f\left(\overline{A}^{\omega b}\right)\subseteq\overline{f(A)}^{\omega b}$

byproposition (1.2.19)since f is ωb́-closed $\overline{f(A)}^{\omega b}\subseteq$

$f\left(\overline{A}^{\omega b}\right)$by Proposition.(2.1.23),thus $\left(\overline{A}^{\omega b}\right) = \overline{f(A)}^{\omega b}$

iii $\longrightarrow$ i

Since $\overline{f(A)}^{\omega b} \subseteq f\left(\overline{A}^{\omega b}\right)$ then f is ωb́-closed since

$f\left(\overline{A}^{\omega b}\right) \subseteq \overline{f(A)}^{\omega b}$ by proposition (1.2.19) thus f is

ωb́-continuous and f bijective therefore f is ωb́-homeomorphism .

2.2 On ωb-Separation Axioms

In this section we introduce the associative separation axioms of the ωb-open sets which already defined in the previous chapter and then give some new propositions about them .

Definition (2.2.1): [15]

A space X is called T_1-space if for each $x \neq y$ in X,there exists open sets U and V such that $x \in U$, $y \notin U$ and $y \in V, x \notin V$.

Definition (2.2.2): [14]

A space X is called bT_1-space if for each $x \neq y$ in X,there exists b-open sets U and V such that $x \in U$, $y \notin U$ and $y \in V, x \notin V$.

Definition (2.2.3):

A space X is called ωbT_1-space if for each $x \neq y$ in X,there exists ωb-open sets U and V such that $x \in U$, $y \notin U$.and. $y \in V, x \notin V$.

Proposition (2.2.4):

Every T_1-space is bT_1-space

Proof

Let (X, τ) be bT_1-space and $x,y \in X \ni x \neq y$,thenthere exists two open sets U,Vsuch that $x \in U$,$y \notin U$ and $y \in V, x \notin V$ Since every open set is b-open set thus,U, V are two b-open set such that $x \in U, y \notin U$ and $y \in V$, $x \notin V$ therefore,(X, τ) be bT_1-space

Remark (2.2.5): [14]

But the converse of (2.2.4) is not true in general,as the example .

Proposition (2.2.6):

Every T_1-space is ωbT_1-space

Proof

Similar to prove of Proposition (2.2.4)

Remark (2.2.7):

but the converse is not true in general, in fact from example [14] it is easy to check that is ωbT_1-space but not T_1-space .

Proposition (2.2.8):

Every bT_1-space is ωbT_1-space.

Proof

Similar to prove of Proposition (2.2.4) .

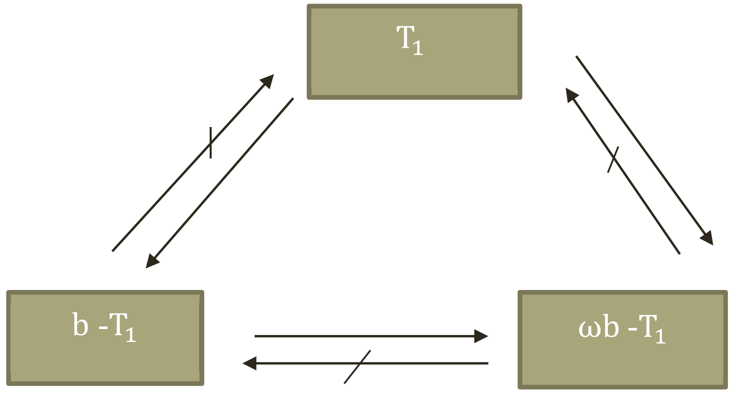
Remark (2.2.9):

But The converse is not true in general as the following

Example (2.2.10):

Let $X = N, \tau = \{G: 1 \in G\} \cup \{\emptyset\}, BO(X) = \{G: 1 \in G\} \cup \{\emptyset\}$ then $1,3 \in N$ is not exists two b-open sets$\ni$ $1 \in U$and $3 \notin U$but $3 \in V$and $1 \notin V$ thus is not bT_1-space since $\omega BO(X) = \{A: A \subseteq N\}$therefor is ωbT_1-space .

The following diagram shows the relations among the difference types of T_1-space



Proposition (2.2.11):

Every open subspace of ωbT_1-space is ωbT_1-space

Proof

Let $x,y \in M \ni x \neq y$ Since X is ωbT_1-space then $\exists$ two ωb-open set U,V such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$, let $A = U \cap M, B = V \cap M$ thus A, B are ωb-open set in Mand $x \in A$ but $y \notin A$ and $y \in B$ but $x \notin B$ therefore,M is ωbT_1-space

Theorem (2.2.12):

Let $f: X \longrightarrow Y$ be a ωb-irresolute,injective function ,if Y is ωbT_1-space then X is ωbT_1-spaces

Proof

Let $x, y \in X \ni x \neq y$ then $f(x), f(y) \in Y$ and $f(x) \neq f(y)$ Since Y is ωbT_1-space then there exists two ωb-open sets U, V inY such that $f(x) \in U$ but $f(y) \notin U$ and f (y) ∈ V but f(x) ∉ V thus x ∈ $f^{-1}(U)$ but y ∉ $f^{-1}(U)$ and y ∈ $f^{-1}(V)$ but $x \notin f^{-1}(V)$ since f is ωb-irresolute hence $f^{-1}(U), f^{-1}(V)$ are ωb-open therefore X is ωbT_1- space

Proposition (2.2.13):

Let X be a topological space then X is ωbT_1-space if and only if {x} is ωb-closed set for each x ∈ X.

Proof

Let X be ωbT_1-space and x ∈ X and let y ∉ {x} Since X is ωb-T_1-space then there exists an ωb-open set V such that y ∈ V, x ∉ V,then $V \cap \{x\} = \emptyset$ it is $(V - \{y\}) \cap \{x\} = \emptyset$ hence $y \notin \{x\}'^{\omega b}$ thus $\{x\}'^{\omega b} \subseteq \{x\}$ and hence $\overline{\{X\}}^{\omega b} = \{x\} \cup \{x\}'^{\omega b} = \{x\}$ so that {x} is ωb-closed set for each x ∈ X by Proposition (1.1.34) (1),(2) .

Conversely:

Assume that $\{x\}$ is ωb-closed set for each $x \in X$,let $x \neq y$ in X,then $X - \{x\} = V$ is ωb-open set such that $y \in V, x \notin V$, let $X - \{y\} = U$ hence U is ωb-open set which is contains x therefore X is ωbT_1-space.

Theorem (2.2.14):

Let $f: X \to Y$ be an bijective ωb-open function,if X is T_1-space then Y is ωbT_1-space

Proof

Let $y_1, y_2 \in Y \ni y_1 \neq y_2$ since f onto function then $x_1, x_2 \in X \ni y_1 = f(x_1), y_2 = f(x_2)$ since X is T_1-space ∃ U,V open sets in X $\ni x_1 \in U$ but $x_2 \notin U$ and $x_2 \in V$ but $x_1 \notin V$ hence f is ωb-open $\ni f(U), f(V)$ two are ωb-open set in Y then , $y_1 = f(x_1) \in f(U)$ but $y_2 = f(x_2) \notin f(U)$ and $y_2 = f(x_2) \in f(V)$ but $y_1 = f(x_1) \notin f(V)$ thus $f(U), f(V)$ are two ωb-open therefore,Y is ωbT_1-space .

Theorem (2.2.15):

Let $f: X \to Y$ be an one-to-one ωb-continuous function,if Y is T_1-space then X is ωbT_1-space

Proof

Let $x_1, x_2 \in X \ni x_1 \neq x_2$,since $f: X \to Y$ is one-to-one function and $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$ and $f(x_1)$, $f(x_2) \in Y$ since Y is T_1-space $\exists$ U, V open sets in Y $f(x_1) \in$ U but $f(x_2) \notin$ U and $f(x_2) \in$ V but $f(x_1) \notin$ since f is ωb-continuous function, then $f^{-1}(U)$, $f^{-1}(V)$ are ωb-open set in X, since $f(x_1) \in$ U thus $x_1 \in f^{-1}(U)$ and since $f(x_2) \notin$ U, then $x_2 \notin f^{-1}(U)$ and since $f(x_2) \in$ V then $x_2 \in f^{-1}(V)$ since $f(x_1) \notin$ V thus $x_1 \in f^{-1}(V)$therefore X is ωbT_1-space.

Theorem (2.2.16):

Let X and Y be ωb́-homeomorphism then X be ωbT_1- space if and only if Y is ωbT_1- space

Proof

Let X and Y be ωb́-homeomorphism topological space and let X be ωbT_1-space to Prove Y is ωbT_1-space,let $y_1, y_2 \in Y \ni y_1 \neq y_2$sinceXandYωb́-homeomorphism topological then $\exists\ f: X \longrightarrow Y$ be ωb́-homeomorphism, since $y_1, y_2 \in Y$ and f ontofunction then $\exists\ x_1, x_2 \in X$ $\ni f(x_1) = y_1, f(x_2) = y_2$, since f is one-to-one, then $x_1 \neq x_2$, since X is ωbT_1-space thus $\exists$ U, V ωb-open set in X $(x_1 \in U, x_2 \notin U)and(x_2 \in V, x_1 \notin V)$such that f is ωb́-open functionthen f(U),f(V)are ωb-open set in Y, since $x_1 \in U$ then $f(x_1) \in f(U)$ since $x_2 \notin U$, thus $f(x_2) \notin f(U)$ and $x_2 \in V$ then $f(x_2) \in f(V)$ hence $x_1 \notin V$ then$f(x_1) \notin f(V)$ therefore Y is ωbT_1-space

Conversely:

Let $x_1, x_2 \in X \ni x_1 \neq x_2$, since $f: X \longrightarrow Y$is ωb́-homeomorphism then f is one-to-one and since $x_1 \neq x_2$ thus $f(x_1) \neq f(x_2)$ hence $f(x_1), f(x_2) \in$

Y,since Y is ωbT_1-space $\exists$ U, V ωb-open sets in X $\ni$ $(f(x_1) \in U, f(x_2) \notin U)$and $(f(x_2)V, f(x_1) \notin V)$since f $\omega\acute{b}$-continuous function thus $f^{-1}(U), f^{-1}(V)$are ωb-open set in X,since $f(x_1) \in U$thus $x_1 \in f^{-1}(U)$ and $f(x_2) \notin U$ thus $x_2 \notin f^{-1}(U)$ and $f(x_2) \in V$, then $x_2 \in f^{-1}(V)$ thus $f(x_1) \notin V$,then $x_1 \notin f^{-1}(V)$ therefore X is ωbT_1-space .

Definition (2.2.17): [21]

Aspace (X,T) is called a door space ifevery subset of X,is either open or closed.

Example (2,2.18): [21]

The space (X, T) for $X = \{a, b\}$ and $\tau = \{X, \emptyset, \{a\}\}$is a door space .

Definition (2.2.19): [24]

A topological space (X,T) is said to be R_0 if every open set contains the closure of each of its singletons.

Definition (2.2.20):

A topological space (X,T) is said to be ωb -R_0if every ωb-open set contains the ωb-closure of each of its singletons.

Theorem (2.2.21):

The topological door space is ωb -R_0 if and only if it is ωbT_1-space

Proof

Let x,y are distinct points in X,Since (X,τ) is door space,then $\{x\}$ is open or closed,if$\{x\}$ is open hence ωb-open in X, let $V = \{x\}$ then $x \in V$ and $y \notin V$since (X, τ) is ωb –R_0space thus, $\overline{(\{X\})}^{\omega b} \subset V$ hence $x \notin X - V$,$y \in X - V$ therefore,$X - V$ is ωb-open subset of X,if $\{x\}$ is closed hence it is ωb-closed $y \in X - \{x\}$ and $X - \{x\}$ is ωb-open set in x, Since $(X,\tau)$is ωb-R_0 space, then$\overline{(\{y\})}^{\omega b} \subset$ $X - \{x\}$,let V= $X - \overline{(\{y\})}^{\omega b}$ thus $x \in V$ but $y \notin V$and V ωb-open set in X therefore (X,T) is ωbT_1-space

Conversely:

Let (X, τ) be ωbT_1-space and, let V be an ωb-open set of x and $x \in V$ for each $y \in X - V$ there is an ωb-open set $V_y$suchthat $x \notin V_y$ but $y \in V_y$ then, $\overline{(\{X\})}^{\omega b} \cap$ $V_y = \emptyset$ foreach $y \in X - V$thus $\overline{(\{X\})}^{\omega b} \cap (\cup_{y\in x-v} V_y)$ $= \emptyset$ hence $y \in V_y$,$X - V \subset \left(\cup_{y\in x-v} V_y\right)$, $\overline{(\{x\})}^{\omega b} \subset V$ therefore (X,τ) is ωb -R_0.

Definition (2.2.22): [15]

A space X is called T_2-space (Hausdorff space) if for each $x \neq y$ in X,there exists disjoint an open sets U and V such that $x \in U, y \in V$.

Definition (2.2.23): **[14]**

A space X is called bT_2-space(b-Hausdorff space) if for each $x \neq y$ in X,there existsdisjoint an b-open sets U, V such that $x \in U, y \in V$.

Definition (2.2.24):

A space X is called ωbT_2-space (ωb-Hausdorffspace) if for each $x \neq y$ in X,there exists disjoint an ωb-open sets U, V such that $x \in U, y \in V$.

Proposition (2.2.25): **.[14]**

It is clear that every Hausdorff space is b-Hausdorff space .

Remark (2.2.26):

But The converse of (2.2.25) is not true in general as the example [14]

Remark (2.2.27):

It is clear that every Hausdorff space is ωbT_2-space but the converse is not true,In general,as the following example [14] it is easy to check that is ωbT_2-space But not T_2-space.

Remark (2.2.28):

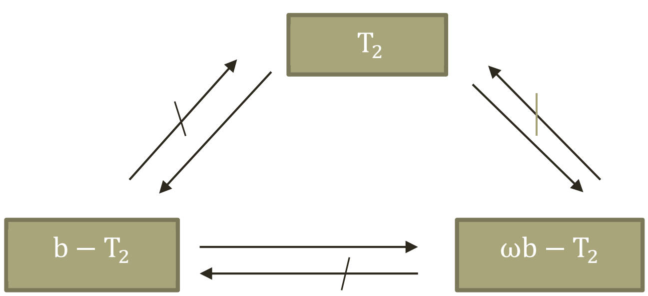
Every bT_2-space is ωbT_2-space but the converse is not true in general in fact from

Example (2.2.29):

Let $X = N, \tau = \{A \subseteq X: A^c \text{finite}\} \cup \emptyset$,$BO(X) =$ $\{G: G \subseteq X, G \text{ is infinite and } G^c \text{ infinite}\} \cup$ $\{G: G \subseteq X, G^c \text{ is finite}\} \cup \{\emptyset\}$ then , $\overline{\{1\}}^{\circ} \cup \overline{\{1\}^{\circ}} = \emptyset$ $\Longrightarrow \{1\} \nsubseteq \overline{\{1\}}^{\circ} \cup \overline{\{1\}^{\circ}} \therefore \{1\}$ is not b-open, let $A =$ $\{1\}$ and $1 \in U = N - \{2\}$ thus , U is b-open set contain 1 since N-A is countable hence A is ωb-open Since $1,2 \in N$ is not exists to b-open sets U, V such that $1 \in U, 2 \in V$ and $U \cap V = \emptyset$, then is not

bT_2-space since $\omega BO(X) = \{A: A \subseteq N\}$ therefore, is ωbT_2-space .

The following diagram shows the relations among the different types of T_2-space .



Let $f: X \longrightarrow Y$ be a bijection function .

1-.If f is ωb-open and X is T_2-space then Y is ωbT_2-space .

2- If f is ωb-continuous and Y is T_2-space then X is ωbT_2-space .

Proof

Let $f: X \longrightarrow Y$ be a bijective

1- Suppose f is ωb-open and X is T_2-space,let $y_1 \neq y_2 \in Y$ since f is bijective thenthere existx_1, x_2 in X such that $f(x_1) = y_1$and $f(x_2) = y_2$ and $x_1 \neq x_2$ since X is T_2-space then there existsdisjoint open sets U and V in X, such that ($x_1 \in U$ and $x_2 \in V$)

Since f ωb-open f(U) and f(V) are ωb-open sets in Y hence $f(x_1) = y_1 \in f(U)$and $y_2 = f(x_2) \in f(V)$ since f is bijective f(U) and f(V) are disjoint in Y

thus Y is ωbT_2 − space .

2- suppose $f: X \longrightarrow Y$ is ωb-continuous and Y is T_2-space,let $x_1, x_2 \in X$ with $x_1 \neq x_2$ let $f(x_1) = y_1$, $f(x_2) = y_2$, since f is one -to-one, since Y is T_2-space then, there exists open sets U andV containg y_1 and $y_2$respect − ively since f is ωb-continuous f^{-1} $(U)$and $f^{-1}(V)$ are disjoint ωb-open containg $x_1$and x_2 respectively, thus X is ωbT_2-space .

Theorem (2.2.31):

Let X and Y be ωb́-homeomorphism topological space then X is ωbT_2-space if and only if Y is ωbT_2-space.

Proof

By Theorem (2.2.30)

Theorem (2.2.32):

Every ωbT_2-space is ωbT_1-space .

Proof

Let (X, τ)be a ωbT_2-space let x and y be two disjoint distinct in X,sinceX isωbT_2-spacethereexists disjoint ωb-open set U and V such that $x \in U$ and $y \in V$,since U and V are disjoint $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$ hence X is ωbT_2-space .

Remark (2.2.33):

But the converse of theorem (2.2.32) is not true in general as the following

Example (2.2.34):

Let.$X = R$ and τ coffinite topology on R,then, $\omega b(X) = \{G: G \subseteq X \text{ is } G^c \text{finite}\} \cup \{G: G \subseteq X \text{ is } G^c \text{ is in finite}\}$ and X is ωb-T_1-space,since foreach $x,y \in X$ such that $X \neq y$there esistsωb-open sets $U, V$and $U = R - \{y\}$,$V = R - \{x\}$ thus $x \in U, y \notin U$ and $y \in V, X \notin V$ but X is no ωbT_2-space since for each $x,y \in X \ni x \neq y$ is not esists disjoint ωb-open U, V hence $x \in U, y \in V$ therefore (R, τ) is not ωbT_2-space .

Theorem (2.2.35):

Let M be open subspace of X then M is ωbT_2-space,if X is ωbT_2-space

Proof

Let $x, y \in M, x \neq y$ then $x, y \in X$ so $\exists\, B_1, B_2$ suchthat $B_1 \cap B_2 = \emptyset \ni x \in B_1,\ y \in B_2$ where B_1, B_2 are ωb-open set in X,let $E_1 = B_1 \cap M$, $E_2 = B_2 \cap M$ are ωb-open set subset in M,and $x \in E_1, y \in E_2$, then $E_1 \cap E_2$ $= (B_1 \cap M) \cap (B_2 \cap M) = (B_1 \cap B_2) \cap M = \emptyset = \emptyset$ hence M is ωbT_2-space .

Theorem (2.2.36):

Let $f: X \longrightarrow Y$ be one-to-one, ωb-irresolute function and Y is ωbT_2-space then ,(X, τ_1) is ωbT_2-space.

Proof

Suppose$f(X, \tau) \longrightarrow (Y, \tau)$ is one-to-one and f is ωb-irresolute and $(Y, \tau_2)$is ωbT_2-space let $x_1, x_2 \in X$

with $x_1 \neq x_2$ since f is one-to-one then, $y_1 = f(x_1) \neq f(x_2) = y_2$ for some $y_1, y_2 \in Y$ since (Y, τ_2) is ωbT_2-space there exists disjoint ωb-open set U and V such that $y_1 = f(x_1) \in U$ and $y_2 = f(x_2) \in V$ then $x_1 = f^{-1}(y_1) \in f^{-1}(U)$, $x_2 = f^{-1}(y_2) \in f^{-1}(V)$ and since f is ωb-irresolute, $f^{-1}(U)$ and $f^{-1}(V)$ are ωb-openset in(X, τ)since f is one-to-one $U \cap V = \emptyset$ hence $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset$ is (X, τ_1) is ωbT_2-space .

<u>Definition (2.2.37):</u> **[24]**

A topological space (X,τ) is said to be R_1space if for x and y in X,with $\overline{(\{X\}} \neq \overline{(\{y\}})$there exists disjoint ωb-open set U and V such that $\overline{\{X\}} \subset U$and $\overline{\{y\}} \subset V$.

<u>Definition (2.2.38):</u>

A Topological space (X,τ) is said to be ωb -R_1space if for x and y in X,with $\overline{(\{X\})}^{\omega b} \neq \overline{(\{y\})}^{\omega b}$ There exists disjoint ωb-open set U and V such that $\overline{(\{X\})}^{\omega b}$ $\subset$ U and $\overline{(\{y\})}^{\omega b} \subset$ V.

Theorem (2.2.39):

The door space is ωb -R_1 if and only if it is ωbT_2-space

Proof

Let x and y be two distinct points in X,Since X is door space for each x in X,the set {X}is open or closed, if {X} is open,since $\{X\} \cap \{y\} = \emptyset$ then $\{X\} \cap \overline{(\{y\})}^{\omega b} = \emptyset$,thus $\overline{(\{X\})}^{\omega b} \neq \overline{(\{y\})}^{\omega b}$ if{X} is closed so it is ωb-closed and $\overline{(\{X\})}^{\omega b} \cap \{y\} = \{X\} \cap \{y\} = \emptyset$ therefore $\overline{(\{X\})}^{\omega b} \neq \overline{(\{y\})}^{\omega b}$ we have $(X, \tau)$is ωb-R_1space so that there are disjoint ωb-open set U and V such that $x \in \overline{(\{X\})}^{\omega b} \subset$ U and Y $\subset$ V, so X is ωbT_2-space .

Conversely:

Let x and y be any points in X,with $\overline{(\{X\})}^{\omega b} \neq \overline{(\{y\})}^{\omega b}$ by theorem (2.2.32) so by Proposition (2.2.13) hence $\overline{(\{X\})}^{\omega b} = \{X\}$ and $\overline{(\{y\})}^{\omega b}$ this implies $x \neq y$ since X is ωbT_2-space, there are two disjoint ωb-open sets U and V Such that $\overline{(\{X\})}^{\omega b} = \{X\} \subset U$ and $\overline{(\{y\})}^{\omega b} = \{y\} \subset V$ this proves X is $\omega b\text{-}R_1$ space .

Corollary (2.2.40):

Let.(X ,τ) be door space then if X is $\omega b\text{-}R_1$space then it is $\omega b\text{-}R_0$ space.

Proof

Let X be an $\omega b\text{-}R_1$door space, then by theorem (2.2.39) is X is ωbT_2-space thus by theorem(2.2.32) such that by Theorem (2.2.21) therefore X is ωb-R_0space .

Definition (2.2.41): **[23]**

A space X is said to be regular space,if for each $x \in X$ and A closed subset X,such that $x \notin A$ there exist disjoint open sets U, Vsuch that $x \in U$ and $A \subseteq V$

Definition (2.2.42): **[14]**

Aspace X is said to be b-regular space,if for eachx $\in X$ and A closed subset of X,such that $x \notin A$,there exists disjoint b-open sets U, V such that $x \in U$ and $A \subseteq V$

Definition (2.2.43): **[14]**

AspaceX is said to be called $\acute{b}$-regular,if foreachxinX andb-closed subsetAsuch that $x \notin A$ there existdisjoint sets U,Vsuch that U openV b-openand $\exists x \in A$, $A \subseteq V$.

Definition (2.2.44):

A space X is said to be ωb-regular space,if for each x in X, and A closed set such that $x \notin A$ there exists disjoin ωb-open sets U, V such that $x \in U$ and $A \subseteq V$.

Definition (2.2.45):

A space X is said to be ωb́-regular space,if for each x in X and ωb-closed set A such that $x \notin A$ there exists disjoint set U,V ∋ U is open V is ωb-open ∋ $x \in U$, $A \subseteq V$

Remark (2.2.46):

1-It is clear that each regular space is b-regular and each b́-regular space is b-regular,However, a b-regular space is not regular in general and a b-regular space is not b́- Regular space . **[3]**

2- It is clear that each regular space is ωb-regular but the converse is not true in general, ωb-regular space is not regular.

Example (2.2.47):

Let $X = \{a, b, c, d\}, \tau = \{X, \emptyset, \{d\}, \{b, c\}, \{b, c, d\}\}$

then ωBO(X)= $\{G: G \subseteq X\}$ and X is ωb-regular space but X is not regular since $\{a, b, c\}$ is closed set d $\notin$ $\{a, b, c\}$ and thus do not exists disjoint open sets which is separate them in X.

Remark (2.2.48):

It is clear that each ωb́-regular space is ωb-regular and each,b-regular space isωb-regular,however ωb-regular space is not ωb́-regular in general and ωb-regular space is not b-regular space as the following

Examples (2.2.49):

1- Let $X = \{1,2,3,4\}, \tau = \{X, \emptyset, \{4\}, \{2,3\}, \{2,3,4\}\}$

then $BO(X) = X, \emptyset, \{2\}, \{3\}, \{3,4\}, \{1,2,4\}, \{4\}, \{2,3\}$ $, \{2,3,4\}\{4\}, \{2,3\}, \{2,3,4\}, \{1,4\}, \{1,2,3\} \{1,3,4\}, \{2,4\}$

Thus BO(X)= $\{G: G \subseteq X\}$ and X is ωb-regular space but is not ωb́-regular Since $\{2,3\}$ is ωb-closed set thus1 $\notin \{2,3\}$ but there exists no disjoint open set U and ωb-open set V such that x $\in$ U,$\{2.3\} \subseteq$ V .

2- Let$X = \{1,2,3\}, \tau = \{X, \emptyset, \{1\}, \{1,2\}\}$ then $\omega b(X) = \{G: G \subseteq X\}$ and $BO(X) = \{X, \emptyset, \{1\}, \{1,2\}, \{1.3\}\}$ and X is ωb-regular but is not b-regular space since$\{2,3\}$ is closed set $\ni$ 1 $\notin \{2,3\}$ there exist nodisjoint b-open set U and V such that 1 $\in$ U,$\{2.3\} \subseteq$ V

Theorem (2.2.50):

Let X and Y be ωb́-homeomorphic topologicalspace if X is ωb́ -regular space then , Y is ωb-regular space .

Proof

Let X and Y be ωb́-homeomorphic topological and let X be ωb́-regular space to prove Y is ωb-regular space let y $\in$Y and A is closed in Y $\ni$ y$\notin$A since X, Y be ωb́

-homeomorphic topological ∃ f: X → Yωb́-homeomor-Phism function,since f onto then there exists x ∈ X f(x) = y since f is ωb-continuous function and A^c open in Ythen $f^{-1}(A^c) = [f^{-1}(A)]^c$ is ωb-open set in X thus $f^{-1}(A)$ is ωb-closed in X and x $\notin f^{-1}(A)$ and X is ωb́-regular space, then there exists open U and ωb-open set V,U ∩ V = ∅ ∋ x ∈ U, $f^{-1}(A) \subseteq$ V then,f(U) is ωb-open set in Y (f ωb-open function) and f (V) is ωb-open set in Y thus,(f ωb́-open function) hence y ∈ f (U), A = f $(f^{-1}(A)) \subseteq$ f (V) Therefore Y,is ωb-regular space.

Proposition (2.2.51)

ATopological space X is ωb-regular space iff for every x ∈ X and each open U in X such that x ∈ Uthere exists an ωb-open set L such that x ∈ L ⊆ $\overline{L}^{\omega b}$ ⊆ U

Proof

Let X be ωb-regular space and $x \in X$,U be open set in X,such that $x \in U$ then U^c is Closed set in X and $x \notin U^c$ thus there exists disjoint ωb-open set L, V hence $x \in L$, $U^c \subseteq V$ therefore $x \in L \subseteq \overline{L}^{\omega b} \subseteq V^c \subseteq U$

Conversely:

Let $x \in X$ and M be a closed set in X,such that $x \notin M$ then M^c is an open set in X,and $x \in M^c$ thus there exists an ωb-open set L such that $x \in L \subseteq \overline{L}^{\omega b} \subseteq M^c$ hence $x \in L, M \subseteq \left(\overline{L}^{\omega b}\right)^c$ and L , $\left(\overline{L}^{\omega b}\right)^c$ are disjoint ωb-open set therefore X is ωb-regular.

Proposition (2.2.52):

ATopological space X is ωb́-regular space,iff for every $x \in X$ and every ωb-open set U in X,such that $x \in U$ there exists an open set V such that $x \in V \subseteq \overline{V}^{\omega b} \subseteq U$

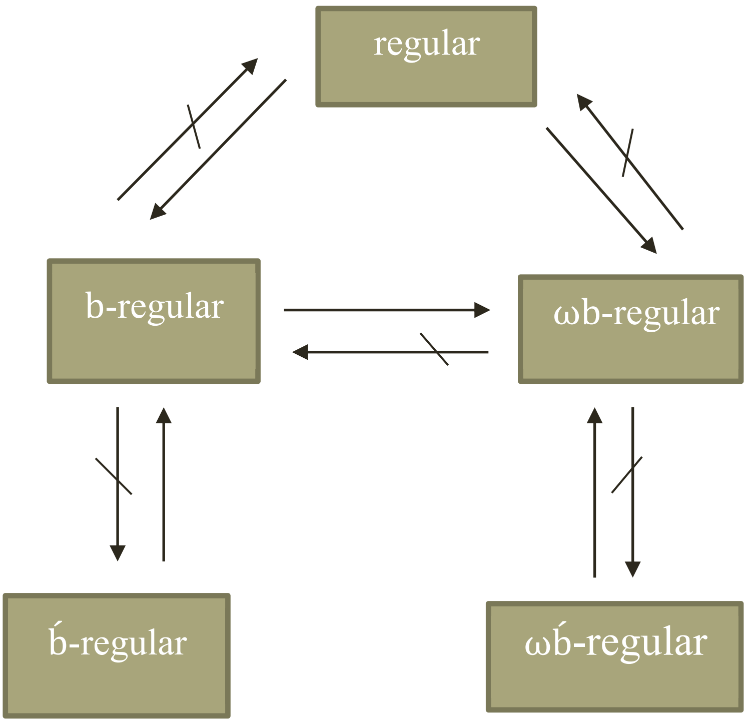
Proof

Assume that X is $\omega\acute{b}$-regular space and $x \in X$, U is ωb-open set in X, such that $x \in U$ thus U^c is ωb-closed set in X and $x \notin U^c$ since X is $\omega\acute{b}$-regular then there exist disjoint V, L such that V is open set, L is ωb-open $x \in V, U^c \subseteq L$ hence $x \in V \subseteq \overline{V}^{\omega b} = L^c \subseteq U$

Conversely:

Let $x \in X$ and M ωb-closed set in X such that $x \notin M$ then M^c an ωb-open set in X, and $x \in M^c$ thus there exist open set L, suchthat $x \in L \subseteq \overline{L}^{\omega b} \subseteq M^c$ hence $x \in L$, $M \subseteq \left(\overline{L}^{\omega b}\right)^c$ and L, $\left(\overline{L}^{\omega b}\right)^c$ are disjoint ωb-open sets therefore X is $\omega\acute{b}$-regular.

The following diagram shows the relations among the difference types of regular space.


regular
b-regular
ωb-regular
b́-regular
ωb́-regular

Definition (2.2.53): [27]

ATopological space X is called normal space,if for every $C_1$and C_2 are disjoint closed subset in X there exists disjointopensets$V_1, V_2$with $C_1 \subseteq V_1$and $C_2 \subseteq V_2$

Definition (2.2.54): [14]

A Topological space X is called b-normal space,if for every disjoint closed set C_1, C_2 there exist disjoint b-open sets V_1, V_2 such that $C_1 \subseteq V_{1,} C_2 \subseteq V_2$.

Definition (2.2.55):

ATopological space X is called b́-normal space,if for every disjoint b-closed sets C_1, C_2 there exists disjoint b-open sets V_1, V_2 such that $C_1 \subseteq V_{1,} C_2 \subseteq V_2$.

Definition (2.2.56):

A topological space X is called ωb-normal space,if for every disjoint closed sets

C_1, C_2 there exist disjoint ωb-open sets V_1, V_2 such that $C_1 \subseteq V_1, C_2 \subseteq V_2$.

Definition (2.2.57):

A Topological space X is called ωb́-normal space,if for every disjoint ωb-closed sets C_1, C_2 there exists disjoint open sets V_1, V_2 such that $C_1 \subseteq V_1, C_2 \subseteq V_2$.

Remark (2.2.58):

1- It is clear that every normal space is b-normal, but the converse is not true in general. **[15]**

2-It is clear that every normal space is ωb-normal but the converse is not true in general .

Example (2.2.59):

Let $X = \{a, b, c, d, e\}, \tau = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{a, b, d, e\}, \{d\}, \{a, d\}\}$ then $\omega BO(X) = \{G: G \subseteq X\}$ it is clear that X is ωb-normal space but is not normal.

in fact the disjoint closed sets $\{c\}\{b, e\}$ cannot be separated by open sets in X .

Remark (2.2.60):

It is clear that every b-normal space is ωb-normal and each b́-normal space is b-normal however, a ωb-normal is not b-normal space in general and b-normal space is not b́-normal as the following .

Examples (2.2.61):

1- Let $X = \{1,2,3, \dots\}$ and τ be cofinite topology on X then $BO(X) = \{G: G \subseteq X, G \text{ is infinite and } G^c \text{ infinite}\} \cup \{G: G \subseteq X, G^c \text{is finite}\} \cup \{\emptyset\}$ and $\omega BO(X) = \{G: G \subseteq X\}$ thus, X is ωb-normal but is not b-normal,since$\{1\}$,$\{2\}$ are disjoint closed sets in X but there is not exists disjoint b-open set U,V such that $\{1\} \subseteq U$,$\{2\} \subseteq V$

2 − Let $X = \{1,2,3,4,5\}, \tau = \{X, \emptyset, \{1\}, \{3,4\}, \{1,3,4\}, \{1,2,4,5\}, \{4\}, \{1,4\}\}$ then

$\mathrm{BO(X)} = \{X, \emptyset, \{1\}, \{3,4\}, \{1,3,4\}, \{1,2,4,5\}, \{4\}, \}$
$\{\{1,2,4\}, \{1,4,5\}, \{1,2,3,4\}, \{1,3,4,5\}, \{2,4,5\}, \{2,3,4,5\}$
$\{1,5\}, \{2,4\}, \{4,5\}\{3,4,5\}, \{2,3,4\}, \{1,2,5\}, \{1,2\}\}$ and

X is b-normal but is not b́-normal space $\{2\}\{1,3\}$ are disjoint b-closed sets in X ,but there is not exists two disjoint b-open set U,Vsuch that $\{2\} \subseteq \mathrm{U}, \{1,3\} \subseteq \mathrm{V}$

Remark (2.2.62):

It is clear that every ωb́-normal space is ωb-normal but the converse is not true in general

Example (2.2.63):

Let $X = \{1,2,3, \ldots\}, \tau = \{G: G \subseteq X, 1 \notin G\} \cup$ $\{G: G \subseteq X, 1 \in G, G^c \text{ is finite }\}$ then $\omega\mathrm{BO(X)} =$ $\{G: G \subseteq X\}$ thus X is ωb-normal but is not ωb́-normal space, since $\{1\}$,$\{2,3,4, \ldots\}$ are disjoint ωb-closed set but has not disjoint open set.

Theorem (2.2.64):

Let X and Y be homeomorphic topological space,If X is ωb́-normal space then, Y is normal space .

Proof

Let X and Y be homeomorphic topological space and X be ωb́-normal space,to prove Y is normal space,let C_1, C_2 are closed in Y, $C_1 \cap C_2 = \emptyset$, since X and Y be homeomorphic topological space then $\exists\ f: X \to Y$ homeomorphism topological,sincef is continuous function $\exists\ {C^c}_1, {C^c}_2$ are open set thus, $f^{-1}({C^c}_1) = [f^{-1}(C_1)]^c, [f^{-1}(C_2)]^c = f^{-1}({C^c}_2)$ are open set in X,hence $f^{-1}(C_1)$,$f^{-1}(C_2)$ are closed set in X,therefore, $f^{-1}(C_1)$, $f^{-1}(C_2)$ are ωb-closed set in X $\ni C_1 \cap C_2 = \emptyset$, then $f^{-1}(C_1) \cap f^{-1}(C_2) = f^{-1}(C_1 \cap C_2) = \emptyset$

Since X is ωb́-normal space, then there exist open sets U, V in X $\ni f^{-1}(C_1) \subseteq U$,$f^{-1}(C_2) \subseteq V$ and $U \cap V = \emptyset$

since f bijective such that $C_1 \subseteq f(U), C_2 \subseteq f(V)$ and $f(U) \cap f(V) = f(U \cap V) = f(\emptyset) = \emptyset$ thus,$f(U)$, $f(V)$ are open set in Y (f is open function)

Therefore,Y is normal .

Theorem (2.2.65):

Let X and Y be ωb́-homeomorphic topological space,if X is ωb́-normal space then Y is ωb-normal space .

Proof

Similar to prove of theorem (2.2.64) .

Proposition (2.2.66): **[23]**

A space X is normal space,iff for every closed set $D \subseteq X$ and each open set U in X,such that $D \subseteq U$ there exists an open set V such that $D \subseteq V \subseteq \overline{V} \subseteq U$.

Proposition (2.2.67):

ATopological space X is ωb-normal space,iff for every closed set $D \subseteq X$ and each open set U in X such that $D \subseteq U$ there exists an ωb-open set V such that

$D \subseteq V \subseteq \overline{V}^{\omega b} \subseteq U$

Proof

Let X be ωb-normal space and let D be closed set and U open set in $X \ni D \subseteq U$ thenD, U^c are disjoint closed sets in X Since X is ωb-normal space thus,there exists disjoint ωb-open sets V, L hence $D \subseteq V, U^c \subseteq L$ therefore $D \subseteq V \subseteq \overline{V}^{\omega b} \subseteq \overline{L^c}^{\omega b} = L^c \subseteq U$

Conversely:

Let D_1, D_2 be disjoint closed sets in X,then D_2^c is open set in X and $D_1 \subseteq D_2^c$ there exists an ωb-open set V such that $D_1 \subseteq V \subseteq \overline{V}^{\omega b} \subseteq D_2^c$ hence $D_1 \subseteq V$

, $D_2 \subseteq \left(\overline{V}^{\omega b}\right)^c$ and V, $\left(\overline{V}^{\omega b}\right)^c$ are disjoint ωb-open sets therefore X is ωb-normal space.

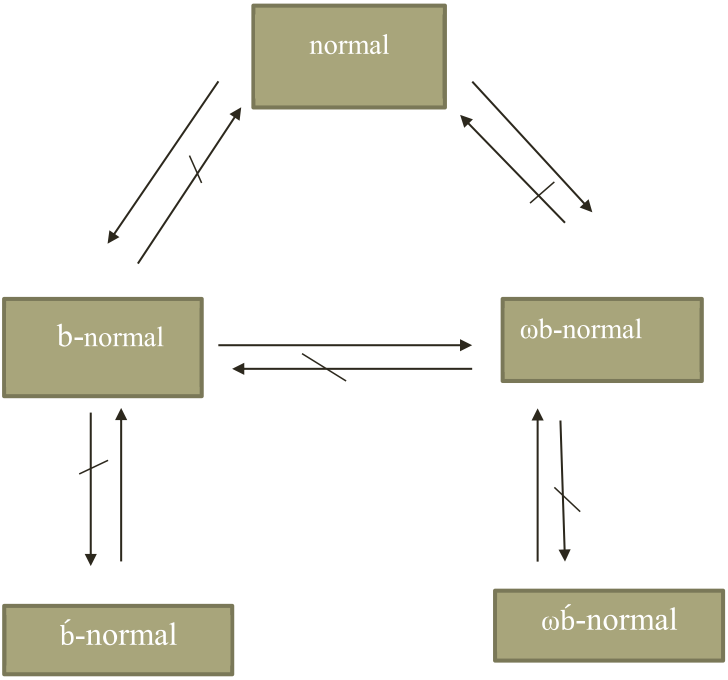
Proposition (2.2.68):

A space X is ωb́-normal space iff for every ωb-closed set C in X and each ωb-open set U in X,hence $C \subseteq U$ there exists an open set V such that $C \subseteq V \subseteq V^{-\omega b} \subseteq U$.

Proof

Similar to prove of theorem (2.2.67) .

The following diagram shows the relations among the disfferent typeof normal space .


normal
b-normal
ωb-normal
b́-normal
ωb́-normal

Proposition (2.2.69):

If X is both ωb-normal space and T_1-space then X is ωb-regular space .

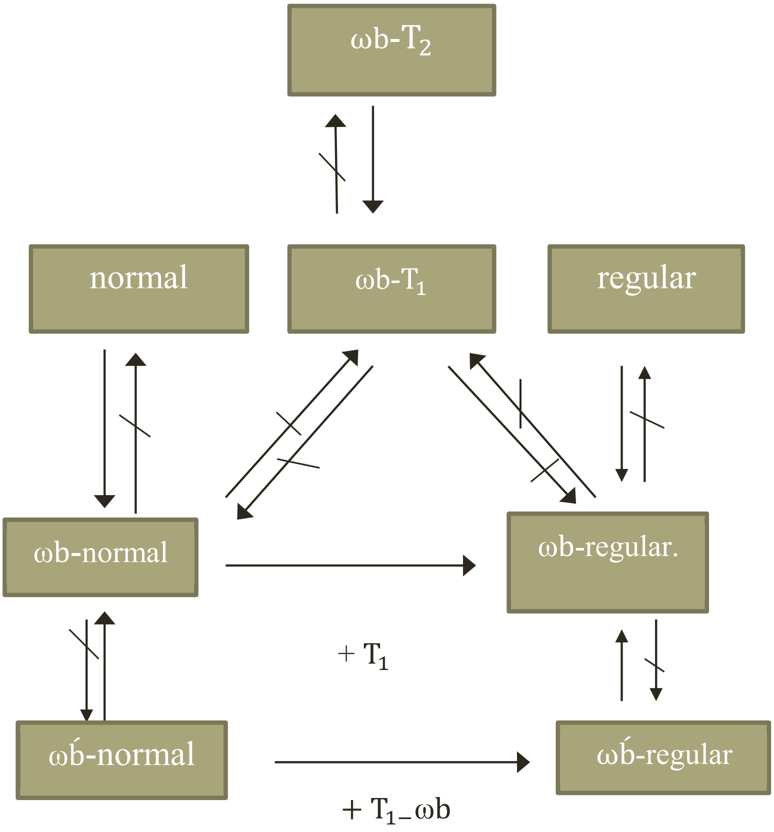
Proof

Let $x \in X$ and L be an open set in X such that $x \in L$ then $\{x\}$ is closed subset of X and $\{x\} \subseteq L$ since X is ωb-normal,thus there exists an ωb-open set V such that $\{X\} \subseteq V \subseteq \overline{V}^{\omega b} \subseteq L$ by proposition (2.2.67) So that $x \subseteq V \subseteq \overline{V}^{\omega b} \subseteq L$ and hence by Propo- sition (2.2.51) therefore X is ωb-regular space .

Corollary (2.2.70):

IF X is both $\omega\acute{b}$-normal space and ωbT_1-space then X is $\omega\acute{b}$-regular space .

The following diagram explains the relationship among these types of ωb-separation axiom


ωb-T_2
normal
ωb-T_1
regular
ωb-normal
ωb-regular.
+ T_1
ωb́-normal
ωb́-regular
+ T_{1-}ωb

2.3 ωb-Connected Space

In this section, we introduce the definition ωb-connected space and with some propositions and remarks related to them that are proved .

We recall that any two subsets A and B of a space (X, τ) are called τ-separated if $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ see [20]

Definition (2.3.1):

Let X be a space two Subsets A and B of a space X are called ωb-separated if

$\overline{A}^{\omega b} \cap B = A \cap \overline{B}^{\omega b} = \emptyset$.

Definition (2.3.2): [9]

Let X be a topological space and $\emptyset \neq A \subseteq X$ then A is called connected set,if A is not union of any two separated sets .

Definition (2.3.3):

Let X be a topological space and $\emptyset \neq A \subseteq X$ then A is called ωb-connected set,if is not union of any two ωb-separated sets.

Definition (2.3.4):

A set is called ωb-clopen if it is ωb-open and ωb-closed .

Proposition (2.3.5):

Let (X, τ)be topological a space,then the following statements are equivalent:

i. X is ωb-connected space.

ii. the only ωb-clopen sets in the space are X and $\emptyset$.

iii.There exist no two disjoint nonempty ωb-open set A and B such that $X=A\cup B$

Proof

(i)→(ii)

Let X be ωb-connected space, suppose that D is ωb-clopen set such that $D \neq \emptyset$ and $D \neq X$, let $E = X - D$ since $D \neq X$,then $E \neq \emptyset$,Since D is ωb-open sets,then E is ωb--closed but $\overline{D}^{\omega b} \cap E = D \cap E = \emptyset$ (since D is ωb-clopen set and E is ωb-closed set) hence $\overline{D}^{\omega b} \cap E = D \cap \overline{E}^{\omega b} = \emptyset$,thus D and E are two ωb-separated sets and $X = D \cup E$,hence X is not ωb-connected space,which is contradiction therefor the only ωb-clopen sets in X are $\emptyset$ and X .

(ii)→(iii)

Suppose the only ωb-clopensets in the space are $\emptyset$ and Xassume that there exists two disjoint non-empty ωb-open A and B such that $X = A \cup B$ since $A = B^c$,then A is ωb-clopen set but $A \neq \emptyset$ and $A \neq X$ which is a contradiction hencethere exists no two disjoint non

-mpty ωb-open set A and B such that $X = A \cup B$

(iii)→(i)

Suppose that X is not ωb-connected space there exist two non − empty ωb-Separated sets A and B such that, $X = A \cup B$ Since $\overline{A}^{\omega b} \cap B = A \cap \overline{B}^{\omega b} = \emptyset$ and the $A \cap B \subseteq \overline{A}^{\omega b} \cap B$ then $A \cap B = \emptyset$ Since $\overline{A}^{\omega b} \subseteq B^c = A$ thus A is ωb-closed set by same way we can see that B is ωb-closed set,since $A = B^c$ thus A and B are twodisjoint non-empty ωb-open sets such that $X = A \cup B$ which is a contradiction therefor X is ωb-connected space .

Remark (2.3.6):

Every ωb-connected space is connected but the converse is not true in general .

Example (2.3.7):

Let X={1,2,3} ,τ ={X, ∅,{1},{2},{1,2}},

$\omega b(X)=\{X, \emptyset, \{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}\}$, it is clear that X is connected space, but X is not ωb-connected since $\{1\},\{2,3\}$ are ωb-open in X such that $\{1\} \cup \{2,3\} = X$ and since $\{1\},\{2,3\}$ are disjoint ωb-open set.

Proposition (2.3.8):

Let A be ωb-connected set and D, E be ωb-separated sets, if $A \subseteq D \cup E$, then either $A \subseteq D$ or $A \subseteq E$.

Proof

Suppose that A be ωb-connected set and D, E are ωb-separated sets and $A \subseteq D \cup E$, let $A \nsubseteq D$ and $A \nsubseteq E$ suppose $A_1 = D \cap A \neq \emptyset$ and $A_2 = E \cap A \neq \emptyset$ since $A \subseteq D \cup E$ then $(D \cup E) \cap A = A$ thus $(D \cap A) \cup (E \cap A) = A$ therefore $A_1 \cup A_2 = A$, since $A_1 = D \cap A$, then $A_1 \subseteq D$ thus $\overline{A_1}^{\omega b} \subseteq \overline{D}^{\omega b}$ since D, E are ωb-separated sets then $\overline{D}^{\omega b} \cap E = \emptyset$, then

$\overline{A_1}^{\omega b} \cap A_2 = \emptyset$ since $A_2 = E \cap A$ thus, $A_2 \subseteq E$ thus $\overline{A_2}^{\omega b} \subseteq \overline{E}^{\omega b}$ thus $A_1 \cap \overline{A_2}^{\omega b} = \emptyset$ and $A = A_1 \cup A_2$ therefore A is a union of two ωb-separated set A_1, A_2 hence A is not ωb-connected set this contradic - tion,then either $A \subseteq D$ or $A \subseteq E$

Proposition (2.3.9):

Let X be a space such that any two elements xand y of X are contained in some ωb-connected subspace of X then X is ωb-connected .

Proof

Let X is not ωb-connected then X is the union of two ωb-separated sets A and B, since A, B non empty sets there exists a, b such that $a \in A$, $b \in B$, let M be ωb-connected subspace of X,which contains a, b thus $M \subseteq A$ or $M \subseteq B$ by Proposition(2.3.8) which is contrad-iction(since $A \cap B = \emptyset$) therefore X is ωb-connected

Proposition (2.3.10):

If D is ωb-connected set and $D \subseteq E \subseteq \overline{D}^{\omega b}$,then E is ωb-connected .

Proof

Let D be ωb-connected set and $D \subseteq E \subseteq \overline{D}^{\omega b}$ suppose E is not ωb-connected then there exists two sets A, B such that $\overline{A}^{\omega b} \cap B = A \cap \overline{B}^{\omega b} = \emptyset$ and $E = A \cup B$ Since $D \subseteq E = A \cup B$ thus either $D \subseteq A$ or $D \subseteq B$,By proposition (2.3.8),if $D \subseteq A$,then $\overline{D}^{\omega b} \subseteq \overline{A}^{\omega b}$,thus $\overline{D}^{\omega b} \cap B = \overline{A}^{\omega b} \cap B = \emptyset$, but $D \subseteq E \subseteq \overline{D}^{\omega b}$, $\overline{D}^{\omega b} \cap B = B$,therefore $B = \emptyset$ which is contradiction,hence E is ωb-connected set by the same way can get a contradiction if $D \subseteq B$ hence E is ωb-connected.

Proposition (2.3.11):

If A is ωb-connected set then $\overline{A}^{\omega b}$ is ωb-connected.

Proof

Suppose A is ωb-connected and $\overline{A}^{\omega b}$ is not ωb-connected then there exists ωb-separated sets D, E such that $\overline{A}^{\omega b} = D \cup E$, since $A \subseteq \overline{A}^{\omega b}$ then $A \subseteq D \cup E$ Since A is ωb-connected then by proposition (2.3.8) either $A \subseteq D$ or $A \subseteq E$, if $A \subseteq D$ then $\overline{A}^{\omega b} \subseteq \overline{D}^{\omega b}$ but $\overline{D}^{\omega b} \cap E = \emptyset$,hence $\overline{A}^{\omega b} \cap E = \emptyset$ Since $\overline{A}^{\omega b} = D \cup E$, thus $E = \emptyset$ this, Contradiction,if $A \subseteq E$ then $\overline{A}^{\omega b} \subseteq \overline{E}^{\omega b}$ but $\overline{D}^{\omega b} \cap E = \emptyset$hence $\overline{A}^{\omega b} \cap D = \emptyset$ since $\overline{A}^{\omega b} = D \cup E$,thus$D = \emptyset$ which is contradiction therefor $\overline{A}^{\omega b}$ is ωb-connected

Definition (2.3.12):

Let X be a space, $A \subseteq X$, A is called ωb-dence set in X,if $\overline{A}^{\omega b} = X$

We mentioned that a space X is said to be hyper connected if, for every nonempty open subset of X is dence see [1]

Remark (2.3.13):

Let X be a topological space and $A \subseteq X$,if A is ωb-connected set in X then,$\overline{A}$ need not is ωb-connected set in X .

Example (2.3.14):

Let$X = \{a,b,c\}$,$\tau = \{X, \emptyset\}$, $\omega b(X)=\{X ,\emptyset \{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\}\}$,let$A=\{a\}$then,A is ωb-connected in X but $\overline{A} = X$ is no ωb-connected since $\exists \{a\},\{b,c\}$ are ωb-open sets in X $\ni \{a\} \cup \{b,c\} = \overline{A}$and $\{a\} \cap \{b,c\} = \emptyset$.

Corollary (2.3.15):

If a Topological space X,contains a ωb-connected subspace Esuch that $\overline{E}^{\omega b} = X$,then X is ωb-connected

Proof

Suppose Ea ωb-connected subspace of a space X,such that $\overline{E}^{\omega b} = X$ since $E \subseteq X = \overline{E}^{\omega b}$ then by proposition (2.3.10) thus,X is ωb-connected .

Proposition (2.3.16):

The ωb-continuous from a topological space (X, τ) onto image of a topological space(Y, τ')ωb-connected space is connected.

Proof

Let $f: (X, \tau) \rightarrow (Y, \tau')$ be ωb-continuous, onto function and X is ωb-connected to prove that Y is connected, suppose Y is not connected space so $Y = A \cup B$ such that $A \neq \emptyset, B \neq \emptyset$ since A,B are disjoint open set, then $f^{-1}(Y) = f^{-1}(A \cup B)$ thus $X = f^{-1}(A) \cup f^{-1}(B)$,since f is ωb-continuous hence $f^{-1}(A)$ and $f^{-1}(B)$ are ωb-open in X,and since that $A \neq \emptyset, B \neq \emptyset$ and f is onto then,$f^{-1}(A) \neq \emptyset, f^{-1}(B) \neq \emptyset$ and $f^{-1}(A) \cap f^{-1}(B) = \emptyset$ hence X is not ωb-connected space which is contradiction therefore, Y is connected .

Corollary (2.3.17):

The ωb́-continuous from atopological space (X, τ),onto image of topological space (Y, τ') ωb-connected space is ωb-connected space

Proof

same to proof of Proposition (2.3.16)

Proposition (2.3.18):

Let X be topological space,and let $Y = \{a, b\}$ have the discrete space,then X is ωb-connected iff there is not ωb-continuous function from X onto Y.

Proof

Suppose $f: (X, \tau) \to (Y, \tau')$ is ωb-continuous, onto function so there exists $x, y \in X$ such that $x \neq y, f(x) = a, f(y) = b$ then,$f^{-1}(\{a\}) = A, A \subseteq X$ and $f^{-1}(\{b\}) = B$,B $\subseteq$ X thus A and B are ωb-open set in X and since f is ωb-continuous, hence $X = A \cup B$ such that $A \cap B = \emptyset, A \neq \emptyset, B \neq \emptyset$ which is a contrad-iction therefore,X is ωb-connected .

Conversely:

Suppose there is no ωb-continuous, onto function and let X is not ωb-connected then,$X = A \cup B$ such that

$A \neq \emptyset, B \neq \emptyset$, $A \cap B = \emptyset$ thus A,B are ωb-open disjoint sets Define $g: (X, \tau) \rightarrow (Y, \tau')$ such that

$g(X) = \begin{cases} a & \forall x \in A \\ b & \forall x \in B \end{cases}$ hence $g^{-1}(a) = A$, $g^{-1}(b) = B$ thus,g is ωb-continuou

which is contradiction therefore X is ωb-connected.

Definition (2.3.19): [15]

Atopological space (X, τ) is said to be locally connected,if for each point $x \in X$,and each open set U such that $x \in U$ there exists connected open V such that $x \in V \subseteq V$

Definition (2.3.20):

A topological space (X, τ) is said to be ωb-locally connected ,if for each point $x \in X$ and each ωb-open set U such that $x \in U$,there exists ωb-connected open V such that $x \in V \subseteq U$.

Proposition (2.3.21):

Every ωb-locally connected space is locally connected space.

Proof

Let X is ωb-locally.connected space,let $x \in X$ and U open set inX $\ni x \in U$ then,there exists a ωb-connected open set Vsuch that $x \in V \subseteq U$since X is ωb-locally connected since that every ωb-connected set is connected, thus V is connected open set in X,such that $x \in V \subseteq U$ therefore,X is locally connected space.

Remark (2.3.22):

But The convers of proposition (2.3.21) is not true in general.

Example (2.3.23):

Let $X = \{1,2,3\}$, $\tau = \{X, \emptyset, \{2,3\}\}$ the ωb-open sets are $X, \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\},$

$\{1,3\},\{2,3\}$ then (X,τ) is locally connected but (X,τ) is no ωb-locally connected,since $1 \in \{1,2\}$ and there exists no V ωb-connected open set such that $1 \in V \subseteq \{1,2\}$.

Definition (2.3.24):

Let (X,τ) be any topological space a maximal ωb-connected of X is said to be ωb-component of X .

Remark (2.3.25):

If (X,τ) is a ωb-locally connected space,then (X,τ) it need not be ωb-connected and the converse is not true in general

Example (2.3.26):

Let.$X = \{a,b,c\}, \tau =$ $\{X,\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\}\}$ the $\omega b(X) = \tau$, it is clear (X,τ) ωb-locally connected but(X,τ) is not ωb-connected since $\{a\},\{b,c\}$are ωb-open sets in X, such that $X = \{a\} \cup \{b, c\}$ and $\{a\}, \{b, c\}$ are disjoint in X .

Theorem (2.3.27):

For a topological space (X, τ) the following condition are equivalent:

1- X is a ωb-locally connected

2- Every ωb-component subset of every ωb-open set is open.

Proof

(1)→(2)

Let X be ωb-locally connected and let Mbe ωb-component of A,such that $x \in M$,since $x \in X$ and A is ωb-open set in X hence $x \in M \subseteq A$,then $x \in A$ and A is ωb-open set in X,since X is ωb-locally connected thus,there exists ωb-connected open set V in X such that $x \in V \subseteq A$ since M is ωb-component,hence

$V \subseteq M$ and $\bigcup_{x \in M} V_x \subseteq M$,$\ni M = \bigcup_{x \in X} \{V_x : x \in M\}$ therefore,M is open set.

(2)→(1)

Let $x \in X$ and U be ωb-open set in X such that $x \in U$ and let M ωb-component of U such that $x \in M \subseteq U$, thus M is open set in X,by (2) Since that M is, ωb-component, hence M is ωb-connected therefore X is a ωb-locally connected .

<u>Proposition (2.3.28):</u>

The ωb-continuous, and open,image of ωb-locally connected space is locally connected .

Proof

Let $f:(X,\tau) \to (Y,\tau')$ be ωb-continuous open and onto function and (X,τ) is ωb-locally connected space to prove (Y,τ') is locally connected,let $y \in Y$ and U is open set in

Y ∋ y ∈ U Since f is onto,then ∃ x ∈ X such that $f(x) = y$ since f is ωb-continuous hence $f^{-1}(U)$ is ωb-open set in X such that $x \in f^{-1}(U)$, since X is ωb-locally connected thus there exist V is ωb-connected open set in X such that x ∈ V ⊆ f^{-1}(U) since X is ωb-locally connected then f(x) ∈ f(V) ⊆ U such that f(V) is open and f(V) is connected by proposition (2.3.16) therefore Y is a locally connected

Corollary (2.3.29):

The ωb́-continuous,open,image of ωb-locally connected space is ωb-locally connected .

Proof

Let $f: (X, \tau) \to (Y, \tau')$ be ωb́-continuous ,open and onto function and (X, τ) is ωb-locally connected space to prove (Y, τ′) is ωb-locally connected,let y ∈ Y and U is ωb − open set in Ysuch that y ∈ U, since f onto there exist x ∈ X such that $f(x) = y$ for each y ∈ Y,since f is ωb́-continuous,hence $f^{-1}(U)$ is ωb-open set in X such that x ∈ $f^{-1}(U)$ Since X is ωb-locally connected then ∃ V ωb-connected open set in X such that that $x \in V \subseteq f^{-1}(U)$ since f is open then $f(V)$ is open set in Y,and $f(V)$ is ωb-connected by corollary (2.3.17) Hence $f(V)$ is ωb-connected open set in Y,such that $y \in f(V) \subseteq U$ therefore,Y is a ωb-locally connected space.

Remark (2.3.30):

The ωb-continuous image of ωb-locally connected need not be ωb-locallyconnected

Example (2.3.31):

Let X = {1, 2 ,3},Y={a, b ,c},τ=D τ′={ Y,∅,{a}} ωb(X)=D , ωb(Y) =

$\{Y, \emptyset, \{a\}, \{b\}, \{c\}, \{b,c\}, \{\{a,b\}, \{a,c\}\}$,Define

$f: (X, \tau) \to (Y, \tau')$ such that $f(1) = a, f(2) = b, f(3) =$ c,is ωb-continuous,ontofunction,it is clear (X,τ) is ωb-locally connected but(Y,τ′)is not ωb-locally connected since b ∈{a ,b} and exists no ωb-connected open setV inY such that $b \in V \subseteq \{a, b\}$

Definition (2.3.32):

A space X is said to be ωb-hyper connected,if for every nonempty ωb-open subset of X is ωb-dence.

Now, we explain the relation between an ωb-hyper connected space and hyper connected space

Proposition (2.3.33):

Every ωb-hyper connected space is hyper connected.

Proof

LetX be ωb-hyper connected space,then for every ωb-open set of X,is ωb-dence in X, then $\overline{A}^{\omega b} = X$,to prove X is hyper connected since $\overline{A}^{\omega b} \subseteq \overline{A}$ and

$\overline{A}^{\omega b} = X$ thus $\overline{A} = X$,Therefore X is hyper connected .

Remark (2.3.34):

The convers of the proposition (2.3.33) is not true in general

Example (2.3.35):

Let $X = \{1,2,3\}$, $\tau = \{X, \emptyset\}$ the ωb-open sets $\omega b(X) = \{X, \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$ it is clear (X, τ) is hyper connected but (X, τ) is not ωb-hyper connected, since $\{1\} \in \omega b(X)$ and $\overline{\{1\}}^{\omega b} = \{1\} \neq X$

Proposition (2.3.36):

Every ωb-hyper connected space is ωb-connected.

Proof

Let X be ωb-hyper connected space and suppose X is not ωb-connected,then there exists A is ωb-clopen set in X such that $A \neq \emptyset$.and $A \neq X$,thus $A = \overline{A}^{\omega b}$ which is a contradiction (since X is ωb-hyper connected) therefore X is ωb-connected.

Lemma (2.3.37):

Let τ and $\acute{\tau}$ be two topological space on the set X,such that $\omega bO(X) \subseteq \omega\acute{b}O(X)$,then $\overline{A}^{\omega\acute{b}} \subseteq \overline{A}^{\omega b}$ $,\forall A \subseteq X$.

Proof

Let $x \in \overline{A}^{\omega\acute{b}}$,then $A \cap \acute{B} \neq \emptyset \ \forall \acute{B} \in \omega\acute{b}O(X) \ni x \in \acute{B}$ by proposition (1.1.24) hence $A \cap \acute{B} \neq \emptyset \ \forall \acute{B} \in \omega\acute{b}O(X) \ni x \in \acute{B}$, thus $x \in \overline{A}^{\omega\acute{b}}$ therefore $\overline{A}^{\omega\acute{b}} \subseteq \overline{A}^{\omega b}$

Proposition (2.3.38):

Let τ and τ' be two topological space on the setX,such that $\tau \subseteq \tau'$ and $\omega bO(X) \subseteq \omega b\acute{} O(X)$,if $(X,\acute{\tau})$ is ωb-hyper connected space then (X,τ) is ωb-hyper connected .

Proof

Let $U \in \omega bO(X)$ then $U \in \omega b\acute{} O(X)$, hence $\overline{U}^{\omega\acute{b}} = X$ (since (X,τ') is ωb-hyper connected) and sincethat $U^{-\omega\acute{b}} \subseteq U^{-\omega b}$ by lemma (2.3.37)thus $x \subseteq U^{-\omega b}$ but $\overline{A}^{\omega b} \subseteq X$ therefore (x,τ) is ωb-hyper connected.

Remark (2.3.39):

The converse of the proposition (2.3.38) is not true in general

Example (2.3.40):

Let X=R,$\tau = \{A \subseteq R: A^c \text{countable}\} \cup \{\emptyset\}$, $\tau\acute{}$=D , $\omega bO(X) = \{A \subseteq R: A \text{ uncountable}\} \cup \{\emptyset\}$, $\omega b\acute{} O(X)$

= D,it is clear(X,τ) that is ωb-hyperconnected, then.

$A \in \omega bO(X), \overline{A}^{\omega b} = X$, but $(X, \tau') = D$ is not ωb-

heper connected thus $\overline{A}^{\omega b}$ =A for every $A \in \omega \acute{b}O(X)$

Definition (2.3.41): [13]

A Topological space (X, τ) is said to be extremally disconnected, if closure of every open subset of X is open in X.

Definition (2.3.42):

A topological space$(X, \tau)$is said to be ωb-extremally disconnected,if the closure of every open subset of X is ωb-open.

Remark (2.3.43):

Every extremally disconnected space is ωbextremally disconnected space and the convers is not true in general.

Example(2.3.44):

Let.X $= \{1,2,3\}$, $\tau = \{X, \emptyset, \{1\}, \{2\}, \{1,2\}\}$ then

$\omega bO(X) = \{X, \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$

is clear that (X, τ) is ωb-extremally disconnected but (X, τ) is not Extremally disconnected since $\overline{\{1\}} = \{1,3\} \notin \tau$.

Theorem (2.3.45):

The a topological space (X,τ) if X is ωb-extremally disconnected then regular closed sub set of X is ωb-open set .

Proof

Let A is regular closed subset of X,since $A \subseteq \overline{A}$,then $A^{\circ} \subseteq \overline{A}^{\circ} \subseteq \overline{\overline{A}^{\circ}}$ and since that $\overline{A}^{\circ}$ is open set in X thus $A \subseteq \overline{\overline{A}^{\circ}} \subseteq \overline{\overline{A}^{\circ}}^{\circ\omega b}$ (since X ωb-extremally disconnectd) but A is closed set thus $\overline{\overline{A}^{\circ}}^{\circ\omega b} = \overline{A^{\circ}}^{\circ\omega b}$

hence $\overline{A^{\circ}}^{\circ\omega b} = A^{\circ\omega b}$ therefore A is ωb-open set.

Proposition (2.3.46):

Every ωb-hyper connected is a ωb-extremally disconnected space but the convers is not true in general.

Proof

Let A be open in X,since X ωb-hyper connected space then $\overline{A}^{\omega b} = X$ since $\overline{A}^{\omega b} \subseteq \overline{A}$,thus. $\overline{A} = X$ hence $\overline{A}$ is ωb-open set (since X is open then X is ωb-open) therefore X is ωb-extremally disconnected.

Remark (2.3.47):

But The converse of Proposition (2.3.47) is not true in general .

Example (2.3.48):

Let X = {1,2,3}, τ =D, then $\omega bO(X) = D$,it is clear that (X ,τ) is ωb-extremally disconnected since the closure of every open subset of X,is ωb-open, but (X ,τ) isnot ωb-hyper connected since $\overline{A}^{\omega b} = A \neq X$ $\forall A \in \omega bO(X)$.

Proposition (2.3.49):

If X is ωb-connected then X is not to ωb-extremely disconnected .

Proof

Let X ωb-connected and X is ωb-extremely disconnected to get the contradiction then for every A open set we get $\overline{A}$ ωb-open Since $\overline{A}$ closed set then $\overline{A}$ ωb-closed thus Xis not ωb-connected byproposition (2.3.5) if X is ωb-connected then the only ωb-clopen sets are ∅,X Therefore,X is not ωb-extremely disconnected but the convers is not true in general as the following .

Chapter Three

On ωb-compact spaces, ωb–lindelof spaces

Introduction

This chapter is divided into two sections In section one, we introduce the definitions of compact,b-com

pact,ω-compactand ωb-compact spaces,we find the relation between them moreover,we give some generalizations on this concept,also we introduce new concept namely nearlyωb-compact space and give useful characterization on this concept,some results about this subject are proved.In section two weintroduce new definition of ωb-lindelof space and nearly ωb-lindelof space to the best of our knowledge we give some results which are related with this subject we introduce the concept ofalmost contra-ωb-continuous function via the notion of ωb-open set and study some set equivalent of almost contra ωb-continuous.

3.1 ωb-compact space

In this section we introduce the definition of ωb-compact space and proved some propositions and remarks which are related to it also we introduce the definition ωb-compact function and we study it is relation to other known classes of generalizedcompact function as ωb-compact function .

Definition (3.1.1): [25]

ATopological space X is said to be compact if every open cover of X, has a finite sub cover.

Definition (3.1.2): [14]

A topological space X is said to be b-compact,if every b-open cover of X has a finit sub cover.

Remark (3.1.3): [3]

Let (X,τ) be topological space then,b-compact space is compact

Remark (3.1.4): [3]

But the converse of (3.1.3) is not true in general as the example .

Definition (3.1.5): [17]

A topological space X is said to be ω-compact,if every ω-open cover of X,has a finite sub cover.

Theorem (3.1.6): [25]

1- Every closed subset of a compact space is compact

2- In any topological space the intersection of a compact subset with closed subset is compact.

3- Every compact subset of a Hausdorff space is closed.

Definition (3.1.7):

A topological space X is said to be ωb-compact,if every ωb-open cover of X has a finite sub cover.

Remark (3.1.8):

1- It is clear that every ωb-compact space is compact.

2- It is clear that every ω-compact space is compact.

Remark (3.1.9):

But the converse of (3.1.8) is not true in general as the example **[17]**

Remark (3.1.10):

1- Every b-compact is not true in general ω-compact .

2- Every b-compact is not true in general ωb-compact as the follows

Example (3.1.11):

Let $\mathrm{X} = \mathrm{Z}$,be the integer number with topological ,$\tau = \{\mathrm{X}, \emptyset, \mathrm{Z}^{+}, \mathrm{Z}^{-}\}$ then $\mathrm{BO(X)} = \{A \subseteq X: 0 \notin A\} \cup \{X\}$ thus, X is b-compact since $\omega\mathrm{o(X)} = \omega\mathrm{BO(X)} = \{\mathrm{A}: \mathrm{A} \subseteq \mathrm{X}\}$ therefore,X is not ω-compact and ωb-compact

Remark (3.1.12):

1- Every ω-compact is not true in general b-compact.

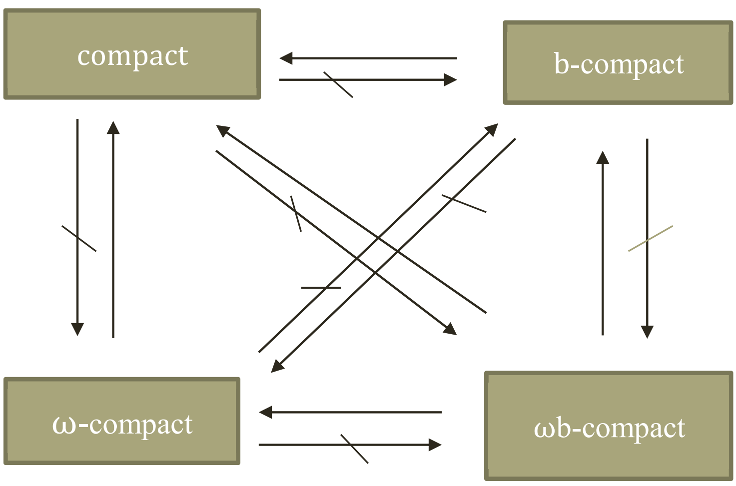
2- Every ω-compact is not true in general ωb-compact.

as the follows

Example (3.1.13)

Let B is an un countable, $X = B\cup\{a\}$,. $a \notin B$ and, $\tau = \{\emptyset, X, \{a\}\}$ then, $\omega o(X) = \{\emptyset, X, \{a\}\} \cup \{G \subseteq X: G^c \text{ is finite }\}$ thus, X is ω-compact since $BO(X) = \{\{a, b\}: b \in B\}$ and $\omega BO(X) = \{A: A \subseteq X\}$ thus, X is not b-compactand ωb-compact .

The following diagram shows the relations amongs the different types of compact space.


compact
b-compact
ω-compact
ωb-compact

Theorem (3.1.14):

Let $f: X \to Y$ be an onto, ωb-continuous function, if X is ωb-compact then,Y is compact.

Proof

Let $\{G_\lambda: \lambda \in I\}$be an open cover of Y then $\{f^{-1}(G_\lambda): \lambda \in I\}$is an ωb-open cover of X, Since X is ωb-compact thus, X has finite subcover say $\{f^{-1}(G_{\lambda i}): i = 1,2,..,n\}$ and $G_{\lambda i} \in \{G_\lambda: \lambda \in I\}$ hence $\{G_{\lambda i}: i = 1,2,\ldots,n\}$ is afinite sub cover of Y therefore,Yis compact

Proposition (3.1.15):

For any topological space X,the following statement are equivalent:

1- X is ωb-compact.

2- Every family of ωb-closed sets $\{V_\alpha: \alpha \in \Lambda\}$ of X such that $\bigcap_{\alpha\in\Lambda} V_\alpha = \emptyset$ then,there exists finite subset $\Lambda_0 \subseteq \Lambda$ such that $\bigcap_{\alpha\in\Lambda_0} V_\alpha = \emptyset$.

Proof

(1)→(2)

Assume that X is ωb-compact,let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of ωb-closed subset of X such that $\bigcap_{\alpha\in\Lambda} V_\alpha = \emptyset$ then,the family $\{X - V_\alpha : \alpha \in \Lambda\}$ is ωb-open cover of the ωb-compact (X, τ)there exists a finite subset Λ_0 of Λ,thus $X = \bigcup\{X - V_\alpha : \alpha \in \Lambda_0\}$ therefore $\emptyset = X - \bigcup\{X - V_\alpha : \alpha \in \Lambda_0\} = \bigcap\{X - (X - V_\alpha) : \alpha \in \Lambda_0\} = \bigcap\{V_\alpha : \alpha \in \Lambda_0\}$

(2)→(1)

Let $U = \{U_\alpha : \alpha \in \Lambda\}$ be an ωb-open cover of the space(X, τ) then,$X - \{U_\alpha : \alpha \in \Lambda\}$ is a family of ωb-closed subset of (X, τ) with $\bigcap\{X - U_\alpha : \alpha \in \Lambda\} = \emptyset$ by assumption, there exists a finite subset Λ_0 of Λ hence $\bigcap\{X - U_\alpha : \alpha \in \Lambda_0\} = \emptyset$,so $X = X - \bigcap\{X - U_\alpha : \alpha \in \Lambda_0\} = \bigcup\{U_\alpha : \alpha \in \Lambda_0\}$ therefore X is ωb-compact.

Theorem (3.1.16):

Let $f: X \rightarrow Y$ be an onto, ωb́-continuous function,if X is ωb-compact then Y is ωb-compact

Proof

Let $\{V_\alpha : \alpha \in \Lambda\}$ be an ωb-open cover of Y then $\{f^{-1}(V_\alpha) : \alpha \in \Lambda\}$ is an ωb-open cover of X,since X is ωb-compact thus X is has finite sub cover say $\{f^{-1}(V_{\alpha i}) : i = 1,2, \ldots, n\}$ and $V_{\alpha i} \in \{V_\alpha : \alpha \in \Lambda\}$ hence $\{V_{\alpha i} : i = 1,2, \ldots, n\}$ is a finite sub cover of Y therefore, Y is ωb-compact.

Definition (3.1.17):

A subset B of a topological space X is said to be ωb-compact relative to X,if every cover of B by ωb-open sets of X,has finite sub cover of B,the subset B is ωb-compact if It is ωb-compact as a subspace.

Proposition (3.1.18):

Let Y be ωb-open subspace of a space X and $B \subseteq Y$ then B is ωb-compact set in Y if and only if B is ωb-compact in X .

Proof

Let B an ωb-compact set inY and let $\{V_\alpha : \alpha \in \Lambda\}$ be ωb-open cover of B in X then $B \subseteq \bigcup_{\alpha\in\Lambda} V_\alpha$,since $B \subseteq Y, B \subseteq \cup \{Y \cap V_\alpha : \alpha \in \Lambda\}$ since $Y \cap V_\alpha$ is ωb-open relative toY thus,$\{Y \cap V_\alpha : \alpha \in \Lambda\}$ is ωb-open cover of Brelative to Y we have $B \subseteq (Y \cap V_\alpha) \cup \ldots \cup \left(Y \cap V_{\alpha_n}\right)$ therefore, B is ωb-compact in X .

Conversely:

Let B be ωb-compact set in X and let $\{U_\alpha : \alpha \in \Lambda\}$ be an ωb-open cover of B inY, then $B \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$,thus there exists $V_\alpha$is ωb-open relative to X such that $U_\alpha = Y \cap V_\alpha, \forall \alpha \in \Lambda$,hence $B \subseteq \bigcup_{\alpha\in\Lambda} V_\alpha$ where$\{V_\alpha : \alpha \in \Lambda\}$ ωb-open cover of B, relative to X,since B is ωb –

compact set in X ,,$\exists\ \alpha_1, \alpha_2, \ldots \alpha_n \in \Lambda$ such that B $\subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$ since B ⊆ Y,B $\subseteq Y \cap \{V_{\alpha_1} \cup V_{\alpha_2}, \ldots \cup V_{\alpha_n}\} = (Y \cap V_{\alpha_1}) \cup \ldots \cup (Y \cap V_{\alpha_n})$, since $Y \cap V_{\alpha_i} = U_i$ therefore B is ωb- compact inY.

Proposition (3.1.19):

If X is a topological space such that every ωb-open subset of X is ωb-compact relativeto X then every subset is ωb-compact relative to X.

Proof

Let B be an arbitrary subset of X and,let $\{V_\alpha : \alpha \in \Lambda\}$ be a cover of B by ωb-open sets of X then the family $\{V_\alpha : \alpha \in \Lambda\}$ is a ωb-open cover of the ωb-open set $\bigcup\{V_\alpha : \alpha \in \Lambda\}$hence by hypothesis there is a finite subfamily $\{V_{\alpha i} : i = 1,2, \ldots n\}$ which covers $\bigcup\{V_\alpha : \alpha \in \Lambda\}$ $\bigcup\{V_\alpha : \alpha \in \Lambda\}$ this sub family is also a cover of the set B .

Theorem (3.1.20):

The following statements are equivalent for any topological space .

1- X is ωb-compact .

2- Every family F of ωb-open sets, if no finite subfamily of F covers X then,F does not cover X .

3-Every family F of ωb-closed sets,if.F satisfies the finite.intersection condition then $\cap\{A : A \in F\} \neq \emptyset$

4- Every family F of subsets of X,if F satisfies the finiteintersection conditionthen$\cap\{\overline{A}^{\omega b} : A \in F\} \neq \emptyset$

Proof

(1) if andonly if (2) and (2) if andonly if(3) are obvious (3)⇒(4)if $F \subset p(X)$ satisfies the finite intersection condition then $\cap\{\overline{A}^{\omega b} : A \in F\}$is a family of ωb-closed sets which obviously satisfies the finite intersection condition .

(4) ⇒(3)

Follows from the fact that $A = \overline{A}^{\omega b}$ for every $\overline{A}^{\omega b}$ ωb-closed set A .

Recall that: a topological space X is called nearly compact if for every regular opencover of X has finite sub cover see **[26]**

Definition (3.1.21):

A topological space X is said to be nearly ωb-compact if every ωb-regular open cover of X, has finite sub cover .

Theorem (3.1.22):

For any topological space X, the following statement are equivalent :

1- X is nearly ωb-compact.

2- Every ωb-open cover $\mu = \{V_\alpha : \alpha \in \Lambda\}$ of X,there exists a finite subset $\Lambda_0 \subseteq \Lambda$ such that $X = \overline{V_{\alpha \in \Lambda o}}^{\omega b \circ \omega b}$.

Proof.

(1)→(2)

Let.$\mu = \{V_\alpha : \alpha \in \Lambda\}$ be ωb-open cover of X then

$\left\{\overline{V_\alpha}^{\omega b\,\circ\omega b} : \alpha \in \Lambda\right\}$ is ωb-regular open cover of the

nearly ωb-compact space X thus,there exists a finite

subset $\Lambda_0 \subseteq \Lambda$ Such that $X = \bigcup_{\alpha\in\Lambda_0} \overline{V_{\alpha\in\Lambda o}}^{\omega b\,\circ\omega b}$.

(2)→(1)

It is clear since ωb-regular open set is ωb-open.

Theorem (3.1.23):

For any topological space X,the following statement are equivalent:

1- X is nearly ωb-compact .

2- Every family of ωb-closed sets $\{V_\alpha : \alpha \in \Lambda\}$ of X,such that $\bigcap_{\alpha\in\Lambda} V_\alpha = \emptyset$ then there exists a finite subset $\Lambda_o \subseteq \Lambda$ hence $\bigcap_{\alpha\in\Lambda} \overline{V_\alpha{}^{\circ\omega b}}^{\omega b} = \emptyset$.

3- Every family of ωb-regular closed sets $\{V_\alpha : \alpha \in \Lambda\}$ of X such that $\bigcap_{\alpha\in\Lambda} V_\alpha = \emptyset$ Then there exists a finite subset $\Lambda_0 \subseteq \Lambda$ hence $\bigcap_{\alpha\in\Lambda} \overline{V_\alpha{}^{\circ\omega b}}^{\omega b} = \emptyset$.

Proof

(1)→(2)

Let $\{V_\alpha : \alpha \in \Lambda\}$ be a family of ωb-closed sets of X,such that $\bigcap_{\alpha\in\Lambda} V_\alpha = \emptyset$,let $C_\alpha = X\text{-}V_\alpha$,the family $\{C_\alpha : \alpha \in \Lambda\}$ is an ωb-open cover of space X,Since X is nearly ωb-compact by theorem (3.1.22) there exists a finite subset $\Lambda_0 \subseteq \Lambda$ such that $X = \cup\left\{\overline{C_\alpha}^{\omega b\,\circ\omega b} : \alpha \in \Lambda_0\right\}$,then $X - \bigcup\left\{\overline{C_\alpha}^{\omega b\,\circ\omega b} : \alpha \in \Lambda_0\right\} =$

$\bigcap_{\alpha\in\Lambda} \overline{V_\alpha{}^{\circ\omega b}}^{\omega b} = \emptyset$

(2)→(3)

Let $\{V_\alpha : \alpha \in \Lambda\}$ bea family of ωb-regular closed set of X,such that $\bigcap_{\alpha\in\Lambda} V_\alpha = \emptyset$, V_α is ωb-closed set by(2) then there exists afinite subset $\Lambda_0 \subseteq \Lambda$ hence

$\bigcap_{\alpha\in\Lambda} \overline{V_\alpha{}^{\circ\omega b}}^{\omega b} = \emptyset$

(3)→(1) .

Let $\{C_\alpha : \alpha \in \Lambda\}$ be a family of ωb-regular open cover of X,then $\{X - C_\alpha : \alpha \in \Lambda\}$ is ωb-regular closed such that $\bigcap_{\alpha\in\Lambda} X - C_\alpha = \emptyset$ there exists a finite subset Λ_0 $\subseteq \Lambda$hence $\cap_{\alpha\in\Lambda_0} \overline{(X - C_\alpha)^{\circ\omega b}}^{\omega b} = \emptyset$,Therefore $X =$ $= \bigcup_{\alpha\in\Lambda_0} \overline{C_\alpha}^{\omega b\,\circ\omega b}$.

Definition (3.1.24): [27]

A function f: $(D, \geq) \longrightarrow X$ from adirect set $(D, \geq)$ to anon-empty setX is called A net on Xand it denoted by$\{X_\alpha\}_{\alpha\in D}$ $\forall\ \alpha \in D\ \exists\ X_\alpha \in X \ni f(\alpha) = X_\alpha$

Definition (3.1.25):

A point.x ∈ X is said to be ωb-cluster point of a net $\{X_\alpha\}_{\alpha\in\Delta}$ if $\{X_\alpha\}_{\alpha\in\Delta}$ is

Frequently in every ωb-open set containing x.We denote by ωb-cp$\{X_\alpha\}_{\alpha\in\Delta}$ the set of all ωb-cluster points of a net $\{x_\alpha\}_{\alpha\in\Delta}$.

Theorem (3.1.26):

A topological space X is ωb-compact,iff each net $\{X_\alpha\}_{\alpha\in\Delta}$ in X, has at least one ωb-cluster point

Proof

Let Xbe aωb-compact space, assume that there exists some net$\{x_\alpha\}_{\alpha\in\Delta}$ in X such that ωb-cp $\{x_\alpha\}_{\alpha\in\Delta}$ is empty,let $x \in X$ then,there exist $G(x)\in$ ωBO (X,x) is not frequently thus,there exists $\alpha(x)$ $\in \Delta$ suchthat $x_\lambda \notin G(x)$, whenever, $\lambda \geq \alpha(x)$ $\lambda \in \Delta$,the family$\{G(x): x \in X\}$is acover of X by ωb-open sets and has finitesub cover say$\{G_k: k = 1,2,\dots n\}$ where, $G_k = G(x_k)$for $k = 1,2,\dots n$ $\{x_k: k =1,2..n$ let us take $\alpha \in \Delta$,hence$\alpha \geq \alpha(x_k)$ for every$k \in \{1,2,\dots n\}$for every$\lambda \in \Delta$ such that$\lambda \geq \alpha$ wehave,$x_\lambda \notin G(x): k = 1,2,..$

nhence $x_\lambda \notin X$ which is contradiction

Conversely :

IF X is not ωb-compact then there exists $\{G_i : i \in I\}$ a cover of X,by ωb-open set which has no finite subcover; let p(I) be the family of every finite subsets of I clear(p(I),⊆) is directed set foreach j∈Jwe may choose$x_j \in X - \cup\{G_i : i \in j\}$Let us consider the net $\{x_j\}_{j\in p(I)}$by hypothesis,the set ωb-cp$\{x_j\}_{j\in p(I)}$is non empty,let $x \in$ ωb-cp$\{x_j\}$ $s_{j\in p(I)}$and let $i_0 \in I$,hence $x \in G_{i0}$,by thedefinition Of ωb-cluster point,for eachJ $\in P(I)$ thus, there exists $J^* \in P(I)$suchthat $J \subset J^*$and$x_{j^*} \in G_{i0}$ for $J = \{i_0\}$,there exists$J^* \in P(I)$ such that $i_0 \in J^*$and $x_{j^*} \in G_{i0}$ but$x_{j^*} \in X - \cup\{G_i : i \in j^*\} \subset X - G_{i0}$ is contradicition therefore,X is ωb-compact .In the following we will give a characterization of ωb-compact by means of filter bases let us mentioned that anonempty family $\mathcal{F}$a of subsets of X is said to be a filterbase on X if $\emptyset \notin \mathcal{F}$ and each intersection of two members of $\mathcal{F}$,contains third member of $\mathcal{F}$notice that each chain in the family of every filter base on X, has an upper bound the union of every members of the chain then by Zorn's lemma,the family of every filter bases on X, has at least one maximal element Similarly the, family of every filterbases on X, containing a given filterbase $\mathcal{F}$ has at least one maximal element .

Definition (3.1.27):

A filterbase $\mathcal{F}$ on a topological space X,is said to be: .

1- ωb-converge to a point $x \in X$,if for each ωb-open set U containing x there exists $B \in \mathcal{F}$ such that $B \subset U$.

2- ωb-accumulate at $x \in X$, if $U \cap B \neq \emptyset$ for every ωb-open set U containing x and every $B \in \mathcal{F}$.

Lemma (3.1.28):

If a maximal filterbase $\mathcal{F}$ ωb-accumulate at, $x \in X$, then $\mathcal{F}$ ωb-converge to x .

Proof

Let $\mathcal{F}$ be a maximal filterbase with ωb –accumulate at $x \in X$, if $\mathcal{F}$ is not ωb- converge to x, then there, exists a ωb-open set U_0 containing x such that $U_0 \cap B \neq \emptyset$ and $(X - U_0) \cap B \neq \emptyset$ for every $B \in \mathcal{F}$, thus $\mathcal{F} \cup \{U_0 \cap B : B \in \mathcal{F}\}$ is a filter base which contains $\mathcal{F}$, which is contradiction.

Theorem (3.1.29):

Let X be topological space then, following statements are equivalent:

1- X is ωb-compact.

2- Every maximal filterbase ωb-converges to some points of X .

3-Every filterbase ωb-accumulates at some points of X

Proof

(1).⇒(2) .

Le $\mathcal{F}_0$be amaximal filterbase on X suppose that $\mathcal{F}_0$is not ωb-converges to any point of X then by lemma (3.1.28), $\mathcal{F}_0$ is not ωb-accumulates at any point ofX, for each x ∈X then ,there exists a ωb-open set $U_x$containing x and$B_x \in \mathcal{F}_0$hence $U_x \cap B_x = \emptyset$the family$\{U_x : x \in X\}$is a cover of X by ωb-open sets,by (1)thus there exists afinite subset $\{x_1, x_2,\dots, x_n\}$of X,henceX $=\cup \{U_{xk}: k\ 1,2, \dots \dots n\}$ since$\mathcal{F}_0$is afilterbase, there exists $B_0 \in \mathcal{F}_0$ suchthat $B_0 \subset \cap \{B_{x_k}: k = 1,2,..n\} = X - \cup\{U_{x_k}: k = 1,2, \dots n\}$ hence $B_0 = \emptyset$ which is contradiction .

(2) ⇒ (3)

Let $\mathcal{F}$ be a filterbase on X then, there exists a maximal filterbase $\mathcal{F}_0$, hence $\mathcal{F} \subset \mathcal{F}_0$ by(2), $\mathcal{F}_0$ is ωb-converges to some point $x_0 \in X$, let $B \in \mathcal{F}$

for every $U \in BO(X,x_0)$ thus there exists $B_U \in \mathcal{F}_0$ such that $B_u \subset U$ hence $U \cap B \neq \emptyset$ Since it is contains the member $B_U \cap B$ of $\mathcal{F}_0$,this that $\mathcal{F}$ωb-accumulates at x_0 .

(3) ⇒ (1)

Let$\{V_i : i \in I\} = \emptyset$ be any family of ωb-closed sets such that $\cap\{V_i : i \in I\} = \emptyset$ we prove that there exists a finite subset $I_0$of I, hence $\cap\{V_i : i \in I\}$

By theorem(3.1.20) (1) let P (I) be the family of finite subsets of I, assume that$\cap\{V_i : i \in J\} = \emptyset$ for every J$\in P(I)$..........∗ thus the family $\mathcal{F} = \{\cap\{V_i : i \in J\} : J \in P(I)\}$is afilter base,onX by (3) $\mathcal{F}$ is ωb-accumulates to some point$x_0 \in X$, Since $\{X - V_i : i \in I\}$ is acover ofX,there existsi_0I,hence $x_0 \in X - V_{i0}$, $X - V_{i0}$ is ωb –open set contains x_0 $V_{i0} \in \mathcal{F}$ and $(X - V_{i0}) \cap V_{i0} = \emptyset$,which is contrad-iction with the fact that $\mathcal{F}$ ωb-accumulates at x_0 shows that (∗) is false .

Definition (3.1.30):

A space X is said to be countably ωb-compact if every countable cover of X,byωb-open sets has a finite subcover.

Definition (3.1.31): [16]

Let $f: X \to Y$ be a function of a space X into a space Ythen f is called a compact function If $f^{-1}(A)$ is compact set in X,for every compact set A in Y.

Definition (3.1.32):

Let $f: X \to Y$ be a function of a space X into space Y,then f is called a ωb-compact Function if $f^{-1}(A)$ is compact set in X,for every ωb-compact set A in Y .

Remark (3.1.33):

Every compact function is an ωb-compact function but the converse is not true
as the following

Example (3.1.34):

Let X=Y=N and τ be the discrete topology on X, $\acute{\tau}$ be the indiscrete topology on Y,The function $f: X \to Y$ which is defined as $f(x) = x, \forall\, x \in N$ is ωb-compact, but it is not compact function .

Proposition (3.1.35):

Let X, Y and Z be spaces and $f: X \to Y$,.$g: Y \to Z$.be functions,then :

1- If f is a compact function and g is an ωb-compact function, then,$g \circ f$ is an ωb- compact function.

2- If $g \circ f$ is an ωb-compact function, f is onto and continuous; then g is ωb-compact function .

3- If $g \circ f$ is an ωb-compact function, g is ωb-irresolute and one-to-one then,f is ωb-compact function .

Proof

1- Let K be a ωb-compact set in Z,since g is an ωb-compact,then $g^{-1}(K)$ is compact set in Y, Since

f is an compact function thus $f^{-1}\left(g^{-1}(K)\right)$ is compact set in X,hence $g \circ f: X \to Z$ is an ωb-compact function.

2- Let K be a ωb-compact in Z then, $(g \circ f)^{-1}(k)$ is compact set in X, since f is continuous then $f[(g \circ f)^{-1}]$ is a compact set in Y, and since f is onto thus $f\ (g \circ f)^{-1}\ (K) = g^{-1}(K)$ is compact set in Y therefore,g is ωb-compact .

3- Let K be an ωb-compact in Y,Since g is an ωb-irresolute then, g(K)is an ωb-compact set in Z thus, $(g \circ f)^{-1}\left(g(K)\right)$is a compact set in X, Since g is one-to-one then $(g \circ f)^{-1}\left(g(K)\right) = f^{-1}(K)$, hence $f^{-1}(K)$ is a compact set in X, therefore f is ωb-compact function .

Proposition (3.1.36):

Let X and Y be two spaces and $f: X \to Y$ be function then :

1- If f is an ωb -compact function and F is closed subset of X then $f|_F: F \to Y$ is an ωb-compact function .

2- If f is an ωb-compact, continuous function and B is a subset of Y then, f_B; $f^{-1}(B) \to B$ is an ωb-compact function .

Proof

1- Let K be an ωb-compact inY, Since f is an ωb-compact function; then $f^{-1}(k)$ is compact in X, by theorem(3.1.6)(2), then f^{-1} (K)F is compact, but $f_{|F}{}^{-1}(K) = f^{-1}(K) \cap F$, then f^{-1} (K) is compact set in F, therefore $f|_F : F \to Y$ωb-compact function .

2- Let K be an ωb-compact a subset of B,then by theorem (3.1.18) K is ωb-Compact in Y,but f is ωb-compact; thus f^{-1} (K) is compact in X, Since $f^{-1}(K) \subseteq f^{-1}(B)$;hence f^{-1} (K) is compact $f^{-1}(B)$,therefore f_B is an ωb-compact function .

3.2 ωb-Lindelof Space

In section we introduce a new definition to the best of our knowledge ωb-lindelof space and a nearly ωb-lindelof and we give some results which are related with this subject.

Definition (3.2.1): [13]

A topological space X is said to be lindelof,if every open cover of X, has a countable sub cover.

Definition (3.2.2): [11]

1- A topological space X is said to be b-lindelof ,if every b-open cover of.X,has a countable sub cover.

2- A subset B of space X is said to be b-lindelof relative to X,if every cover of B by b-open sets of X has a countable sub cover of B.

Remark (3.2.3):

It is clear that every b-lindelof space is lindelof but the converse is not true in general as the following example shows

Example (3.2.4):

LetA be uncountable set $\ni b \notin A, X = A \cup \{b\}$,let $\tau = \{X, \emptyset, \{b\}\}$be atopology on X such that (X, τ) is lindelof ,where is not a b-lindelof ,Since $\{\{b, a\}: a \in A\}$ is a b-open cover of X which has no countable sub cover.

Definition (3.2.5):

A topological space X is said to be ω-lindelof,if every ω-open cover of X has a countable sub cover.

Theorem (3.2.6): [22]

If X is a space such that every b-open subset of X,is b-lindelöf relative to X,then every subset is b-Lindelöf relative to X .

Definition (3.2.7):

A topological space X is said to be ωb-lindelof,if every ωb-open cover of X,has a countable sub cover.

Remark (3.2.8):

1- Every ωb-lindelof space is lindelof .

2- Every ω-lindelof space is lindelof .

Remark (3.2.9):

But the converse of (3.2.8) is not true in general as the example **[17] .**

Remark (3.2.10):

1- Every b-lindelof is not true in general ω-lindelof .

2- Every b-lindelof is not true in general ωb-lindelof .

as the following

Example (3.2.11):

Let (X, τ) be a space such that X= R and $\tau = \{r_a : a \in R\} \cup \{\emptyset\} \cup \{X\}$ be definition on X.such that $r_a = \{x: x \in R ∞ x \geq a\}$ then,BO(X) $= \{A \subseteq$

R^+ : A is infinite $\} \cup \{r_a : a \in R\} \cup \{\emptyset, X\}$ thus b-lindelof Since $\omega O(X) = \{(a, \infty), [a,\infty), a \in R\}$ hence $\bigcup_{a\in R}(a, \infty)$ is ω-open cover of X,but has not countable subcover, therefore R is not ω-lindelof and ωb-lindelof.

Remark (3.2.12):

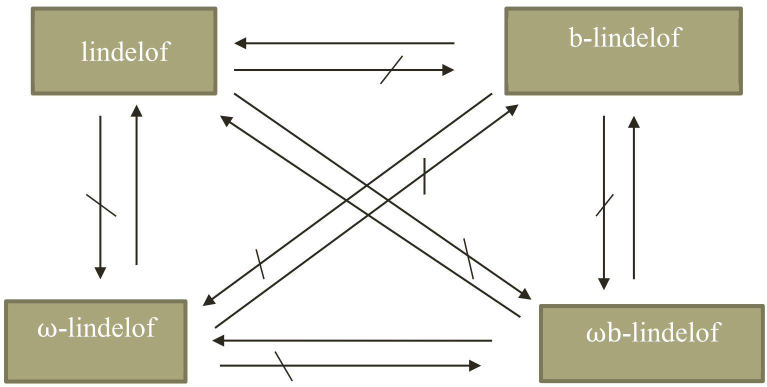
1- Every ω-lindelof is not true in general b-lindelof .

2- Every ω-lindelof is not true in general ωb-lindelof as the following

Example (3.2.13):

Let A is an uncountable ,X = A∪{b},b $\notin$ A, and , $\tau = \{\emptyset, X, \{b\}\}$ then, $\omega O(X) = \{\emptyset, X, \{b\}\} \cup \{G \subseteq X : G^c$ is countable $\}$, thus X is ω-lindelof, since $BO(X) = \{\{b, a\}: a \in A\}$, $\omega BO(X)\{A: A \subseteq X\}$ therefore, X is not b-lindelof and ωb-lindelof .

The following diagram shows the relations among the different types of lindelof space.



Theorem (3.2.14): [22]

For any space X, the following properties are equivalent:

1- X is b-Lindelof

2- Every ωb-open cover of X,has a countable subcover .

Corollary (3.2.15):

A topological space X is b-lindelof,iff for every ωb^*-open cover of X, has a Countable sub cover.

Proof

It is clear Since every b-open and ωb^*-open is ωb-open

Proposition (3.2.16):

A Topological space X is ωb-lindelof,if and only if for every family $\{F_\alpha : \alpha \in \Lambda\}$ of ωb-regular closed sets with countable intersection property $\cap_{\alpha\in\Lambda} F_\alpha \neq \emptyset$

Proof.

Let X be a ωb-lindelof space and suppose that $\{F_\alpha : \alpha \in \Lambda\}$ be a family of ωb-regular closed subsets of X,with countable intersection property suppose that $\cap_{\alpha\in\Lambda} F_\alpha = \emptyset$ Let us consider the ωb-regular open sets $V_\alpha = \{X - F_\alpha : \alpha \in \Lambda\}$,the family $\{V_\alpha : \alpha \in \Lambda\}$is an ωb-regular open cover of space X, Since X is ωb-lindelof the cover $\{V_\alpha : \alpha \in \Lambda\}$has a countable subcover $\{V_{\alpha_i}: \mathrm{i} \in \mathrm{N}\}$,hence X= $\cup \{V_{\alpha_i}: \mathrm{i} \in \mathrm{N}\} = \cup \{(\mathrm{X} - \mathrm{F}_{\alpha_i}): \mathrm{i} \in \mathrm{N}\} = \mathrm{X} - \cap \{\mathrm{F}_\alpha: \mathrm{i} \in \mathrm{N}\}$whence $\cap \{\mathrm{F}_{\alpha_i}: \alpha_i \in \mathrm{N}\} = \emptyset$ then,if the Family.$\{\mathrm{F}_\alpha: \alpha \in \Lambda\}$ of ωb-regular closed sets with countable intersection property thus $\cap_{\alpha\in\Lambda} F_\alpha \neq \emptyset$

conversely:

Let $\{\mathrm{V}_\alpha: \alpha \in \Lambda\}$ be an ωb-regular open cover of X,and suppose that for every family $\{X - \mathrm{F}_\alpha : \alpha \in \Lambda\}$ of ωb-regular closed sets with countable intersectionproperty $\cap_{\alpha\in\Lambda} \mathrm{F}_\alpha \neq \emptyset$then X= $\cup \{\mathrm{V}_\alpha: \alpha \in \Lambda\}$thus $\emptyset \neq \mathrm{X} - \mathrm{X} = \cap \{(\mathrm{X} - \mathrm{V}_\alpha): \alpha \in \Lambda\}$ and $\{(\mathrm{X} - \mathrm{V}_\alpha): \alpha \in \Lambda\}$ is afamily of ωb-regular closed sets with an empty intersection by thehypothesis there exists a countable subset $\{(\mathrm{X} - \mathrm{V}_{\alpha_i}): \mathrm{i} \in \mathrm{N}\}$, hence $\cap (X - \mathrm{V}_{\alpha\mathrm{i}}) = \emptyset$ such that X$-\{\cap (\mathrm{X} - \mathrm{V}_{\alpha\mathrm{i}}) : \mathrm{i} \in \mathrm{N}\}$= X = $\cup \{V_{\alpha i}: i \in N\}$ therefore ,X is ωb-lindelof.

Proposition (3.2.17): [22]

Every ωb-closed subsetof a b-lindelof space X,is b-lindelöf relative to X .

Corollary (3.2.18):

Every ωb^* -closed subsets of a b-lindelof space X, is b-lindelof relative to X.

Proof

It is clear since every ωb^* -closed is ωb-closed .

Corollary (3.2.19): [22]

If a space X is b-lindelof and A is ω-closed or (b-closed) then A is b-lindelof

Relative to X .

Theorem (3.2.20): [22]

Let f be an ωb-continuous function from a space X onto a space Y,if X is b-lindelof then Y is lindelöf.

Theorem (3.2.21): [22]

If $f: X \to Y$ is an ωb-closed, onto such that $f^{-1}(Y)$ is b-lindelof relative to X,and Y is b-Lindelof then,X is b-lindelof .

Corollary (3.2.22):

If $f: X \to Y$ is an ωb^{*}-closed,onto such that $\mathrm{f}^{-1}(\mathrm{Y})$ is b-lindelof relative to X and Y is b-lindelof then X is b-Lindelof .

Proof

It is clear since every ωb^{*}-closed is ωb-closed .

Theorem (3.2.23):

If a Topological space (X, τ) is countable union of open ωb-lindelof subspaces, then it is ωb-lindelof.

Proof

Assume that $\mathrm{X} = \bigcup\{C_n : \mathrm{n} \in \mathrm{N}\}$,where (C_n, τ_n) is an ωb-lindelof subspace,foreach $n \in \mathbb{N}$,suppose $\mathcal{A}$ be a ωb-open cover of the space (X, τ) for each $n \in \mathbb{N}$,the family $\{A \cap C_n : A \in \mathcal{A}\}$ is ωb-open cover of the ωb-lindelof subspace (C_n, τ_n)we find acountabl sub-family $\mathcal{A}_n$of $\mathcal{A}$,hence $C_n = \bigcup\{A \cap C_n : A \in \mathcal{A}_n\}$ put $R = \{A_n : n \in N\}$ then $\mathcal{R}$ is a countable sub family of $\mathcal{A}$, thus $X = \cup\{C_n : n \in N\} \bigcup_{n \in N}\{A \cap C_n : A \in \mathcal{A}_n\} \subseteq \{A : A \in \mathcal{R}\} \subseteq X$,that is $X = \bigcup\{A : A \in \mathcal{R}\}$ therefore (X, τ) is ωb-lindelof .

Definition (3.2.24): [6]

A Topological space X is said to be nearly lindelof if every regular open cover of X has a countable sub cover

Definition (3.2.25): [3]

A Topological space X is said to be nearly b-lindelof if every b-regular open cover of X has a countable sub cover.

Definition (3.2.26):

A topological space is X said to be nearly ωb-lindelof if every ωb-regular open cover of X has a countable sub cover.

Theorem(3.2.27):

For any topological space X,the following statements are equivalent:

1- X is nearly b-lindelof.

2- Every ωb-regular open cover of X has a countable sub cover.

Proof

(1)→(2)

Let $\{U_\alpha: \alpha \in \Lambda\}$ be any ωb-regular open cover of X,for each $x \in X$,there exists $\alpha(x) \in \Lambda$ such that $x \in U_{\alpha(x)}$,Since $U_{\alpha(x)}$ is ωb-regular open cover, there exists a b-regular open.set $V_{\alpha(x)}$,then $x \in V_{\alpha(x)}$and $V_{\alpha(x)} - U_{\alpha(x)}$ is a countable,the family $\{V_{\alpha(x)}: x \in X\}$ is ab-regular open coverof X,since Xis nearlyb-lindelof there exists accountable subset says $\alpha(x_1), \alpha(x_n)$ such that $X = \bigcup\{V_{\alpha(x_i)}: i \in \mathbb{N}\}$,now we have $X = \bigcup_{i\in\mathbb{N}}\{(V_{\alpha(x_i)} - U_{\alpha(x_i)}) \cup U_{\alpha(x_I)}\}$ = ($\bigcup_{i\in N}(V_{\alpha(x_i)} - U_{\alpha(x_i)})$) $\bigcup_{i\in N}(\bigcup_{i\in N} \ U_{\alpha(x_i)})$,for each $\alpha(x_i)$since $V_{\alpha(x_i)}$ $-U_{\alpha(x_i)}$ is a countable set and thus there exists accountable subset $\Lambda_{\alpha(x_i)}$of Λ,such that $V_{\alpha(x_i)} - U_{\alpha(x_i)} \subseteq \bigcup\{U_\alpha: \alpha \in \ \Lambda_{\alpha(x_i)}\}$,therefore we have $X \subseteq [\bigcup_{i\in\mathbb{N}} U_\alpha: \alpha \in \ \Lambda_{\alpha(x_i)}] \cup [\bigcup_{i\in N} U_{\alpha(x_i)}]$.

(2) →(1)

Since every b-regular open set is ωb-regular open the proof is obvious .

Definition (3.2.28):

A function $f:X \to Y$ is said to be almost contra-ωb-continuous,if $f^{-1}(A)$ is ωb-open set in X,for every regular closed subset A in Y.

Proposition (2.2.29):

Atopologicalspace X is nearly ωb-lindelof iff for every family $\{C_\alpha : \alpha \in \Lambda\}$of ωb-regular closed sets with Countable eintersection property then $\bigcap_{\alpha\in\Lambda} C_\alpha \neq \emptyset$.

Proof

same proof of Proposition (3.2.16)

Definition (3.2.30): [10]

A space X is said to be S-Lindelof if every cover of X,by regular closed sets hasa countable subcover.

Definition (3.2.31):

A space X is said to be:

1- S-closed if every regular closed cover of X has a finite subcover **[19]** .

2- countably S-closed if every countable cover of X by regular closed sets has a finite subcover; **[9]** .

Theorem (3.2.32):

Let $f: X \rightarrow Y$ be an almost contra-ωb-continuous ,onto the following statement:

1- if X is ωb-compact,then Y is S-closed .

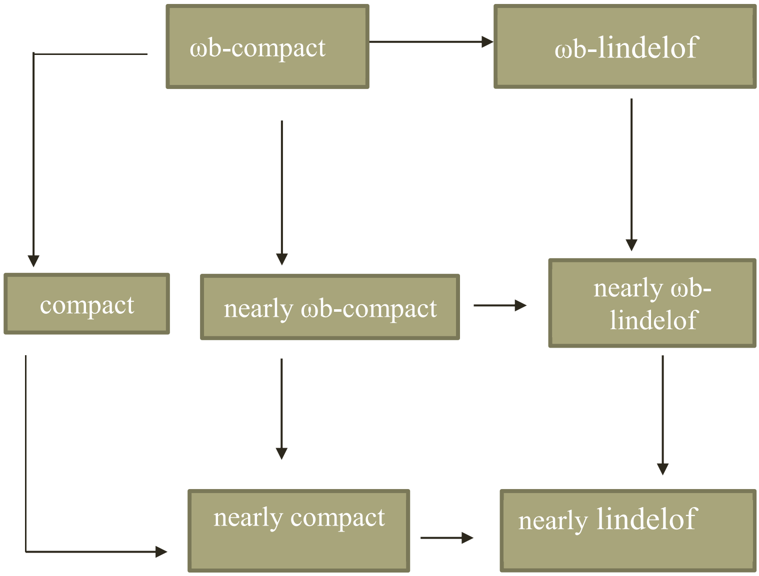
2- if X is ωb-compact,then Y is S-Lindelof .

3- if X is countably ωb-compact,then Y is countably S-closed.

Proof

We prove only (1),let$\{U_\alpha$: α ∈ I } be any regular closed cover of Y, Since f is almost contra- ωb-continuous then { $f^{-1}(U_\alpha)$: α ∈ I}is an ωb-open cover of X and thus there exists a finite subset I_0 of I,hence X = ∪{ $f^{-1}(U_\alpha)$: α ∈ I_0 }therefore Y = ∪$\{U_\alpha$: α ∈ $I_0\}$ and Yis S-closed..We prove (2) and (3) It is clear

The following diagram shows the relations among the different types of lindelof space and compact .


ωb-compact
ωb-lindelof
compact
nearly ωb-compact
nearly ωb-
lindelof
nearly compact
nearly lindelof

ABSTRACT

The main aim of this work is to expand and study some types of topological spaces by ωb-open sets . In this work,we extend these concepts by using ωb-open sets to new definitions for ωb-connected space ωb-compact space,countably ωb-compact,ωb-cluser Point,ωb-lindelof space,then we study the relations between the above mentioned with other concepts like ωb-T_1,ωb-T_2,ωb-regular,ωb-normal,During the work someimportant and new concepts have been illustrated including nearly ωb-compact,nearly ωb-lindelof in addition studing the behavior of these qualities under the in-fluence of certain types of functions we also dealt with the concepts of ωb-closed, ωb-open functions ,ωb-continuous the properties of these functions .

the following are among our main results:

1- Let $f:X \longrightarrow Y$ be a bijective function .

i- If f is ωb-open and X is T_2-space then Y isωbT_2 -space.

ii- If f is ωb-continuous and Y is T_2-space then X isωbT_2-space .

2- The door space is ωb-R_0 if and only if it is ωbT_1-space.

3- The door space is ωb-R_1 if and only if it isωbT_2-space

4- Let X be topological space, then the following statements are.equivalent:

i- X is ωb-compact.

ii- Every maximal filterbase ωb-converges to some points of X.

iii- Every filterbase ωb-accumulates at some points of .X.

5- A topological space X is ωb-compact if and only if each net $\{X_\alpha\}$_($\alpha \longrightarrow \Delta$) in X, has at least one ωb-cluster point .

6- Let f:X $\longrightarrow$ Y be an.almost contra-ωb-continuous, onto the following statement are equivalent

i- if X is ωb-compact, then Y is S-closed.

ii- if X is ωb-compact, then Y is S-Lindelof .

iii- if X is countably ωb-compact,thenY is countably S-closed.