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Approximate Solutions for Non-linear Iterative Fractional Differential Equations

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Abstract. This paper establishes approximate solution for non-linear iterative fractional differential equations:

$$\frac{d^\gamma v(s)}{ds^\gamma} = \aleph(s, v, v(v)),$$

where $\gamma \in (0, 1]$, $s \in I := [0, 1]$. Our method is based on some convergence tools for analytic solution in a connected region. We show that the suggested solution is unique and convergent by some well known geometric functions.

INTRODUCTION

The use of differential equations is the best way to study mathematical models in biology, which is the reason it has attracted the interest of many researchers. It can also be generalized and extended to fractional differential equations. Both these areas have particular kinds of equations called iterative differential equations and fractional iterative differential equations. . There are many scholars who have their undertaken research in these areas [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. These equations are major models in studying infection and are associated with studying the motion of charged particles with delayed interaction.

In this paper we find approximate solutions to some examples using numerical analysis and our work deals with a singular non-linear fractional differential equation in the sense of Riemann-Liouville operators in an analytical category. One of the tools utilized is the theory fractional calculus as provided by the Riemann-Liouville operators. Furthermore, this operator has the advantage of rapid convergence, greater stability and greater precision of various numerical algorithm[11, 12] .

PRELIMINARIES

Recall the following preliminaries:

Definition 2.1[13, 14]

The definition of fractional(arbitrary) order derivative function $\psi(s)$ of order $0 < \gamma < 1$ for **Riemann-Liouville** is

$$D_a^\gamma \phi(s) = \frac{d}{ds}\int_a^s \frac{(s-\beta)^{-\gamma}}{\Gamma(1-\beta)}\phi(\beta)d\beta = \frac{d}{ds} I_a^{1-\gamma}\phi(s), \tag{1}$$

$$(\kappa - 1) < \gamma < \kappa,$$

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in which κ is a whole number and γ is a real number.

Definition 2.2[13, 14]

The fractional (arbitrary) order integral of the function $\phi(s)$ of order $\gamma > 0$ is introduced by

$$I_a^{\gamma}\phi(s) = \int_a^s \frac{(s-\beta)^{\gamma-1}}{\Gamma(\gamma)}\phi(\beta)d\beta. \tag{2}$$

While $a = 0$, it becomes $I_a^{\gamma}\phi(s) = \phi(s) * \Upsilon_{\gamma}(s)$, wherever $(*)$ signify the convolution product

$$\Upsilon_{\gamma}(s) = \frac{s^{\gamma-1}}{\Gamma(\gamma)}$$

and $\Upsilon_{\gamma}(s) = 0, \;\; s \leq 0$ and $\Upsilon_{\gamma} \to \delta(s) a s \gamma \to 0$ wherever $\delta(s)$ is the delta function.

Remark 2.1 From Definitions 1.1 and 1.2, we get

$$D^{\gamma}s^{r} = \frac{\Gamma(r+1)}{\Gamma(r-\gamma+1)}s^{r-\gamma}, \;\; r > -1, \;\; 0 < \gamma < 1$$

and

$$I^{\gamma}s^{r} = \frac{\Gamma(r+1)}{\Gamma(r+\gamma+1)}s^{r+\gamma}, \;\; r > -1, \;\; \gamma > 0.$$

Sufficient conditions to have unique and convergent solution of the equation

$$\frac{d^{\gamma}v(s)}{ds^{\gamma}} = \aleph(s, v, v(v)). \tag{3}$$

Rewrite Eq.3 as follows.

$$\frac{d^{\gamma}v(s)}{ds^{\gamma}} = \aleph(s, v, u), \tag{4}$$

where $s \in I$, $v : I \to \mathbb{R}$ and $u : \mathbb{R} \to \mathbb{R}$, $u = v(v)$ are continuous functions, satisfying the initial condition $v(0) = 0, \;\; s \in I := [0, 1], \;\; v(s)$ is an unknown function and $\aleph(s, v, u)$ is a nonlinear function with respect to the variables $(s, v, u) \in I \times C^2$. To achieve our result, we need the following assumptions:

(a1) $\aleph(s, v, u)$ is analytic function introduced in a neighborhood of the origin $(0, 0, 0) \in I \times C^2$,

(a2) v is a analytic function,

(a3) $\aleph(0, 0, 0) \equiv 0$ near $s = 0$, where $v(0) = 0, \; u(0) = 0$.

Let $\aleph(s, v, u)$ can be written as:

$$\aleph(s, v, u) = as + bv + cu + R_1(s, v, u). \tag{5}$$

We put $cu + R_1(s, v, u) = R_2(s, v, u)$. Then Eq. 5 becomes

$$\aleph(s, v, u) = as + bv + R_2(s, v, u). \tag{6}$$

MAIN RESULTS

We have the following result and existence of an unique and convergent solution.
Theorem 3.1 Let the hypothesis $(a1)-(a3)$ be achieved. if $b \neq \frac{\Gamma(r+1)}{\Gamma(r+1-\gamma)}$, then the Eq3 has an unique and convergent solution $v(s)$ near the origin satisfying $v(0)=0$.

Proof. We recognize that Eq3 has a formal solution, which is:

$$v(s)=\sum_{r=0}^{\infty} v_r s^r, \quad s \in I. \tag{7}$$

We will compensate Eq.7 on the left side of Eq.3 we get

$$\begin{aligned}\frac{d^\gamma v(s)}{ds^\gamma} &= \frac{d^\gamma}{ds^\gamma}\sum_{r=0}^{\infty} v_r s^r,\\ &= \sum_{r=0}^{\infty} v_r \frac{d^\gamma}{ds^\gamma} s^r,\\ &= \sum_{r=0}^{\infty} v_r \frac{\Gamma(r+1)}{\Gamma(r-\gamma+1)} s^{r-\gamma}.\end{aligned}$$

Also, we will compensate Eq.7 on the right side of Eq.6 we get

$$as+bv+R_2(s,v,u) := as+b\sum_{r=0}^{\infty} v_r s^r + R_2(s,v,u).$$

After formally entering the number of solution7 into Eq.3, and comparing the coefficients in s^r in both sides of equation income

$$\begin{aligned}\tfrac{\Gamma(2)}{\Gamma(2-\gamma)} v_1 &= a+bv_1+\Psi_1(s,v,u),\\ \tfrac{\Gamma(3)}{\Gamma(3-\gamma)} v_2 &= bv_2+\Psi_2(s,v,u),\\ \tfrac{\Gamma(4)}{\Gamma(4-\gamma)} v_3 &= bv_3+\Psi_3(s,v,u),\\ &\vdots\\ \tfrac{\Gamma(r+1)}{\Gamma((r+1_-\gamma)} v_r &= bv_r+\Psi_r(s,v,u).\end{aligned} \tag{8}$$

In this way the following formula is obtained

$$\left[\frac{\Gamma(r+1)}{\Gamma(r+1_-\gamma)} - b\right] v_r = \Psi_r(s,v,u), \tag{9}$$

i.e

$$v_r = \frac{\Psi_r(s,v,u)}{\left[\frac{\Gamma(r+1)}{\Gamma((r+1_-\gamma)} - b\right]}. \tag{10}$$

Consequently, we get

$$\|v_r\|_\kappa \leq \left\|\frac{\Psi_r(s,v,u)}{\left[\frac{\Gamma(r+1)}{\Gamma((r+1_-\gamma)} - b\right]}\right\|_\kappa, \tag{11}$$

in which $\|\Psi\|_\kappa = \max_{|s|\leq\kappa} |\Psi(.)|$ Presently manufactures to demonstrate that the formal series solution 7 converges in to solution $(0,0) \in (I,V)$. To discuss the convergence of the reminder $(R_2(s,v,u))$, we expand Eq(6) in the Taylor series of (t,v,u), that is,

$$R_2(s,v,u) = \sum_{m+p+q} b_{m,p,q} s^m v^p u^q$$

so that

- $b_{m,p,q}$ is a constant satisfying $b_{m,p,q} \leq \frac{1}{8}$, while $v = \frac{1}{1-\kappa}$ is harmonic function in I this requires that $v(v) = u$ is also analytic and harmonic in I because $u := v(v) \Rightarrow v(\frac{1}{1-\kappa}) = 1 - \frac{1}{\kappa}$.
- $|b_{m,p,q}| \leq B_{m,p,q}$, $B_{m,p,q} > 0$ on I.
- $\sum_{m+p+q\geq 2} B_{m,p,q} s^m U^{p+q}$ converges in (s, U) where $U > 0$ satisfies $|v| \leq U$ and $|u| \leq U$.

Since Eq.9, is observed and the fact that $b_{m,p,q} \leq 1/8$, we have

$$\begin{aligned}[\tfrac{\Gamma(2)}{\Gamma(2-\alpha)} - b]v_1 &= a,\\ &\vdots,\\ [\tfrac{\Gamma(\sigma+1)}{\Gamma(\sigma+1-\alpha)} - b]v_\sigma &= \textstyle\sum_{m+p+q\geq 2}[\sum_{m+\sigma_1+\ldots+\sigma_p+J_1+\ldots+J_q=\sigma} b_{m,p,q} \times v_{\sigma_1} \times v_{\sigma_2} \times \ldots \times v_{\sigma_p} \times \ldots \times U_{r_p}].\end{aligned} \tag{12}$$

Without loss of generality, we can suppose which there is constant $M > 0$ so that

$$|v_1(s)| \leq M.$$

Next we propose the following formula

$$U(s) = Ms + \frac{1}{1-\kappa} \sum_{m+p+q\geq 2} \frac{B_{m,p,q}}{(1-\kappa)^{m+p+q-2}} s^m U^p U^q, \quad s \in [0, 1] = I, \tag{13}$$

where κ is a parameter with $0 < \kappa < 1$. For the Eq13 is an analytic functional equation in U then in note of the implied function theorem, the Eq. 13 has a unique analytic solution $U(s)$ in a neighborhood of $s = 0$ with $U(0) = 0$. Expanding $U(s)$ into Taylor series in s we get

$$U(s) = \sum_{r\geq 1} U_r s^r \tag{14}$$

where

$$\begin{aligned}U_r &= \textstyle\sum_{m+p+q\geq 2}[\sum_{m+\sigma_1+\ldots+\sigma_p+J_1+\ldots+J_q=\sigma} \frac{B_{m,p,q}}{(1-\kappa)^{m+p+q-2}} \times U_{r_1} \times U_{r_2} \times \ldots \times v_{r_p} \times \ldots \times U_{r_p}],\\ &= \tfrac{1}{(1-\kappa)^{r-1}},\\ &> 0.\end{aligned} \tag{15}$$

Hence, R_2 is bounded by a positive harmonic function. Thus $\aleph$ is convergent.
We apply theorem 3.1 in some examples

Examples

In this subsection we will give some illustrations of theorem 3.1 and check its conditions. Therefore we study the behavior of solutions converging in to 0 and obtain the approximate solutions for them.

Example 4.1

Consider the initial value problem

$$D^{\frac{1}{2}} v(s) = 0.2s + 3v(s) + v(v(s)), \quad v(0) = 0. \tag{16}$$

Therefore, we apply the conditions above; Theorem 3.1 for example 4.1 is the following: $a = 0.2$ is right because $a = M$, $|v_1| \leq M$, $v_1 = -0.12 \Rightarrow |v_1| = 0.12 < a$, $b = 3 \neq \frac{\Gamma(r+1)}{\Gamma(r+1-\gamma)}$, and $c = 1$, $c \leq \frac{1}{(1-\kappa)^{r-1}}$ where $\kappa = \frac{1}{2}$, therefore the example 4.1 satisfies Theorem 3.1 so Eq17 has solution of the form:

$$v(s) = \sum_{r=1}^{\infty} v_r s^r,$$

so that we can re-write it as

$$v_r = \frac{\Psi_r(s, v, u)}{\left[\frac{\Gamma(r+1)}{\Gamma((r+1-\frac{1}{2}))} - 1\right]},$$

where $\Psi_r(s,v,u) = max_r \frac{1}{(\frac{1}{2})^{r-1}}$ and $r = 2,3,4,\ldots$

$$v_2 = \frac{\frac{1}{(\frac{1}{2})^1}}{\left[\frac{\Gamma(3)}{\Gamma((3-\frac{1}{2})} - 3\right]} = -1.11111,$$

$$v_3 = \frac{\frac{1}{(\frac{1}{2})^2}}{\left[\frac{\Gamma(4)}{\Gamma((4-\frac{1}{2})} - 3\right]} = -2.1538462,$$

$$v_4 = \frac{\frac{1}{(\frac{1}{2})^3}}{\left[\frac{\Gamma(5)}{\Gamma((5-\frac{1}{2})} - 3\right]} = -4.23529412,$$

$$v_5 = \frac{\frac{1}{(\frac{1}{2})^4}}{\left[\frac{\Gamma(6)}{\Gamma((6-\frac{1}{2})} - 3\right]} = -8.3809524,$$

$$\vdots$$

We take at $r = 5$ we get the solution as

$$v = -0.12s - 1.11111s^2 - 2.1538462s^3 - 4.23529412s^4 - 8.3809524s^5, \quad 0 < s < 1.$$

Example 4.2

Consider the initial value problem

$$D^{\frac{1}{2}}v(s) = s - 0.3v(s) + v(v(s)), \quad v(0) = 0. \tag{17}$$

Now we apply the conditions above theorem 3.1 for example 4.2 is the following: $a = 1$ is right because $a = M$, $|v_1| \leq M$, $v_1 = 0.612245 \Rightarrow |v_1| = 0.612245 < a$, $b = -0.3 \neq \frac{\Gamma(r+1)}{\Gamma(r+1-\gamma)}$, and $c = 1$, $c \leq \frac{1}{(1-\kappa)^{r-1}}$ where $\kappa = \frac{1}{2}$, therefore the example 4.2 satisfies theorem 3.1 so Eq17 has a solution of the form:

$$v(s) = \sum_{r=1}^{\infty} v_r s^r,$$

so that we can re-write it as

$$v_r = \frac{\Psi_r(s,v,u)}{\left[\frac{\Gamma(r+1)}{\Gamma((r+1-\frac{1}{2}))} - 1\right]},$$

where $\Psi_r(s,v,u) = max_r \frac{1}{(\frac{1}{2})^{r-1}}$ and $r = 2,3,4,\ldots$

$$v_2 = \frac{\frac{1}{(\frac{1}{2})^1}}{\left[\frac{\Gamma(3)}{\Gamma((3-\frac{1}{2})} + 0.3\right]} = 1.33333,$$

$$v_3 = \frac{\frac{1}{(\frac{1}{2})^2}}{\left[\frac{\Gamma(4)}{\Gamma((4-\frac{1}{2})} + 0.3\right]} = 2.77228,$$

$$v_4 = \frac{\frac{1}{(\frac{1}{2})^3}}{\left[\frac{\Gamma(5)}{\Gamma((5-\frac{1}{2})} + 0.3\right]} = 5.66929,$$

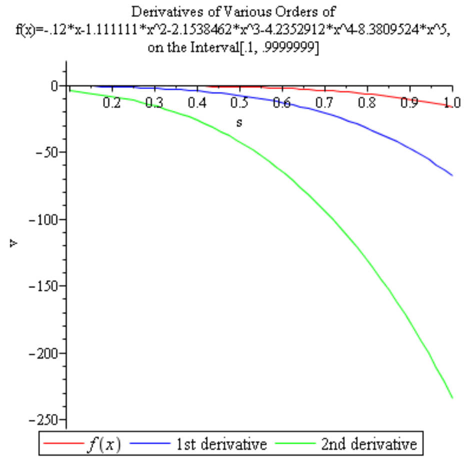

FIGURE 1. This fig explains and demonstrates example 4.1, and we will note all solutions converge in to $(0,0)$.

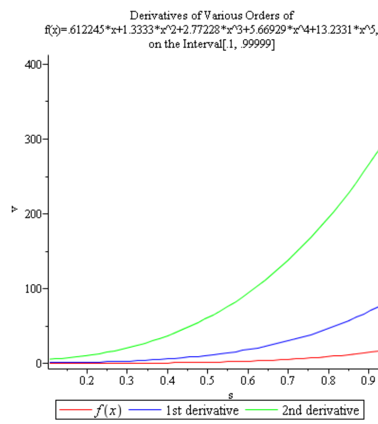

FIGURE 2. This fig explains and demonstrates example 4.2, and we will note all solutions converge in to $(0,0)$.

$$v_5 = \frac{\frac{1}{(\frac{1}{2})^4}}{\left[\frac{\Gamma(6)}{\Gamma((6-\frac{1}{2})} + 0.3\right]} = 13.2331$$

$$\vdots$$

We take at $r = 5$ we get the solution as

$$v = 0.612245 + 1.33333s^2 + 2.77228s^3 + 5.66929s^4 + 13.2331s^5, \quad 0 < s < 1.$$

We find that the behavior of the functions includes the original point $(0,0)$, so that the point $(0,0)$ is an attractive point. Figures 1,2, explain how behavior solutions converge in to $(0,0)$.

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