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Optimal control of some hyperbolic equations with missing data

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Contents

Abstract

The objective of this thesis is to study the optimal control of distributed systems with incomplete data, particularly distributed hyperbolic systems. No-regret control or equivalently Pareto control are used by J. L. Lions to solve the optimal control problems associated with distributed systems with incomplete data. Averaged control was introduced recently by Zauazua to control systems depending upon an unknown parameter.

We present some distributed systems with missing data, we control them via no-regret control and low-regret control methods, and we obtain some optimality systems that characterize the optimum.

The main idea in our work is to apply the notions of no-regret control and averaged control to a hyperbolic equation with unknown parameter and a missing boundary condition. The considered model is motivated by an application in biomedicine. We have introduced the notion of averaged no-regret control to control distributed systems with two kinds of incomplete data, on contrary with previous works, where authors have considered only one kind of missing data. The averaged no-regret control will be characterized by an optimality system.

Keywords: Optimal control, systems with incomplete data, no-regret control, low-regret control, Pareto control, averaged control, averaged no-regret control, hyperbolic equation, fractional diffusion equation, age structured population dynamics, ill-posed wave equation, regularization, electromagnetic wave equation.

الملخص

الهدف من هذه الرسالة هو دراسة التحكم الأمثل في الأنظمة التوزيعية ذات معطيات غير المكتملة، وبالخصوص الأنظمة الزائدية.

استخدم مفهوم التحكم دون ندم أو مكافئه تحكم باريتو من طرف ج. ل. ليونس لحل مسائل التحكم الأمثل المرتبطة بالأنظمة التوزيعية ذات بيانات ناقصة. تم إنشاء مفهوم التحكم المتوسط من قبل زوازوا للتحكم في الأنظمة المرتبطة بوسيط مجهول.

نقدم بعض الانظمة التوزيعية مع البيانات المفقودة، و نتحكم بها من خلال مفهومي التحكم دون ندم و التحكم منخفض الندم، ونتحصل على بعض الأنظمة التي تميز التحكم الأمثل.

تتمثل الفكرة الرئيسية في عملنا في تطبيق مفهومي التحكم دون ندم والتحكم المتوسط على معادلة زائدية ذات وسيط مجهول وشرط حدي مفقود. ويتم تحفيز النموذج المدروس بتطبيق في الطب الحيوي. نقدم مفهوم التحكم المتوسط دون ندم على الأنظمة التوزيعية مع نوعين من البيانات غير المكتملة، على النقيض من الأعمال السابقة، حيث لم يعتبر المؤلفون سوى نوع واحد فقط من البيانات الناقصة. سيميز التحكم المتوسط دون ندم بنظام استمثالي

الكلمات المفتاحية : التحكم الأمثل، الأنظمة ذات البيانات غير مكتملة، التحكم دون ندم، التحكم المنخفض الندم، تحكم باريتو، التحكم المتوسط ، التحكم المتوسط دون ندم، المعادلة الزائدية، معادلة الانتشار ذات مشتقات كسرية، الأنظمة الحركية العمرية للسكان، معادلة الأمواج المعتلة، التعديل، معادلة الأمواج الكهرومغناطيسية.

Dedication

This thesis is dedicated to:
The sake of Allah, my Creator and my Master,
My great teacher and messenger, Mohammed (May Allah bless
and grant him), who taught us the purpose of life,
My great parents, who never stop giving of themselves in countless
ways,
My beloved brothers and sisters,
To all my family, the symbol of love and giving,
My friends who encourage and support me,
And all the people in my life who touch my heart,
I dedicate this research.

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Notations & abbreviations

$\mathbb{R}_+$	Set of positive real numbers.
$\|.\|_H$	A norm in Banach space H.
$(.,.)_H$	A scalar product in Hilbert space H.
$\langle .,. \rangle_{H',H}$	Duality product between H and H'.
$\vert . \vert_H$	A semi-norm in H.
C^2	The class of functions with continuous first and second derivative.
$\frac{\partial y}{\partial \nu} = \nabla y.\nu$	The conormal derivative.
$\Delta = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}$	The laplacien operator.
$\nabla = \left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n} \right)^T$	The gradient operator.
div	Divergence.
$\overset{\circ}{G}$	The interior of G.
$\mathcal{A}^*$	The adjoint operator of $\mathcal{A}$.
$d\Gamma$	Lebesgue measure on boundary Γ.
χ_ω	Characteristic function of the set ω.
$\mathcal{L}(\mathcal{Y}, \mathcal{Z})$	The space of linear bounded operators from $\mathcal{Y}$ to $\mathcal{Z}$.
$\mathcal{D}(Q)$	The space of functions in C^∞ with a compact support tin Q.
${}^{RL}D_t^\alpha$	Reimann-Liouville fractional derivative of order α with respect to time.
w.r.t.	With respect to.
PDE	Partial differential equation.
s.t.	Such that.
iff	If and only if.
a.e.	Almost every where.

List of scientific activities

Publications

- Hafdallah, A. & Ayadi, A.(2018) Optimal control of electromagnetic wave displacement with an unknown velocity of propagation, International Journal of Control, DOI:10.1080/00207179.201
- Hafdallah, A., Laouar, C. & Ayadi, A. (2017) No-regret optimal control characterization for an Ill-posed wave equation. International Journal of Mathematics Trends and Technology (IJMTT).Vol 41(3), 283-288.

Unpublished papers

- Hafdallah, A. & Ayadi, A.(2016) Averaged optimal control for some evolution problems with missing parameter. DOI: 10.13140/RG.2.1.3002.8563.
- Hafdallah, A. & Ayadi, A. (2016) No-regret optimal control redefinition and consequences. hal-01312321.

Conference presentations

- Hafdallah, A. and Ayadi, A., Optimal control for wave equation with incomplete data. Talk presented at the First International Conference on Modeling and Scientific Computing in Mathematical Engineering (MOCASIM), Marrakesh, Morocco from 19 to 22 November 2014.
- A. Hafdallah, A. Ayadi. About optimal control of problems with missed data. Talk presented at the Seventh Workshop of Modelling, Analysis and Control of Systems (MACS-7), El Jadida, Morocco 29, and 30 October 2015.
- A. Hafdallah, A. Ayadi. Optimal control of thermoelasticity coupled systems with incomplete data. Talk presented at scientific day, Oum El Bouaghi, Algeria, June 1st 2016.

Posters

- A. Hafdallah & A. Ayadi. (2016) About optimal control of problems with missed data. Poster presented at the International Conference of the Euro-Maghreb Laboratory of Mathematics and their Interactions (LEM2I), Hammamet, Tunisia, April 27 to May 1st 2016.

Introduction

The issues relating to ecology, environment, and climate are now the focus of many scientists, citizens, political parties, businesses, and states because of its global effect and interest. The climate plays a key role at all levels, especially its profound change "global warming". It is well-known that air and water are real sources of life for flora, fauna, and humans. Thus, as their natures are corrupted by environmental attacks, they become dangers for living beings. This may include vegetative disorders for flora and intoxication or even cases of diseases for humans. Scientists are working to determine the best palliatives for the protection and sanitation of these natural resources. They can not, therefore, succeed without interdisciplinary cooperation. From data observed by naturalists to models of equations designed by mathematicians to the expertise of computer engineers, digital simulation plays a very important role in the mediation between scientific disciplines.

Actually, when modeling those phenomenon and other phenomenon (for example in physics, statics, dynamic population and many other specialties) we encounter some missing data, because of inaccessibility of some data or because of other reasons, for example, in almost all the problems of meteorology or oceanography, we never know the initial data, we have a great variety of possibilities when choosing the initial moment. Same thing for the problems of pollution in a lake, a river or in an estuary, ...

In addition, boundary conditions may also be unknown or only partially known on a part of the boundary that may, for example, be inaccessible to measurements whether biomedical situations or situations corresponding to accidents. The same goes for source terms that can be difficult to access, the same for the structure of the domain, which can also be imperfectly known, for example in oil well management where part of the boundary of the domain is unknown.

Hence, their modeling leads to PDEs with some incomplete data (or missing data). In system analysis, incomplete data means that the initial conditions, boundary conditions, second member of the equation or some of the parameters in the main operator in the system are unknown. One of the objectives in the study of those problems is to control her regardless of the missing terms in their associated mathematical model. Throughout this thesis, we use the terms 'missing data', 'incomplete data' or 'uncertainty' equivalently.

In this thesis, we are interested in the optimal control of distributed systems with incomplete data by using the notion of *no-regret control* and by the notion of *averaged control*.

The first study of those problems was by the famous mathematician Jacques Louis Lions (1986a, 1986b) where he introduced the notion of *Pareto control* for stationary and evolutionary systems, and the general linear abstract equations (1987), the original idea of *Pareto control* was in statistics many years later by Savage (1972). Afterward, Lions (1992) introduced the notion of least regret or low-regret control which consists of transferring the optimal control problem with missing data to a classical optimal control problem (low-regret control), by approximating the no-regret control by a sequence of low-regret controls converging to the unique no-regret control. In several works, Lions applied these notions like (Gabay & Lions, 1994; Lions, 1994 and 1999). Nakoulima and al. (2002) extended the definition of no-regret control and low-regret control to nonlinear distributed systems with missing data.

Later, few studies have been published, like (Nakoulima, Omrane & Velin, 2003) where authors discussed the no-regret and low-regret control for many types of distributed systems (elliptic, parabolic and hyperbolic) with incomplete data. Afterwards, many authors applied these notions to control PDEs with incomplete data, for example in (Dorville, Nakoulima, & Omrane, 2004) authors studied an ill-posed heat equation by approximating her by approaching her by a sequence of elliptic equations with missing data (Dorville & Omrane, 2006 and 2007), in (Dorville, and Omrane, 2006) they treated a forward-backward heat coupled systems by a similar way. Jacob and Omrane (2010) studied an age structured population with missing initial distribution, Berhail and Omrane (2015) studied a Cauchy elliptic problem by regularization into a problem with incomplete data. Recently, Baleanu, Joseph, and Mophou (2016) start studying problems with fractional derivatives by a fractional wave equation then a fractional diffusion equation with missing data in (Mophou, G., 2017). In (Mahoui et al., 2017) authors studied the case of a pointwise control for diffusion equation with incomplete data, unfortunately, they didn't give a characterization for the low-regret control.

However, average control is a new concept in control theory introduced by E. Zuazua (2014) to control systems containing an unknown parameter. A natural idea is to solve those problems is to look for a robust control i.e. looking for control independently of the unknown parameter. Simply, the idea consists on controlling the average of state with respect to the unknown parameter to be equal or closed to a fixed target, then in (Lazar & Zuazua, 2014) authors studied the problem of averaged controllability and observability both for a wave equation, and in (Lohéac & Zuazua, 2017) authors treated the problem of averaged controllability for a general control systems.

Note that in all previous studies of optimal control problems with missing data authors take into account only one kind of incomplete data either a missed boundary condition or an unknown parameter in the main equation. For the first, they applied the no-regret control and for the second they applied the average control.

In our case, we treat a more complicated case where the considered model contains two different kinds of missing data i.e., a missed Dirichlet boundary condition and an unknown velocity of propagation datum. We introduce the notions of *averaged no-regret control* and *averaged low-regret control* to study such kind of optimal control problems with missing data. A motivating example could be found in biomedicine, where the X-rays could damage cells to avoid their harmful effects we have to make the displacement and consequently the energy suitable for the burden of living cells. In our study, we give optimality systems characterizing low-regret control then no-regret control.

Moreover, we control an ill-posed wave equation by a regularization method we get a control problem of a well-posed equation with incomplete data which could be treated via no-regret control and low-regret control notions.

Thesis' overview

This thesis is divided into three chapters:

In the first chapter, we give a brief overview of the classical theory of optimal control for distributed systems (Lions, 1971; Hinze & al., 2008). Then, we outline the notions of no-regret control, low-regret control and the equivalent notion of Pareto control with a characterization of each one in the case of an abstract equation. We finish the chapter by the presenting the concept of average control (Zuazua, 2014).

In the second chapter, we present a variety of optimal control problems with missing data (abstract parabolic equation, a fractional diffusion equation, an age structured population dynamics equation, all with incomplete data).

In the last chapter, we treat the optimal control problem for an ill-posed wave equation, by a regularization we make her well-posed with incomplete data where we characterize the optimal control after taking some limits in suitable spaces. For the main part of the chapter, we study the optimal control of an electromagnetic wave equation with an unknown velocity of propagation and with an unknown Dirichlet boundary condition motivated by an application in X-rays and biomedicine. To solve her we introduce the concept of *averaged no-regret control.*

We end thesis with a conclusion and perspectives describing main obtained results and perspectives for further research projects on the topic.

Chapter 1

Generalities and basic concepts on optimal control of distributed systems with incomplete data

In this primary chapter, we give some basic concepts and results concerning optimal control of distributed systems with incomplete data, starting by the classical theory of optimal control of distributed systems with complete data (Lions, 1971), then we study the notions of no-regret control, low-regret control and Pareto control for optimal control of distributed systems with incomplete data. Moreover, we present the new concept in control theory introduced in (Zuazua, 2014) to control systems with an unknown parameter.
No-regret control is introduced in (Lions, 1992) to study the optimal control problems where considered models contain missing data, this notion was developed in (Nakoulima, Omrane & Velin, 2003). Afterward, many authors applied this notion to study various kinds optimal control problems with incomplete data. Also, we present the notion of Pareto control and we prove its equivalent with the no-regret control notion. In the end, we present the new notion in control theory, it's the averaged control introduced by Zuazua (2014) to control systems depending on an unknown parameter.

1.1 Optimal control of distributed parameters systems

In this section, we recall some classical results about optimal control of distributed parameter systems, i.e., systems defined by PDEs, in this situation the dimension of space of solutions is infinite for this many authors called her "optimal control in infinite dimensional spaces". Many authors interested to this subject like (Lions, 1971) and (Hinze et al., 2008).

Let $\mathcal{Y}$ and $\mathcal{U}$ be real Hilbert spaces of states and controls resp., and let $\mathcal{A}$ be a linear partial differential operator stationary or evolutionary makes an isomorphism on $\mathcal{Y}'$ identified to $\mathcal{Y}$, $\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ the control operator, $\mathcal{U}_{ad} \subset \mathcal{U}$ a non empty closed convex subset of admissible controls, f is a source function in $\mathcal{Y}$. Consider the well-posed abstract linear partial differential equation :

$$\mathcal{A}y(v) = f + \mathcal{B}v \tag{1.1.1}$$

The state equation $(1.1.1)$ must be associated with a boundary conditions and initial conditions in case of evolutionary equations. Let $\mathcal{Z}$ be a Hilbert space of observations, and let $C \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ be the observations operator, we consider the following quadratic cost function :

$$J(v) = \|Cy(v) - z_d\|_{\mathcal{Z}}^2 + (Nv, v)_{\mathcal{U}} \quad \text{for every } v \in \mathcal{U}_{ad} \tag{1.1.2}$$

where z_d is a fixed observation in $\mathcal{Z}$, N is a symmetric definite positive operator bounded on $\mathcal{U}$. The optimal control problem consists of determining u optimal control that minimizes J on $\mathcal{U}_{ad}$, in other words, we search u solution of

$$\inf_{v \in \mathcal{U}_{ad}} J(v) \ \text{ such that } \ (v, y(v)) \ \text{ verifies } \ (1.1.1)\,. \tag{1.1.3}$$

A control-state pair $(u, y(u))$ is called optimal pair if it solves $(1.1.1) - (1.1.3)$.

1.1.1 Existence and uniqueness of optimal control

Here we shall give a general strategy to prove the existence and uniqueness of optimal control for $(1.1.1) - (1.1.3)$ as follows (for more details see (Lions, 1971) and (Hinze et al., 2008)):

Existence

Since $v \to J(v)$ is continuous (which implies that $v \to J(v)$ is lower weakly semi-countinous) convex function on $\mathcal{U}_{ad}$ and even strictly convex, coercive because $(Nv, v)_{\mathcal{U}} \geq \alpha \|v\|_{\mathcal{U}}^2$, $\alpha > 0$ for every $v \in \mathcal{U}_{ad}$, then there exists u solution to $(1.1.1)-(1.1.3)$. Therefore, we consider a minimizing sequence (v_n), i.e., (v_n) verifies

$$\mathcal{A}y(v_n) = f + \mathcal{B}v_n \text{ and } J(v_n) \to J(u)\,,$$

we prove that (v_n) is bounded then by a a compactness argument there exists a subsequence still be denoted by (v_n) converges weakly in $\mathcal{U}$ to u.

Uniqueness

Usually, uniqueness results are obtained from the strict convexity of cost function J and the linearity of equation $(1.1.1)$.

1.1.2 Optimality systems (Optimal control characterization)

Theorem 1.1 *(Hinze et al., 2008) The cost function J is Gateaux-differentiable function (see Appendices Definition 1; Theorem 2), then the following necessary and sufficient optimality condition holds:*

$$\left(J'(u), v-u\right)_{\mathcal{U}} = 2\left(Cy(u) - z_d, Cy(v-u)\right)_{\mathcal{Z}} + 2\left(Nu, v-u\right)_{\mathcal{U}} \geq 0 \quad \forall v \in \mathcal{U}_{ad}. \tag{1.1.4}$$

Remark 1.1 *A condition of the form* (1.1.4) *is called variational inequality.*

Let's try to rewrite optimality condition (1.1.4) as follows: let $C^* \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ the adjoint of C, and introduce the adjoint state $p = p(u)$ given by

$$\mathcal{A}^* p = C^*\left(Cy(u) - z_d\right), \tag{1.1.5}$$

where $\mathcal{A}^*$ is the adjoint operator of $\mathcal{A}$, then

$$\begin{aligned}\left(Cy(u) - z_d, Cy(v-u)\right)_{\mathcal{Z}} &= \left(C^*\left(Cy(u) - z_d\right), y(v-u)\right)_{\mathcal{Y}} \\ &= \left(\mathcal{A}^* p, y(v-u)\right)_{\mathcal{Y}} = \left(p, \mathcal{A}y(v-u)\right)_{\mathcal{Y}} \\ &= \left(p, \mathcal{B}(v-u)\right)_{\mathcal{Y}} = \left(\mathcal{B}^* p, v-u\right)_{\mathcal{U}},\end{aligned}$$

so, the optimality condition (1.1.4) is equivalent to

$$\left(\mathcal{B}^* p + Nu, v-u\right)_{\mathcal{U}} \geq 0 \quad \forall v \in \mathcal{U}_{ad}. \tag{1.1.6}$$

Summarizing by saying that: the optimal control problem $(1.1.1) - (1.1.3)$ has a unique solution u characterized by the following optimality system

$$\begin{cases} \mathcal{A}y(u) = f + \mathcal{B}u, \\ \mathcal{A}^* p = C^*\left(Cy(u) - z_d\right), \\ \left(\mathcal{B}^* p + Nu, v-u\right)_{\mathcal{U}} \geq 0 \quad \forall v \in \mathcal{U}_{ad}, \\ u \in \mathcal{U}_{ad}. \end{cases} \tag{1.1.7}$$

In (1.1.7), the first two equations must be associated to some appropriate boundary conditions.

Remark 1.2 *In the case where $\mathcal{U}_{ad} = \mathcal{U}$ i.e., the case we have no constraints on control, by space structure of $\mathcal{U}$ we deduce that we also have $\left(\mathcal{B}^* p, v-u\right)_{\mathcal{U}} \leq 0$ for every $v \in \mathcal{U}_{ad}$, then in this case the optimality condition* (1.1.6) *takes the form*

$$\mathcal{B}^* p + Nu = 0 \ \text{ in } \mathcal{U}.$$

1.1.3 Remarks on numerical methods to solve optimal control problems

Classification of resolution methods for optimal control problems of distributed systems is hard to establish. Nevertheless, in general there is two principal approaches:

Direct approach (approach then control): the principle of the methods of this approach is based on the notion of discretization or approximation whose idea consists in getting an approximation of the problem in infinite dimension by another in finite dimension, in other words, approaching the PDEs by a set of ODEs. In this case, the methods developed for the dimension systems may be applied.

Indirect approach (control then approach): the principle of the methods of this approach is to obtain the conditions of optimality directly by considering the problem in infinite dimension without making any approximation. Once the conditions of optimality are obtained, methods of approximation of equations or solutions are used to solve them.

1.1.4 Examples

Optimal control of an elliptic equation

Let Ω be a bounded domain in $\mathbb{R}^n$ with boundary Γ of class C^2. Let $\mathcal{U} = L^2(\Omega)$ be the space of controls, $\mathcal{U}_{ad}$ is the set of admissible controls non-empty closed and convex, $\mathcal{B}$ a bounded operator from $L^2(\Omega)$ to $\mathcal{Y} = H_0^1(\Omega)$. Consider the following optimal control problem

$$\inf_{v \in \mathcal{U}_{ad}} J(v), \tag{1.1.8}$$

such that

$$J(v) = \|Cy(v) - z_d\|^2_{L^2(\Omega)} + (Nv, v)_{\mathcal{U}} \quad \text{for every } v \in \mathcal{U}_{ad}, \tag{1.1.9}$$

where $z_d \in L^2(\Omega)$, N is a positive operator on $\mathcal{U}$, C is the canonical injection from $H_0^1(\Omega)$ to $L^2(\Omega)$. Associate to $(1.1.8)$ $(1.1.9)$ the following second order elliptic PDE

$$\begin{cases} \mathcal{A}(x)\, y = f + \mathcal{B}v & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma, \end{cases} \tag{1.1.10}$$

where

$$\mathcal{A}(x)\, y = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial y}{\partial x_j}\right) + a_o(x)\, y, \tag{1.1.11}$$

and a_{ij} are given functions in Ω with

$$\begin{cases} a_{ij}, a_o \in L^\infty(\Omega), \\ \sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\xi_j \geq \alpha \sum_{i=1}^{n} \xi_i^2 \text{ a.e in } \Omega,\ \alpha > 0,\ \xi_i \in \mathbb{R} \text{ for every } i \in \{1, ..., n\}, \\ a_o(x) \geq \alpha \text{ a.e in } \Omega. \end{cases} \tag{1.1.12}$$

Suppose that $(1.1.10)$ is well posed, then:

Theorem 1.2 *(Lions, 1971) The optimal control problem* $(1.1.8)-(1.1.10)$ *has a unique solution* u *characterized by the following optimality system*

$$\begin{cases} \mathcal{A}(x)\,y(u) = f + u, \\ \mathcal{A}^*(x)\,p = y(u) - z_d & \text{in } \Omega, \\ y(u) = 0,\ p = 0 & \text{on } \Gamma, \end{cases}$$

with

$$\int_\Omega (\mathcal{B}^* p + Nu)(v-u)\,dx \geq 0 \text{ for every } v \in \mathcal{U}_{ad}.$$

Proof. It's similar to the one mentioned in subsection 1.1.2. ■

Optimal control of a parabolic equation

Let Ω be a bounded domain in $\mathbb{R}^n$ with boundary Γ of class C^2, $T > 0$. Consider the space-time cylinder $Q = \Omega \times (0,T)$, and her lateral boundary $\Sigma = \Gamma \times (0,T)$. Let $\mathcal{U} = L^2(Q)$ be the space of controls, $\mathcal{U}_{ad}$ is the set admissible controls non-empty closed and convex, $\mathcal{B}$ is a bounded operator from $L^2(Q)$ to $\mathcal{Y} = L^2(0,T; H_0^1(\Omega))$. Consider the following optimal control problem

$$\inf_{v \in \mathcal{U}_{ad}} J(v), \tag{1.1.13}$$

where

$$J(v) = \|Cy(v) - z_d\|_{\mathcal{Z}}^2 + (Nv, v)_{\mathcal{U}} \text{ for every } v \in \mathcal{U}_{ad}, \tag{1.1.14}$$

with the following second order parabolic PDE

$$\begin{cases} \frac{\partial y}{\partial t} + \mathcal{A}(x)\,y = f + \mathcal{B}v & \text{in } Q \\ y(x,t) = 0 & \text{on } \Sigma \\ y(x,0) = y_0(x) & \text{in } \Omega \end{cases} \tag{1.1.15}$$

where $f \in L^2(Q)$, $y_0 \in L^2(\Omega)$, $z_d \in L^2(Q)$, N is a positive operator on $\mathcal{U}$ and $\mathcal{A}(x,t)$ is a second order elliptic operator.

Suppose that $(1.1.10)$ is well posed, then, to derive an optimality system for $(1.1.13)-(1.1.15)$ we discuss two famous cases in applications depending on observation operator C.

First case: C is the canonical injection from $L^2(0,T;H_0^1(\Omega)) \to L^2(Q)$, so $\mathcal{Z} = L^2(Q)$.

Then, the optimal pair $(u, y(u))$ for $(1.1.13)-(1.1.15)$ is characterized by (Lions, 1971)

$$\begin{cases} \frac{\partial y(u)}{\partial t} + \mathcal{A}(x)\, y(u) = f + \mathcal{B}u, & \\ -\frac{\partial p}{\partial t} + \mathcal{A}^*(x)\, p = y(u) - z_d & \text{in } Q, \\ y(x,t) = 0,\ p(x,t) = 0 & \text{on } \Sigma, \\ y(u)(x,0) = y_0(x),\ p(x,T) = 0 & \text{in } \Omega, \\ \int_0^T \int_\Omega (\mathcal{B}^* p + Nu)(v-u)\, dx dt \geq 0 \text{ for every } v \in \mathcal{U}_{ad}. & \end{cases}$$

Second case: A final observation operator i.e. $C : L^2(0,T;H_0^1(\Omega)) \to L^2(\Omega)$ with $Cy(x,t) = y(x,T)$, so $\mathcal{Z} = L^2(\Omega)$. Then, the optimal pair $(u, y(u))$ for $(1.1.13)-(1.1.15)$ is characterized by (Lions, 1971)

$$\begin{cases} \frac{\partial y(u)}{\partial t} + \mathcal{A}(x)\, y(u) = f + \mathcal{B}u, & \\ -\frac{\partial p}{\partial t} + \mathcal{A}^*(x)\, p = 0 & \text{in } Q, \\ y(x,t) = 0,\ p(x,t) = 0 & \text{on } \Sigma, \\ y(u)(x,0) = y_0(x),\ p(x,T) = y(u)(x,T) - z_d & \text{in } \Omega, \\ \int_0^T \int_\Omega (\mathcal{B}^* p + Nu)(v-u) \geq 0, \text{ for every } v \in \mathcal{U}_{ad}. & \end{cases}$$

Optimal control of a hyperbolic equation

Let Ω, Γ, $T>0$, Q, Σ as in the last subsection. Let $\mathcal{U} = L^2(Q)$ be the space of controls, $\mathcal{U}_{ad}$ is the set admissible controls non-empty closed and convex, $\mathcal{B}$ is a bounded operator from to $\mathcal{Y} = L^2(0,T;H_0^1(\Omega))$. Consider the following optimal control problem

$$\inf_{v \in \mathcal{U}_{ad}} J(v), \tag{1.1.16}$$

with

$$J(v) = \|Cy(v) - z_d\|_{\mathcal{Z}}^2 + (Nv, v)_{\mathcal{U}} \text{ for every } v \in \mathcal{U}_{ad}, \tag{1.1.17}$$

with the following second order hyperbolic PDE

$$\begin{cases} \frac{\partial^2 y}{\partial t^2} + \mathcal{A}(x)\, y = f + \mathcal{B}v & \text{in } Q, \\ y(x,t) = 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x),\ \frac{\partial y}{\partial t}(x,0) = y_1(x) & \text{in } \Omega, \end{cases} \tag{1.1.18}$$

where $f \in L^2(Q)$, $y_0 \in H_0^1(\Omega)$, $y_1 \in L^2(\Omega)$, $z_d \in L^2(Q)$, N is a positive operator on $\mathcal{U}$ and $\mathcal{A}(x,t)$ is a second order elliptic operator.

First case: C is the canonical injection from $L^2\left(0,T;H_0^1\left(\Omega\right)\right) \rightarrow L^2\left(Q\right)$, then $\mathcal{Z} = L^2\left(Q\right)$. Then, the optimal pair $\left(u, y\left(u\right)\right)$ for $(1.1.16) - (1.1.18)$ is characterized by (Lions, 1971)

$$\begin{cases} \frac{\partial^2 y(u)}{\partial t} + \mathcal{A}\left(x\right) y\left(u\right) = f + \mathcal{B}u, & \\ \frac{\partial^2 p}{\partial t^2} + \mathcal{A}^*\left(x\right) p = y\left(u\right) - z_d & \text{in } Q, \\ y\left(x,t\right) = 0,\ p\left(x,t\right) = 0, & \text{on } \Sigma, \\ y\left(u\right)\left(x,0\right) = y_0\left(x\right),\ \frac{\partial y}{\partial t}\left(x,0\right) = y_1\left(x\right) & \\ p\left(x,T\right) = 0,\ \frac{\partial p}{\partial t}\left(x,T\right) = 0 & \text{in } \Omega, \\ \int_0^T \int_\Omega \left(\mathcal{B}^* p + Nu\right)\left(v - u\right) dxdt \geq 0 \text{ for every } v \in \mathcal{U}_{ad}. & \end{cases}$$

Second case: A final observation operator i.e., $C : L^2\left(0,T;H_0^1\left(\Omega\right)\right) \rightarrow L^2\left(\Omega\right)$ with $Cy\left(x,t\right) = y\left(x,T\right)$, so $\mathcal{Z} = L^2\left(\Omega\right)$. Then, the optimal pair $\left(u, y\left(u\right)\right)$ for $(1.1.16) - (1.1.18)$ is characterized by (Lions, 1971)

$$\begin{cases} \frac{\partial^2 y(u)}{\partial t} + \mathcal{A}\left(x\right) y\left(u\right) = f + \mathcal{B}u, & \\ \frac{\partial^2 p}{\partial t^2} + \mathcal{A}^*\left(x\right) p = 0 & \text{in } Q, \\ y\left(x,t\right) = 0,\ p\left(x,t\right) = 0 & \text{on } \Sigma, \\ y\left(u\right)\left(x,0\right) = y_0\left(x\right),\ \frac{\partial y}{\partial t}\left(x,0\right) = y_1\left(x\right) & \\ p\left(x,T\right) = 0,\ \frac{\partial p}{\partial t}\left(x,T\right) = y\left(u\right)\left(x,T\right) - z_d & \text{in } \Omega, \\ \int_0^T \int_\Omega \left(\mathcal{B}^* p + Nu\right)\left(v - u\right) dxdt \geq 0 \text{ for every } v \in \mathcal{U}_{ad}. & \end{cases}$$

1.2 No-regret control, low-regret control and Pareto control

In this section, we make an initiation to the theory of optimal control of problems with incomplete data, where we introduce the concepts of no-regret control, low-regret control and Pareto control. Moreover, we give existence, uniqueness, and characterization for each one with a few remarks. All the following results could be extended to nonhomogeneous boundary values problems and evolutionary equations (as we will see in the next chapters).

1.2.1 Position of problem

We keep the same theoretical framework as mentioned in paragraph 1.1, in addition to introducing a new operator $\beta \in \mathcal{L}\left(F, \mathcal{Y}\right)$, where F is a Hilbert space of uncertainties (incomplete data), G is a non-empty closed subspace of F. Let's consider the following controlled abstract equation with missing data

$$\mathcal{A}y\left(v,g\right) = f + \mathcal{B}v + \beta g, \tag{1.2.1}$$

the equation $(1.2.1)$ is well-posed in $\mathcal{Y}$, has a unique solution $y = y(v,g)$. For every uncertainty $g \in G$, associate to $(1.2.1)$ the following cost function

$$J(v,g) = \|Cy(v,g) - z_d\|_{\mathcal{Z}}^2 + N\|v\|_{\mathcal{U}}^2 \ , v \in \mathcal{U}_{ad}, \tag{1.2.2}$$

where $z_d \in \mathcal{Z}$ and $N > 0$. In this case, we are concerned by the optimal control problem

$$\inf_{v\in\mathcal{U}_{ad}} J(v,g) \text{ for every } g \in G, \tag{1.2.3}$$

with respect to $(1.2.1)$, this problem has no sense when G is an infinite dimensional space, the celebrated mathematician J. L. Lions used many notions to solve this problem like no-regret control (Lions, 1992) and Pareto control (Lions, 1986), their equivalents is proved in (Nakoulima, Omrane & Velin, 2003). Lions thought to take

$$\inf_{v\in\mathcal{U}_{ad}} \left(\sup_{g\in G} J(v,g)\right), \tag{1.2.4}$$

but $J(v,g)$ hasn't an upper bound because $\sup_{g\in G} J(v,g) = +\infty$.

Remark 1.3 *When $G = \{0\}$, then $J(v,g) = J(v,0)$. Therefore, the problem $(1.2.3)$ becomes a standard optimal control problem, i.e., find $\inf_{v\in\mathcal{U}_{ad}} J(v,0)$.*

1.2.2 No-regret control

To avoid difficulty arises in $(1.2.4)$ Lions thought to (this idea was originated in statistics in (Savage, 1972) look only for controls v such that

$$J(v,g) \leq J(0,g) \text{ for every } g \in G. \tag{1.2.5}$$

Note that the optimal control verifies the last equality, otherwise the optimum is $u = 0$.

Definition 1.1 *(Lions, 1992) We say that $u \in \mathcal{U}_{ad}$ is a no-regret control for $(1.2.1)$ $(1.2.2)$ if u solves*

$$\inf_{v\in\mathcal{U}_{ad}} \sup_{g\in G} (J(v,g) - J(0,g)). \tag{1.2.6}$$

Remark 1.4 *Of course, the problem $(1.2.6)$ is defined only for controls such that*

$$\sup_{g\in G} (J(v,g) - J(0,g)) < \infty. \tag{1.2.7}$$

Lemma 1.1 *(Nakoulima, Omrane & Velin, 2003) For every $v \in \mathcal{U}_{ad}$ and every $g \in G$ we have*

$$J(v,g) - J(0,g) = J(v,0) - J(0,0) + 2\langle S(v), g\rangle_{G',G}, \tag{1.2.8}$$

where $S(v) = \beta^* \zeta(v)$ and $\zeta(v)$ solves

$$\mathcal{A}^* \zeta(v) = C^* C (y(v,0) - y(0,0)).$$

Proof. $\mathcal{A}$ is an isomorphism so $y(v,g) = y(v,0) + y(0,g) - y(0,0)$, then

$$\begin{aligned} J(v,g) - J(0,g) &= J(v,0) - J(0,0) + 2(C(y(v,0) - y(0,0)), C(y(0,g) - y(0,0)))_{\mathcal{Z}} \\ &= J(v,0) - J(0,0) + 2(C^*C(y(v,0) - y(0,0)), y(0,g) - y(0,0))_{\mathcal{Y}} \end{aligned} \tag{1.2.9}$$

Introduce $\zeta(v)$ given by

$$\mathcal{A}^* \zeta(v) = C^* C (y(v,0) - y(0,0)),$$

we can write $(1.2.8)$ as

$$\begin{aligned} J(v,g) - J(0,g) &= J(v,0) - J(0,0) + 2(\mathcal{A}^*\zeta(v), y(0,g) - y(0,0))_{\mathcal{Y}} \\ &= J(v,0) - J(0,0) + 2(\zeta(v), \mathcal{A}(y(0,g) - y(0,0)))_{\mathcal{Y}} \\ &= J(v,0) - J(0,0) + 2(\zeta(v), \beta g)_{\mathcal{Y}} \\ &= J(v,0) - J(0,0) + 2\langle \beta^*\zeta(v), g\rangle_{G',G}, \end{aligned}$$

the last equation leads to $(1.2.8)$. ∎

Remark 1.5

1. *By $(1.2.8)$ you can see that condition $(1.2.7)$ holds iff $v \in K$, where*

$$K = \left\{ v \in \mathcal{U}_{ad} : \langle S(v), g\rangle_{G',G} = 0 \ \forall g \in G \right\}$$

   is a closed subspace of U. Then, u is a no-regret control iff $u \in K$.

2. *The notion of no-regret control could be generalized to no-regret control related to any a fixed control $u_0 \in U_{ad}$, i.e., we want controls v s.t.*

$$J(v,g) \leq J(u_0,g) \text{ for every } g \in G.$$

Definition 1.2 *We say that $u \in \mathcal{U}_{ad}$ is a no-regret control related to $u_0 \in \mathcal{U}_{ad}$ for $(1.2.1)$ $(1.2.2)$ if u solves*

$$\inf_{v \in \mathcal{U}_{ad}} \sup_{g \in G} (J(v,g) - J(u_0,g)). \tag{1.2.10}$$

Unfortunately, the main difficulty with no-regret control arises when we want to characterize the set K, for this reason we shall approximate the no-regret control by a sequence of controls called low-regret controls.

1.2.3 Low-regret control

One thought to relax $(1.2.5)$ by making some quadratic perturbation on $J(0,g)$ (Lions, 1992), in other words, we search controls v such that

$$J(v,g) \leq J(0,g) + \gamma \|g\|_G^2 \text{ for every } g \in G, \gamma > 0.$$

Definition 1.3 *(Lions, 1992) We say that $u_\gamma \in \mathcal{U}_{ad}$ is a low-regret control for $(1.2.1)\,(1.2.2)$ if u solves*

$$\inf_{v\in\mathcal{U}_{ad}} \sup_{g\in G} \left(J(v,g) - J(0,g) - \gamma \|g\|_G^2\right), \gamma > 0. \tag{1.2.11}$$

Take $(1.2.8)$ into account to get the equivalence between $(1.2.11)$ and

$$\inf_{v\in\mathcal{U}_{ad}} \left(J(v,0) - J(0,0) + \sup_{g\in G}\left(2\left\langle S(v), g\right\rangle_{G',G} - \gamma \|g\|_G^2\right)\right),$$

and thanks to Legendre transform (Aubin, 1984: p. 49) for

$$\sup_{g\in G}\left(2\left\langle S(v), g\right\rangle_{G',G} - \gamma \|g\|_G^2\right) = \frac{1}{\gamma}\|S(v)\|_G^2,$$

then,

$$\inf_{v\in\mathcal{U}_{ad}} \mathcal{J}^\gamma(v), \tag{1.2.12}$$

where

$$\mathcal{J}^\gamma(v) = J(v,0) - J(0,0) + \frac{1}{\gamma}\|S(v)\|_G^2. \tag{1.2.13}$$

Low-regret control existence and uniqueness

Proposition 1.1 *The problem $(1.2.1)\,(1.2.12)\,(1.2.13)$ has a unique solution u_γ.*

Proof. It's clear that

$$\mathcal{J}^\gamma(v) \geq -J(0,0) \quad \forall v \in \mathcal{U}_{ad},$$

then $d_\gamma = \inf_{v\in\mathcal{U}_{ad}} \mathcal{J}^\gamma(v)$ exists. Let (v_n^γ) be a minimizing sequence s.t. $d_\gamma = \lim_{n\to\infty} \mathcal{J}^\gamma(v_n^\gamma)$, we have

$$-J(0,0) \leq \mathcal{J}^\gamma(v_n^\gamma) = J(v_n^\gamma, 0) - J(0,0) + \frac{1}{\gamma}\|S(v_n^\gamma)\|_G^2 \leq d_\gamma + 1,$$

which gives the bounds

$$\|v_n^\gamma\|_{\mathcal{U}} \leq C_\gamma, \frac{1}{\sqrt{\gamma}}\|S(v_n^\gamma)\|_G \leq C_\gamma, \ \|Cy(v_n^\gamma)\|_{\mathcal{Z}} \leq C_\gamma, \tag{1.2.14}$$

where C_γ is a constant independent of n. From $(1.2.14)$ we deduce that there exists $u_\gamma \in \mathcal{U}_{ad}$ s.t. $v_n^\gamma \rightharpoonup u_\gamma$ weakly in $\mathcal{U}_{ad}$. ■

Theorem 1.3 *The sequence of low-regret controls converges weakly in $\mathcal{U}_{ad}$ when $\gamma \to 0$ to the unique no-regret control u solution to $(1.2.1)\,(1.2.2)$.*

Proof. Let u_γ be the unique low-regret control solution to $(1.2.1)\,(1.2.12)\,(1.2.13)$. Then,

$$J(u_\gamma, 0) - J(0,0) + \frac{1}{\gamma}\left\|S(u_\gamma)\right\|_G^2 \leq J(v,0) - J(0,0) + \frac{1}{\gamma}\left\|S(v)\right\|_G^2 \text{ , } \forall v \in \mathcal{U}_{ad},$$

take $v = 0$ to get

$$J(u_\gamma, 0) + \frac{1}{\gamma}\left\|S(u_\gamma)\right\|_G^2 \leq J(0,0) = \text{constant.}$$

Remember the definition of $J(v,g)$ in $(1.2.2)$ to find

$$\left\|Cy(u_\gamma,0) - z_d\right\|_{\mathcal{Z}}^2 + N\left\|u_\gamma\right\|_{\mathcal{U}} + \frac{1}{\gamma}\left\|S(u_\gamma)\right\|_G^2 \leq C, \tag{1.2.15}$$

where C is a constant independent of γ. From $(1.2.15)$ we deduce that (u_γ) is bounded in $\mathcal{U}_{ad}$, then there exists a subsequence still be denoted (u_γ) converges weakly to $u \in \mathcal{U}_{ad}$. Let's prove that u is the unique no-regret control solution to $(1.2.1)\,(1.2.2)$ as follows:
For $v \in \mathcal{U}_{ad}$, we have

$$J(u,g) - J(u,0) - \gamma\left\|g\right\|_G^2 \leq J(v,g) - J(0,g) \text{ for every } g \in G,$$

then

$$J(u,g) - J(u,0) - \gamma\left\|g\right\|_G^2 \leq \sup_{g \in G}\left(J(v,g) - J(0,g)\right) \text{ for every } g \in G,$$

pass to limit $\gamma \to 0$ to get

$$J(u,g) - J(u,0) \leq \sup_{g \in G}\left(J(v,g) - J(0,g)\right) \text{ for every } g \in G,$$

which means that u is a no-regret control. ■

Approximated abstract optimality system (Optimality system of low-regret control)

In the following proposition, we give an optimality system characterizing low-regret control u_γ.

Proposition 1.2 *The low-regret control u_γ, solution to $(1.2.1)\,(1.2.12)\,(1.2.13)$ is characterized by the following optimality system*

$$\begin{cases} \mathcal{A}y_\gamma = f + \mathcal{B}u_\gamma, \\ \mathcal{A}^*\zeta_\gamma = C^*C\left(y_\gamma - y(0,0)\right), \\ \mathcal{A}\rho_\gamma = \frac{1}{\gamma}\beta\beta^*\zeta_\gamma, \\ \mathcal{A}^*p_\gamma = C^*\left(Cy_\gamma - z_d\right) + C^*C\rho_\gamma, \\ \left(\mathcal{B}^*p_\gamma + Nu_\gamma, v - u_\gamma\right)_{\mathcal{U}} \geq 0 \quad \forall v \in \mathcal{U}_{ad}. \end{cases} \tag{1.2.16}$$

Proof. Let u_γ be solution to (1.2.1) (1.2.12) (1.2.13). A first order necessary condition gives for every $v \in \mathcal{U}_{ad}$

$$\begin{aligned}\left(\mathcal{J}^{\gamma'}(u_\gamma), v-u_\gamma\right)_{\mathcal{U}} &= \left(C^*\left(Cy(u_\gamma,0)-z_d\right), y(v-u_\gamma,0)-y(0,0)\right)_{\mathcal{Z}} + N\left(u_\gamma, v-u_\gamma\right)_{\mathcal{U}} \\ +\frac{1}{\gamma}\left(S(u_\gamma), S(v-u_\gamma)\right)_G &\geq 0. \end{aligned} \quad (1.2.17)$$

Denote $y_\gamma = y(u_\gamma, 0)$, $\zeta_\gamma(v) = \beta S(v)$, by definition we have

$$\mathcal{A}^*\zeta_\gamma = C^*C\left(y_\gamma - y(0,0)\right).$$

Also, let ρ_γ be the solution of

$$\mathcal{A}\rho_\gamma = \frac{1}{\gamma}\beta\beta^*\zeta_\gamma.$$

Now, introduce the adjoint state $p_\gamma = p(u_\gamma, 0)$ defined by

$$\mathcal{A}^*p_\gamma = C^*\left(Cy_\gamma - z_d\right) + C^*C\rho_\gamma,$$

then

$$\begin{aligned}\frac{1}{\gamma}\left(S(u_\gamma), S(v-u_\gamma)\right)_G &= \frac{1}{\gamma}\left(\beta^*\zeta_\gamma(u_\gamma), \beta^*\zeta(v-u_\gamma)\right)_G \\ &= \frac{1}{\gamma}\left(\beta\beta^*\zeta_\gamma(u_\gamma), \zeta(v-u_\gamma)\right)_{\mathcal{Y}} \\ &= \left(\mathcal{A}\rho_\gamma, \zeta(v-u_\gamma)\right)_{\mathcal{Y}} \\ &= \left(\rho_\gamma, \mathcal{A}^*\zeta(v-u_\gamma)\right)_{\mathcal{Y}} \\ &= \left(\rho_\gamma, C^*C\left(y(v-u_\gamma,0)-y(0,0)\right)\right)_{\mathcal{Y}} \\ &= \left(C^*C\rho_\gamma, y(v-u_\gamma,0)-y(0,0)\right)_{\mathcal{Y}} \\ &= \left(\mathcal{A}^*p_\gamma - C^*\left(Cy_\gamma - z_d\right), y(v-u_\gamma,0)-y(0,0)\right)_{\mathcal{Y}} \\ &= \left(p_\gamma, \mathcal{A}\left(y(v-u_\gamma,0)-y(0,0)\right)\right)_{\mathcal{Y}} - \left(C^*\left(Cy_\gamma - z_d\right), y(v-u_\gamma,0)-y(0,0)\right)_{\mathcal{Y}}.\end{aligned}$$

Finally,

$$\frac{1}{\gamma}\left(S(u_\gamma), S(v-u_\gamma)\right)_G = \left(p_\gamma, \mathcal{B}(v-u_\gamma)\right)_{\mathcal{Y}} - \left(C^*\left(Cy_\gamma - z_d\right), y(v-u_\gamma,0)-y(0,0)\right)_{\mathcal{Y}}.$$

Hence, we deduce that the optimality condition (1.2.17) is equivalent to

$$\left(\mathcal{B}^*p_\gamma + Nu_\gamma, v-u_\gamma\right)_{\mathcal{U}} \geq 0 \text{ for every } v \in \mathcal{U}_{ad}.$$

■

Abstract optimality system (Optimality system of no-regret control)

Let's introduce

$$P: \text{orthogonal projection operator of } F \text{ on } G,$$

then, $v \in K$ iff

$$P\beta^*\zeta(v) = 0, \tag{1.2.18}$$

Finding a no-regret control u is equivalent to

$$\inf J(0,g), \, v \text{ subject to } (P\beta^*\zeta(v) = 0).$$

Approach by a penalty argument and define

$$J_\varepsilon(v) = J(v,0) + \frac{1}{\varepsilon}\|P\beta^*\zeta(v)\|_F^2, \, \varepsilon > 0, \tag{1.2.19}$$

and consider the following problem

$$\inf_{v\in\mathcal{U}_{ad}} J_\varepsilon(v), \tag{1.2.20}$$

this problem has a unique solution u_ε such that

$$u_\varepsilon \to u, \text{ in } \mathcal{U}_{ad}.$$

Set

$$y(u_\varepsilon) = y_\varepsilon, \, \zeta(u_\varepsilon) = \zeta_\varepsilon, \, \lambda_\varepsilon = \frac{1}{\varepsilon}P\beta^*\zeta_\varepsilon.$$

The control u_ε is characterized by

$$(Cy_\varepsilon - z_d, Cy(v-u_\varepsilon,0))_{\mathcal{Z}} + N(u_\varepsilon, v-u_\varepsilon)_{\mathcal{U}} + (\lambda_\varepsilon, P\beta^*\zeta(v-u_\varepsilon))_F \geq 0 \;\; \forall v \in \mathcal{U}_{ad}. \tag{1.2.21}$$

Also,

$$\mathcal{A}y_\varepsilon = \mathcal{B}u_\varepsilon, \, \mathcal{A}^*\zeta_\varepsilon = C^*Cy_\varepsilon.$$

Introduce

$$\mathcal{A}^*p_\varepsilon = C^*(Cy_\varepsilon - z_d) + C^*C\rho_\varepsilon, \, \mathcal{A}\rho_\varepsilon = \beta\lambda_\varepsilon.$$

Then

$$\begin{aligned}
& (\mathcal{A}^*p_\varepsilon, y(v-u_\varepsilon,0))_{\mathcal{Y}} + (\mathcal{A}\rho_\varepsilon, \zeta(v-u_\varepsilon))_{\mathcal{Y}} \\
= \;& (Cy_\varepsilon - z_d, Cy(v-u_\varepsilon,0))_{\mathcal{Z}} + (C\rho_\varepsilon, Cy(v-u_\varepsilon,0))_{\mathcal{Z}} + (\beta\lambda_\varepsilon, \zeta(v-u_\varepsilon))_F \\
= \;& (p_\varepsilon, \mathcal{B}(v-u_\varepsilon))_{\mathcal{Y}} + (\rho_\varepsilon, \mathcal{A}^*\zeta(v-u_\varepsilon))_{\mathcal{Y}} \\
= \;& (p_\varepsilon, \mathcal{B}(v-u_\varepsilon))_{\mathcal{Y}} + (C\rho_\varepsilon, Cy(v-u_\varepsilon,0))_{\mathcal{Z}}.
\end{aligned}$$

Optimality condition $(1.2.21)$ is reduced to

$$\left(\mathcal{B}^* p_\varepsilon + N u_\varepsilon, v - u_\varepsilon\right)_{\mathcal{U}} \geq 0 \quad \forall v \in \mathcal{U}_{ad}. \tag{1.2.22}$$

A difficulty lies in obtaining a priori estimate on λ_ε. Introduce $\hat{p}_\varepsilon$, σ_ε s.t.

$$\mathcal{A}^* \hat{p}_\varepsilon = C^* \left(C y_\varepsilon - z_d\right), p_\varepsilon \in \mathcal{Y},$$

$$\mathcal{A}^* \sigma_\varepsilon = C^* C \rho_\varepsilon, \sigma_\varepsilon \in \mathcal{Y}.$$

Then $p_\varepsilon = \hat{p}_\varepsilon + \sigma_\varepsilon$. Make $\varepsilon \to 0$, since $u_\varepsilon \to u$ in $\mathcal{U}_{ad}$, we also know that $y_\varepsilon \to y$, $\zeta_\varepsilon \to \zeta$ and $\hat{p}_\varepsilon \to \hat{p}$ all in $\mathcal{Y}$, with

$$\mathcal{A} y = \mathcal{B} u,\ \mathcal{A}^* \zeta = C^* C y,\ \mathcal{A}^* \hat{p} = C^* \left(C y - z_d\right).$$

Now, optimality condition $(1.2.22)$ is equivalent to

$$\left(\mathcal{B}^* \hat{p}_\varepsilon + \mathcal{B}^* \sigma_\varepsilon + N u_\varepsilon, v - u_\varepsilon\right)_{\mathcal{U}} \geq 0 \quad \forall v \in \mathcal{U}_{ad}.$$

When $\varepsilon \to 0$ we get

$$\left(\mathcal{B}^* \hat{p} + \mathcal{B}^* \sigma + N u, v - u\right)_{\mathcal{U}} \geq 0 \quad \forall v \in \mathcal{U}_{ad}. \tag{1.2.23}$$

Consider

$$\mathcal{A} \rho = \beta g,\ \mathcal{A}^* \sigma = C^* C \rho,$$

then, introduce

$$|||g||| = \left\| \mathcal{B}^* \sigma \right\|_{\mathcal{U}}, \tag{1.2.24}$$

$(1.2.24)$ defines a semi-norme on $\overset{\circ}{G}$, also we construct the quotient space still denoted by $\overset{\circ}{G}$ associated to $g_1 \sim g_2$ iff $|||g_1||| = |||g_2|||$, and we define $\widehat{G}$ the completion of G (on the quotient space) with respect to the norm $|||.|||$ topology.
Then,

$$\lambda_\varepsilon \text{ remains in a bounded set of } \widehat{G}. \tag{1.2.25}$$

Now, we can announce the following theorem characterizing no-reget control for $(1.2.1)\,(1.2.2)$.

Theorem 1.4 *(Lions, 1987) Suppose that $(1.2.25)$ holds, then the no-regret control u solution to $(1.2.1)\,(1.2.2)$ is characterized by the following optimality system*

$$\begin{cases} \mathcal{A} y = f + \mathcal{B} u, \\ \mathcal{A}^* \zeta = C^* C \left(y - y\left(0,0\right)\right), \\ \mathcal{A} \rho = \beta \lambda, \\ \mathcal{A}^* p = C^* \left(C y - z_d\right) + C^* C \rho, \\ \left(\mathcal{B}^* p + N u, v - u\right)_{\mathcal{U}} \geq 0 \quad \forall v \in \mathcal{U}_{ad}, \\ \lambda \in \widehat{G}. \end{cases} \tag{1.2.26}$$

Remark 1.6 *In (Nakoulima, Omrane & Velin, 2003), authors made the following hypothesis: First, solve*

$$\mathcal{A}\rho = \mathcal{B}g,\ g \in G,\ \rho \in \mathcal{Y},$$

then,

$$\mathcal{A}^{*}\sigma = C^{*}C\rho,\ \sigma \in \mathcal{Y},$$

and set $\mathcal{R}g = \mathcal{B}^{*}\sigma$. We make the assumption that

$$\|\mathcal{R}g\|_{\widehat{G}} \geq c\,\|g\|_{G},\ c > 0, \text{ for every } g \in G.$$

The space $\widehat{G}$ is the closure of G for some topology $(H, \|.\|_{H})$, where H is a subspace of F. In applications the space $\widehat{G}$ will be specified.

1.2.4 Pareto control

Definition 1.4 *(Nakoulima, Omrane & Velin, 2003) We say that $u \in \mathcal{U}_{ad}$ is a Pareto control for (1.2.1) (1.2.2) iff*

$$J(u, g) \leq J(v, g) \text{ for every } v \in \mathcal{U}_{ad} \text{ and every } g \in G,$$

and if there exists $g_0 \in G$ s.t.

$$J(u, g_0) < J(v, g_0) \text{ for every } v \in \mathcal{U}_{ad}.$$

Definition 1.5 *(Nakoulima, Omrane & Velin, 2003) We say that $u \in \mathcal{U}_{ad}$ is a Pareto control related to a control $u_0 \in \mathcal{U}_{ad}$ if*

$$J(u, g) \leq J(u_0, g) \text{ for every } g \in G.$$

Remark 1.7

1. *When we take $u_0 = 0$, the last definition becomes no-regret control definition, then Pareto control related to a control $u_0 \in U_{ad}$ is a generalization of no-regret control.*

2. *We can easily generalize, identities concerning no-regret control to Pareto control related to a control $u_0 \in U_{ad}$ as the following identity*

$$J(v, g) - J(u_0, g) = J(v, 0) - J(u_0, 0) + 2\left\langle S(v - u_0), g\right\rangle_{G',G}. \tag{1.2.27}$$

Proposition 1.3 *There exists a unique Pareto control u related to $u_0 \in \mathcal{U}_{ad}$. Moreover, u is the unique element of $K + u_0$ that minimizes $J(v, 0)$ on $K + u_0$.*

Proof. First, let us see when

$$J(v,g) \leq J(w,g), \forall g \in G. \tag{1.2.28}$$

From $(1.2.27)$ we know that

$$J(v,g) - J(w,g) = J(v,0) - J(w,0) + 2\langle S(v-w), g\rangle_{G',G},$$

since G is a vector space, $(1.2.28)$ is equivalent to

$$\langle S(v-w), g\rangle_{G',G} = 0 \; \forall g \in G,$$

and

$$J(v,0) \leq J(w,0), \tag{1.2.29}$$

i.e., $(1.2.28)$ is equivalent to $v - w \in K$ and $(1.2.29)$. In this situation we get :

$$J(v,g) - J(w,g) = J(v,0) - J(w,0),$$

take $v = u$, $w = u_0$, it will be a simple matter to verify that u is a Pareto control related to u_0 iff

$$\begin{cases} u \in K + u_0, \\ J(u,0) \leq J(v,0), \; \forall v \in K + u_0, \end{cases} \tag{1.2.30}$$

this means that u minimizes $J(v,0)$ on $K + u_0$. ■

Remark 1.8

1. *Conditions* $(1.2.30)$ *prove that finding* u *a Pareto control related to* u_0*, is equivalent to an optimal control problem with constraint on the state. This constraint is*

$$\left(C\left(y\left(v-u_0,0\right)-y\left(0,0\right)\right), C\left(y\left(0,g\right)-y\left(0,0\right)\right)\right)_{\mathcal{Z}} = 0, \; \forall g \in G. \tag{1.2.31}$$

2. *When* $G = \{0\}$ *condition* $(1.2.31)$ *is verified and one deals with a standard optimal control problem (i.e., an optimal control problem with complete data).*

Equivalence between no-regret control and Pareto control

In this paragraph, the following theorem proves the equivalence between the notions of Pareto control and no-regret control both related to the same control.

Theorem 1.5 *Let* $u_0 \in \mathcal{U}_{ad}$ *a fixed given control. Then, a control* $u \in \mathcal{U}_{ad}$ *is a Pareto control related to* u_0 *iff* u *is a no-regret control related to* u_0*.*

Proof. Let $u \in \mathcal{U}_{ad}$ be a Pareto control related to u_0 and $v \in K + u_0$. Then for every $g \in G$

$$\langle S(u-u_0), g\rangle_{G',G} = 0 = \langle S(v-u_0), g\rangle_{G',G}$$

also from Proposition 1.3 we know that $J(u,0) \leq J(v,0)$. Use (1.2.27) to get

$$\begin{aligned} J(u,0) - J(u_0,0) + 2\langle S(u-u_0), g\rangle_{G',G} &\leq \sup_{g\in G}\left(J(v,g) - J(u_0,g)\right) \quad \forall g \in G \\ &\Rightarrow \sup_{g\in G}\left(J(u,g) - J(u_0,g)\right) \leq \sup_{g\in G}\left(J(v,g) - J(u_0,g)\right). \end{aligned}$$

Then,

$$\sup_{g\in G}\left(J(u,g) - J(u_0,g)\right) = \inf_{v\in K+u_0}\left(\sup_{g\in G}\left(J(v,g) - J(u_0,g)\right)\right).$$

Now, let $v \in \mathcal{U}_{ad} \setminus K + u_0$. There exists $g_0 \in G$ such that $\langle S(v-u_0), g\rangle_{G',G} \neq 0$. So, we have

$$\sup_{g\in G}\left(J(v,g) - J(u_0,g)\right) = J(v,0) - J(u_0,0) + 2\sup_{g\in G}\langle S(v-u_0), g\rangle_{G',G} = +\infty.$$

Note that G is a linear space, so, we only have the following two possibilities: $\sup_{g\in G}\langle S(w), g\rangle_{G',G} = 0$ or $\sup_{g\in G}\langle S(w), g\rangle_{G',G} = +\infty$. In this case, $\lim_{t\to+\infty}\langle S(v-u_0), tg\rangle_{G',G} = +\infty$.
Also, u is a Pareto control related to u_0 i.e. $J(u,g) - J(u_0,g) \leq 0 \ \forall g \in G$, then

$$J(u,g) - J(u_0,g) \leq 0 \leq \sup_{g\in G}\left(J(v,g) - J(u_0,g)\right) \quad \forall g \in G,$$

we deduce that

$$\sup_{g\in G}\left(J(u,g) - J(u_0,g)\right) = \inf_{v\in\mathcal{U}_{ad}\setminus K+u_0}\left(\sup_{g\in G}\left(J(v,g) - J(u_0,g)\right)\right).$$

Finally, we conclude that u is a Pareto control related to u_0.
Conversely, let u be a Pareto control related to u_0. We have

$$\sup_{g\in G}\left(J(u,g) - J(u_0,g)\right) \leq \sup_{g\in G}\left(J(v,g) - J(u_0,g)\right) \text{ for every } v \in \mathcal{U}_{ad},$$

choose $v = u_0$ and take (1.22) into account to find

$$J(u,0) + 2\sup_{g\in G}\langle S(u-u_0), g\rangle_{G',G} \leq J(u_0,0) = \text{constant},$$

$J(u,0)$ is nonnegative, then $\sup_{g\in G}\langle S(u-u_0), g\rangle_{G',G} \leq$ constant. We deduce that $\sup_{g\in G}\langle S(u-u_0), g\rangle_{G',G} = 0$. Therefore, $\langle S(u-u_0), g\rangle_{G',G} \leq 0 \ \forall g \in G$ and hence $\langle S(u-u_0), g\rangle_{G',G} = 0$. Then $u \in K + u_0$ with

$$J(u,0) \leq J(v,0) \quad \forall v \in K + u_0$$

i.e., u is a Pareto control related to u_0. ■

Remark 1.9

1. *In proposition 1.3, we get the uniqueness of the Pareto control related to u_0, and we know that it's the unique minimizer of $J(v,0)$ on $K+u_0$. In the second part of theorem 1.5, we proved that the no-regret control related to u_0 if it exists minimizes $J(v,0)$ on $K+u_0$. Indeed, the no-regret control related to u_0 and Pareto control are the same as proved in Theorem 1.5.*

2. *All previous notions (i.e., no-regret control, low-regret control and Pareto control) could be generalized to nonlinear distributed systems with missing data. For more details see (Nakoulima, Omrane & Velin, 2002).*

1.3 Examples and applications to some elliptic distributed systems with missing data

1.3.1 A system with an internal control, boundary missing data and boundary observation

Let Ω be an open bounded set of $\mathbb{R}^n$ with smooth boundary Γ. Consider the following distributed system

$$\begin{cases} -\Delta y + y = f + v & \text{in } \Omega, \\ \frac{\partial y}{\partial \nu} = g & \text{on } \Gamma, \end{cases} \tag{1.3.1}$$

where $v \in \mathcal{U} = L^2(\Omega)$, $g \in G \subset F = L^2(\Gamma)$, $f \in L^2(\Omega)$ and G is a closed subset of F. The equation $(1.3.1)$ has a unique solution $y(v,g) \in H^{\frac{3}{2}}(\Omega)$ (Lions, 1972).

Associate to $(1.3.1)$ the following cost function

$$J(v,g) = |y(v,g) - z_d|^2_{L^2(\Gamma)} + N \|v\|^2_{L^2(\Omega)} \ , v \in L^2(\Omega), \tag{1.3.2}$$

where $z_d \in L^2(\Gamma)$ and $N > 0$.

We want to get some existence, uniqueness and characterization results of a no-regret control for $(1.3.1)$ $(1.3.2)$. Start by

$$J(v,g) - J(0,g) = J(v,0) - J(0,0) + 2\left(y(v,0) - y(0,0), y(0,g) - y(0,0)\right)_{L^2(\Gamma)} \tag{1.3.3}$$

Use Green formula (see Appendices, Theorem 3) to find

$$\int_\Gamma (y(0,g) - y(0,0)) \frac{\partial \zeta(v)}{\partial \nu} d\Gamma = \int_\Gamma \zeta(v)\, g d\Gamma$$

where $\zeta(v)$ solves

$$\begin{cases} -\Delta\zeta+\zeta=0 & \text{in } \Omega, \\ \frac{\partial\zeta}{\partial\nu}=y(v,0)-y(0,0) & \text{on } \Gamma. \end{cases} \tag{1.3.4}$$

Moreover, we have the following regularity result $y(0,g)-y(0,0)\in H^{\frac{3}{2}}(\Omega)$ because $\frac{\partial}{\partial\nu}(y(0,g)-y(0,0))\in L^2(\Gamma)$, also $\zeta(v)\in H^2(\Omega)$ then $\frac{\partial\zeta}{\partial\nu}=y(v,0)-y(0,0)\in H^{\frac{3}{2}}(\Gamma)$.

Low-regret control: Existence and characterization

The low-regret control associated to $(1.3.1)$ $(1.3.2)$ is a solution to the following optimal control problem:

$$\inf_{v\in\mathcal{U}_{ad}}\mathcal{J}^{\gamma}(v) \tag{1.3.5}$$

where

$$\mathcal{J}^{\gamma}(v)=J(v,0)-J(0,0)+\frac{1}{\gamma}\|\zeta(v)\|^2_{L^2(\Gamma)}. \tag{1.3.6}$$

Lemma 1.2 *The optimal control problem* $(1.3.5)$ $(1.3.6)$ *has unique solution* u_γ*.*

Proof. Since $\mathcal{J}^{\gamma}$ is continuous, coercive and strictly convex on $\mathcal{U}_{ad}$. Then, the problem $(1.3.5)$ $(1.3.6)$ has a unique solution. ■

The following proposition gives an approximated optimality for low-regret control.

Proposition 1.4 *The low-regret control* u_γ*, solution to* $(1.3.5)$ $(1.3.6)$ *is characterized by the following optimality system*

$$\begin{cases} -\Delta y_\gamma+y_\gamma=f+u_\gamma, & \\ -\Delta\zeta_\gamma+\zeta_\gamma=0, & \\ -\Delta\rho_\gamma+\rho_\gamma=0, & \\ -\Delta p_\gamma+p_\gamma=0 & \textit{in } \Omega, \\ \frac{\partial y_\gamma}{\partial\nu}=0,\ \frac{\partial\zeta_\gamma}{\partial\nu}=y_\gamma-y(0,0), & \\ \frac{\partial\rho_\gamma}{\partial\nu}=\frac{1}{\gamma}\zeta_\gamma,\ \frac{\partial p_\gamma}{\partial\nu}=y_\gamma+z_d-\rho_\gamma, & \textit{on } \Gamma, \\ p_\gamma+Nu_\gamma=0 \quad \textit{in } L^2(\Omega), & \end{cases} \tag{1.3.7}$$

with $u_\gamma\in L^2(\Omega)$ *and* $y_\gamma,\zeta_\gamma,\rho_\gamma,p_\gamma\in H^{\frac{3}{2}}(\Omega)$*.*

Proof. Let u_γ be a solution to $(1.3.5)$ $(1.3.6)$. Then, for every $v\in L^2(\Omega)$

$$\left(\mathcal{J}^{\gamma\prime}(u_\gamma),v-u_\gamma\right)_{L^2(\Omega)}=(y(u_\gamma,0)-z_d,y(v-u_\gamma,0)-y(0,0))_{L^2(\Gamma)}+N(u_\gamma,v-u_\gamma)_{L^2(\Omega)}$$

$$+\frac{1}{\gamma}(\zeta(u_\gamma),\zeta(v-u_\gamma))_{L^2(\Gamma)}=0. \tag{1.3.8}$$

Denote $y_\gamma = y(u_\gamma, 0)$, $\zeta_\gamma = \zeta(u_\gamma)$ solution to

$$\begin{cases} -\Delta\zeta_\gamma + \zeta_\gamma = 0 & \text{in } \Omega, \\ \frac{\partial \zeta_\gamma}{\partial \nu} = y_\gamma - y(0,0) & \text{on } \Gamma, \end{cases}$$

and ρ_γ solution to

$$\begin{cases} -\Delta\rho_\gamma + \rho_\gamma = 0 & \text{in } \Omega, \\ \frac{\partial \rho_\gamma}{\partial \nu} = \frac{1}{\gamma}\zeta_\gamma & \text{on } \Gamma. \end{cases}$$

Use Green formula to get

$$\begin{aligned} 0 &= \left(\zeta(v-u_\gamma), -\Delta\rho_\gamma + \rho_\gamma\right)_{L^2(\Omega)} - \left(-\Delta\zeta(v-u_\gamma) + \zeta(v-u_\gamma), \rho_\gamma\right)_{L^2(\Omega)} \\ &= \int_\Gamma \zeta(v-u_\gamma)\frac{\partial\rho_\gamma}{\partial\nu} d\Gamma - \int_\Gamma \rho_\gamma \frac{\partial\zeta(v-u_\gamma)}{\partial\nu} d\Gamma, \end{aligned}$$

in other words

$$\begin{aligned} \left(\rho_\gamma, \frac{\partial\zeta(v-u_\gamma)}{\partial\nu}\right)_{L^2(\Gamma)} &= \left(\zeta(v-u_\gamma), \frac{\partial\rho_\gamma}{\partial\nu}\right)_{L^2(\Gamma)} \\ &= \left(\zeta(v-u_\gamma), \frac{1}{\gamma}\zeta_\gamma\right)_{L^2(\Gamma)}. \end{aligned}$$

Substitute in $(1.3.8)$ to find

$$\left(y_\gamma - z_d + \rho_\gamma, y(v-u_\gamma, 0)\right)_{L^2(\Gamma)} + N(u_\gamma, v-u_\gamma)_{L^2(\Omega)} = 0 \text{ for every } v \in L^2(\Omega).$$

Now, introduce p_γ solution to

$$\begin{cases} -\Delta p_\gamma + p_\gamma = 0 & \text{in } \Omega, \\ \frac{\partial p_\gamma}{\partial \nu} = y_\gamma - z_d + \rho_\gamma & \text{on } \Gamma. \end{cases}$$

Then, optimality condition $(1.3.8)$ is equivalent to

$$\left(\frac{\partial p_\gamma}{\partial\nu}, y(v-u_\gamma, 0) - y(0,0)\right)_{L^2(\Gamma)} + N(u_\gamma, v-u_\gamma)_{L^2(\Omega)} = 0 \text{ for every } v \in L^2(\Omega),$$

but

$$\left(\frac{\partial p_\gamma}{\partial\nu}, y(v-u_\gamma, 0) - y(0,0)\right)_{L^2(\Gamma)} = (p_\gamma, v-u_\gamma)_{L^2(\Omega)}.$$

We deduce that

$$(p_\gamma + Nu_\gamma, v-u_\gamma)_{L^2(\Omega)} = 0 \ \forall\, v \in L^2(\Omega).$$

Hence,

$$p_\gamma + Nu_\gamma = 0 \quad \text{in } L^2(\Omega).$$

■

Optimality system for no-regret control

The following theorem gives a characterization of no-regret control for (1.3.1) (1.3.2).

Theorem 1.6 *The no-regret control u, solution to* (1.3.1) (1.3.2) *is characterized by the following optimality system*

$$\begin{cases} -\Delta y + y = f + u, \\ -\Delta \zeta + \zeta = 0, \\ -\Delta \rho + \rho = 0, \\ -\Delta p + p = 0 & \text{in } \Omega, \\ \frac{\partial y}{\partial \nu} = 0,\ \frac{\partial \zeta}{\partial \nu} = y - y(0,0), \\ \frac{\partial \rho}{\partial \nu} = \lambda,\ \frac{\partial p}{\partial \nu} = y + z_d - \rho & \text{on } \Gamma, \\ p + Nu = 0 \quad \text{in } L^2(\Omega), \end{cases} \tag{1.3.9}$$

with $u \in L^2(\Omega)$ and $y_\gamma, \zeta_\gamma, \rho_\gamma, p_\gamma \in H^{\frac{3}{2}}(\Omega)$.

Proof. Adapt the proof of Theorem 1.3 to this example to get the main results. ■

1.3.2 A distributed system with boundary control, boundary missing data and boundary observation

Let Ω be an open bounded set of $\mathbb{R}^n$ with smooth boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \varnothing$, Γ_1 and Γ_2 both smooth. Consider the following distributed system

$$\begin{cases} -\Delta y + y = 0 & \text{in } \Omega, \\ \frac{\partial y}{\partial \nu} = v & \text{on } \Gamma_1, \\ \frac{\partial y}{\partial \nu} = g & \text{on } \Gamma_2, \end{cases} \tag{1.3.10}$$

where $v \in \mathcal{U} = L^2(\Gamma_1)$, $g \in G \subset F = L^2(\Gamma_2)$, $f \in L^2(\Omega)$ and G is a closed subset of F. The equation (1.3.10) has a unique solution $y(v,g) \in H^{\frac{3}{2}}(\Omega)$ (Lions & Magenes, 1972).
Associate to (1.3.10) the following cost function

$$J(v,g) = |y(v,g) - z_d|^2_{L^2(\Gamma_1)} + N|v|^2_{L^2(\Gamma_1)},\ v \in L^2(\Gamma_1). \tag{1.3.11}$$

Start by no-regret control existence and uniqueness result.

Lemma 1.3 *There exist a unique no-regret control $u \in L^2(\Gamma_1)$ for* (1.3.10) (1.3.11).

As usually, it's easy to find that

$$J(v,g) - J(0,g) = J(v,0) - J(0,0) + 2(\zeta(v), g)_{L^2(\Gamma_1)}.$$

Where $\zeta(v)$ solves

$$\begin{cases} -\Delta\zeta(v) + \zeta(v) = 0 & \text{in } \Omega, \\ \frac{\partial\zeta(v)}{\partial\nu} = y(v,0) & \text{on } \Gamma_1, \\ \frac{\partial\zeta(v)}{\partial\nu} = 0 & \text{on } \Gamma_2. \end{cases}$$

Low-regret control: Existence and characterization

Consequently, the low-regret control for (1.3.10) (1.3.11) is a solution of

$$\inf_{v\in\mathcal{U}_{ad}} \mathcal{J}^\gamma(v), \tag{1.3.12}$$

where

$$\mathcal{J}^\gamma(v) = J(v,0) - J(0,0) + \frac{1}{\gamma}|\zeta(v)|^2_{L^2(\Gamma_1)}. \tag{1.3.13}$$

It's easy to prove that:

Lemma 1.4 *There exist a unique low-regret control $u_\gamma \in L^2(\Gamma_1)$ solution to (1.3.12) (1.3.13) converges weakly to the unique no-regret control solution to (1.3.10) (1.3.11).*

The next proposition gives an approximate optimality system characterizing the low-regret control u_γ.

Proposition 1.5 *The low-regret control u_γ, solution to (1.3.12) (1.3.13) is characterized by the following optimality system:*

$$\begin{cases} -\Delta y_\gamma + y_\gamma = 0, \\ -\Delta\zeta_\gamma + \zeta_\gamma = 0, \\ -\Delta\rho_\gamma + \rho_\gamma = 0, \\ -\Delta p_\gamma + p_\gamma = 0, & \text{in } \Omega, \\ \frac{\partial y_\gamma}{\partial\nu} = u_\gamma,\ \frac{\partial\zeta_\gamma}{\partial\nu} = y_\gamma, \\ \frac{\partial\rho_\gamma}{\partial\nu} = 0,\ \frac{\partial p_\gamma}{\partial\nu} = y_\gamma + z_d - \rho_\gamma, & \text{on } \Gamma_1, \\ \frac{\partial y_\gamma}{\partial\nu} = 0,\ \frac{\partial\zeta_\gamma}{\partial\nu} = 0, \\ \frac{\partial\rho_\gamma}{\partial\nu} = \frac{1}{\gamma}\zeta_\gamma,\ \frac{\partial p_\gamma}{\partial\nu} = 0, & \text{on } \Gamma_2, \\ p_\gamma + Nu_\gamma = 0 \quad \text{in } L^2(\Gamma_1). \end{cases}$$

Where $u_\gamma \in L^2(\Gamma_0)$, $y_\gamma \in H^{\frac{3}{2}}(\Omega)$, $\zeta_\gamma \in H^{\frac{5}{2}}(\Omega)$, $\rho_\gamma \in H^{\frac{7}{2}}(\Omega)$ and $p_\gamma \in H^{\frac{1}{2}}(\Omega)$.

Proof. A first order necessary optimality condition for (1.3.12) (1.3.13) gives for

$$(\mathcal{J}'^\gamma(u_\gamma), v - u_\gamma)_{L^2(\Omega)} = (y(u_\gamma,0) - z_d, y(v-u_\gamma,0))_{L^2(\Gamma_1)} + N(u_\gamma, v-u_\gamma)_{L^2(\Gamma_1)}$$

$$+\frac{1}{\gamma}\left(\zeta\left(u_\gamma\right),\zeta\left(v-u_\gamma\right)\right)_{L^2(\Gamma_1)}=0. \tag{1.3.14}$$

Let ρ_γ be a solution of

$$\begin{cases} -\Delta\rho_\gamma+\rho_\gamma=0 & \text{in } \Omega, \\ \frac{\partial\rho_\gamma}{\partial\nu}=0 & \text{on } \Gamma_1, \\ \frac{\partial\rho_\gamma}{\partial\nu}=\frac{1}{\gamma}\zeta\left(u_\gamma\right) & \text{on } \Gamma_2. \end{cases} \tag{1.3.15}$$

Thanks to Green formula

$$\begin{aligned} 0 &= \int_\Omega \zeta\left(v-u_\gamma\right)\left(-\Delta\rho_\gamma+\rho_\gamma\right)dx-\int_\Omega\left(-\Delta\zeta\left(v-u_\gamma\right)+\zeta\left(v-u_\gamma\right)\right)\rho_\gamma dx \\ &= \int_\Gamma \zeta\left(v-u_\gamma\right)\frac{\partial\rho_\gamma}{\partial\nu}d\Gamma-\int_\Gamma\rho_\gamma\frac{\partial\zeta\left(v-u_\gamma\right)}{\partial\nu}d\Gamma. \end{aligned} \tag{1.3.16}$$

Remember that $\Gamma=\Gamma_1\cup\Gamma_2$, and ρ_γ verifies $(1.3.15)$, then $(1.3.16)$ becomes

$$\left(\frac{1}{\gamma}\zeta\left(u_\gamma\right),\zeta\left(v-u_\gamma\right)\right)_{L^2(\Gamma_1)}=\left(\rho_\gamma,y\left(0,v-u_\gamma\right)\right)_{L^2(\Gamma_1)}.$$

And $(1.3.14)$ writes for every $v\in L^2\left(\Gamma_1\right)$

$$\left(y_\gamma+z_d-\rho_\gamma,y\left(v-u_\gamma,0\right)\right)_{L^2(\Gamma_1)}+N\left(u_\gamma,v-u_\gamma\right)_{L^2(\Gamma_1)}=0. \tag{1.3.17}$$

Introduce p_γ solution of

$$\begin{cases} -\Delta p_\gamma+p_\gamma=0 & \text{in } \Omega, \\ \frac{\partial p_\gamma}{\partial\nu}=y_\gamma+z_d-\rho_\gamma & \text{on } \Gamma_1, \\ \frac{\partial p_\gamma}{\partial\nu}=0 & \text{on } \Gamma_2. \end{cases} \tag{3.1.18}$$

Then, $(1.3.17)$ is equivalent to

$$\left(\frac{\partial p_\gamma}{\partial\nu},y\left(v-u_\gamma,0\right)\right)_{L^2(\Gamma_1)}+N\left(u_\gamma,v-u_\gamma\right)_{L^2(\Gamma_1)}=0 \qquad \forall v\in L^2\left(\Gamma_1\right).$$

Again by Green formula, we get

$$\begin{aligned} 0 &= \int_\Omega p_\gamma\left(-\Delta y\left(v-u_\gamma,0\right)+y\left(v-u_\gamma,0\right)\right)dx-\int_\Omega\left(-\Delta p_\gamma+p_\gamma\right)y\left(v-u_\gamma,0\right)dx \\ &= \int_\Gamma p_\gamma\frac{\partial y\left(v-u_\gamma,0\right)}{\partial\nu}d\Gamma-\int_\Gamma y\left(v-u_\gamma,0\right)\frac{\partial p_\gamma}{\partial\nu}d\Gamma, \end{aligned}$$

we conclude that

$$\left(p_\gamma,\frac{\partial y\left(v-u_\gamma,0\right)}{\partial\nu}\right)_{L^2(\Gamma_1)}=\left(\frac{\partial p_\gamma}{\partial\nu},y\left(0,v-u_\gamma\right)\right)_{L^2(\Gamma_1)}, \tag{1.3.19}$$

but p_γ solves $(1.3.18)$, then $(1.3.19)$ becomes

$$(p_\gamma, v - u_\gamma)_{L^2(\Gamma_1)} = \left(\frac{\partial p_\gamma}{\partial \nu}, y\,(0, v - u_\gamma)\right)_{L^2(\Gamma_1)}.$$

Finally, optimality condition $(1.3.14)$ is equivalent to

$$p_\gamma + N u_\gamma = 0 \quad \text{in } L^2\,(\Gamma_1)\,.$$

■

Optimality system for no-regret control

To get a no-regret control characterization we pass to limit when $\gamma \to 0$, and we announce the following theorem.

Proposition 1.6 *The no-regret control u, solution to $(1.3.10)$ $(1.3.11)$ is characterized by the following optimality system:*

$$\begin{cases} -\Delta y + y = 0, & \\ -\Delta \zeta + \zeta = 0, & \\ -\Delta \rho + \rho = 0, & \\ -\Delta p + p = 0, & \text{in } \Omega, \\ \frac{\partial y_\gamma}{\partial \nu} = u,\ \frac{\partial \zeta_\gamma}{\partial \nu} = y, & \\ \frac{\partial \rho}{\partial \nu} = 0,\ \frac{\partial p}{\partial \nu} = y + z_d - \rho, & \text{on } \Gamma_1, \\ \frac{\partial y_\gamma}{\partial \nu} = 0,\ \frac{\partial \zeta_\gamma}{\partial \nu} = 0, & \\ \frac{\partial \rho_\gamma}{\partial \nu} = \lambda,\ \frac{\partial p}{\partial \nu} = 0, & \text{on } \Gamma_2, \\ p + Nu = 0 \quad \text{in } L^2\,(\Gamma_1)\,. & \end{cases}$$

Where $u \in L^2\,(\Gamma_0)$, $y \in H^{\frac{3}{2}}\,(\Omega)\,,$ $\zeta \in H^{\frac{5}{2}}\,(\Omega)$, $\lambda \in \widehat{G}$ completion of G in $H^{-2}\,(\Gamma)\,,$ $\rho_\gamma \in H^{\frac{7}{2}}\,(\Omega)$ and $p_\gamma \in H^{\frac{1}{2}}\,(\Omega)$.

Proof. See (Lions, 1987) or (Nakoulima, Omrane& Velin, 2003). ■

1.4 Averaged control

In some distributed systems, often parameters are not fully known, in this situation to control such kind of systems we look for robust control strategies, independent of the unknown parameters. To control these systems, Zuazua (2014) introduced the notion of “averaged control”, its the main idea is to control the average of the state with respect to the unknown parameter instead of controlling the state itself.

1.4.1 Averaged controllability of distributed systems

To clarify the idea of averaged controllability, let's take the following example:
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary Γ, $Q = \Omega \times (0,T)$ where $T > 0$, $\Sigma = \Gamma \times (0,T)$ and ω an open non-empty subset of Ω. Consider the following controlled heat equation depending on a parameter:

$$\begin{cases} \frac{\partial y}{\partial t} - \operatorname{div}\left(a\left(x,\sigma\right)\nabla y\right) = v\chi_\omega & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y\left(x,0\right) = y_0\left(x\right) & \text{in } \Omega. \end{cases} \tag{1.4.1}$$

The coefficient of diffusion $a\left(x,\sigma\right)$ supposed to be measurable in x, and depend continuously on a the uncertainty parameter $\sigma \in (0,1)$. We assume that $y_0 \in L^2\left(\Omega\right)$ and $v = v\left(x,t\right) \in L^2\left(Q\right)$, then $(1.4.1)$ has a unique solution (Lions, 1971)

$$y = y\left(x,t;\sigma\right) \in C\left(\left[0,T\right], L^2\left(\Omega\right)\right) \cap L^2\left(0,T; H_0^1\Omega\right), \text{ for every } \sigma \in \left(0,1\right).$$

Here, for simplicity, we consider the following problem of averaged null controllability :
Find a control $u \in L^2\left(Q\right)$ s.t. y solution to $(1.4.1)$ verifies

$$\int_0^1 y\left(x,T;\sigma\right) d\sigma = 0. \tag{1.4.2}$$

One can show that the averaged null controllability $(1.4.1)\,(1.4.2)$ is equivalent to an averaged observability inequality for the following backward equation (Zuazua, 2014)

$$\begin{cases} -\frac{\partial \varphi}{\partial t} + \operatorname{div}\left(a\left(x,\sigma\right)\nabla \varphi\right) = 0 & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi\left(x,T\right) = \varphi_0\left(x\right) & \text{in } \Omega, \end{cases} \tag{1.4.3}$$

the required observability inequality has the form

$$\left\| \int_0^1 \varphi\left(x,0;\sigma\right) d\sigma \right\|_{L^2(\Omega)}^2 \leq C \int_0^T \int_\omega \left| \int_0^1 \varphi\left(x,0;\sigma\right) d\sigma \right|^2 dxdt, \tag{1.4.4}$$

where C is a constant independent of φ. To get an inequality of the form $(1.4.4)$ we need the so-called Carleman inequalities (Fursikov and Imanuvilov, 1996), it's a very challenging issue.

1.4.2 Optimal averaged control for distributed systems depending upon a unknown parameter

In this subsection, we shall focus on optimal averaged control of may types of distributed systems depending upon an uncertainty parameter. In every case, we shall give an optimality system characterizing the averaged optimal control as in classical distributed systems, for more examples and details we refer to (Hafdallah & Ayadi, 2016).

Optimal Averaged control for elliptic distributed systems depending upon a unknown parameter

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary Γ, and ω an open non-empty subset of Ω. Consider the following controlled elliptic equation depending on a parameter:

$$\begin{cases} -\operatorname{div}\left(a\left(x,\sigma\right)\nabla y\right) = v\chi_\omega & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma, \end{cases} \tag{1.4.5}$$

the coefficient of diffusion $a\left(x,\sigma\right)$ is measurable in x and depend on σ in a measurable manner, $v \in L^2\left(\Omega\right)$. For every $\sigma \in (0,1)$ the equation $(1.4.5)$ has a unique solution $y = y\left(x,\sigma\right)$ in $H_0^1\left(\Omega\right)$. We seek to control the average of the state with respect to σ

$$z\left(x\right) = \int_0^1 y\left(x,\sigma\right) d\sigma.$$

Associate to $(1.4.5)$ the following quadratic cost function

$$J\left(v\right) = \left\|z - z_d\right\|^2_{L^2(\Omega)} + N\left\|v\right\|^2_{L^2(\Omega)}, \; v \in L^2\left(\Omega\right), \tag{1.4.6}$$

where $z_d \in L^2\left(\Omega\right)$ and $N > 0$. Then, we want to solve

$$\inf_{v \in L^2(\Omega)} J\left(v\right) \; \text{s.t. } (1.4.5). \tag{1.4.7}$$

Theorem 1.7 *The unique averaged optimal control u solution to for $(1.4.5)-(1.4.7)$ is characterized by*

$$\begin{cases} -\operatorname{div}\left(a\left(x,\sigma\right)\nabla y\left(u\right)\right) = \int_0^1 \varphi\left(u;\sigma\right) d\sigma \chi_\omega, & \\ -\operatorname{div}\left(a\left(x,\sigma\right)\nabla \varphi\right) = \int_0^1 y\left(u;\sigma\right) d\sigma - z_d, & \text{in } \Omega, \\ y\left(u\right) = 0, \; \varphi = 0 & \text{on } \Gamma, \end{cases}$$

with the variational inequality

$$\int_0^1 \varphi\left(u;\sigma\right) d\sigma + Nu = 0 \text{ in } \Omega.$$

Proof. The functional $J : \mathcal{U}_{ad} \to \mathbb{R}$ is a lower semi-continuous function, strictly convex, and coercive. Hence there is a unique admissible control u solution to $(1.4.5) - (1.4.7)$.
A First order Euler condition for gives

$$\int_{\Omega} (z(u) - z_d)(z(v) - z(u))\, dx + N \int_{\Omega} u(v-u)\, dx = 0 \;\forall v \in L^2(\Omega) \tag{1.4.8}$$

where $z(v) = \int_0^1 y(\nu;\sigma)\, d\sigma$, such that $y(\nu;\sigma)$ is the unique solution of

$$\begin{cases} -\operatorname{div}(a(x,\sigma)\nabla y) = v\chi_\omega & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma. \end{cases}$$

Let φ be the $\sigma-$dependent adjoint state given by

$$\begin{cases} -\operatorname{div}(a(x,\sigma)\nabla \varphi) = z(u) - z_d & \text{in } \Omega, \\ \varphi = 0 & \text{on } \Gamma. \end{cases}$$

Now, let's rewrite first order Euler condition $(1.4.8)$ by Green formula

$$\begin{aligned} \int_{\Omega} (z(u) - z_d)(z(v) - z(u))\, dx &= \int_{\Omega}\int_0^1 (z(u) - z_d)(y(\nu;\sigma) - y(u;\sigma)) d\sigma dx \\ &= \int_{\Omega}\int_0^1 \operatorname{div}(a(x,\sigma)\nabla\varphi)(y(\nu;\sigma) - y(u;\sigma)) d\sigma dx \\ &= \int_{\Omega}\int_0^1 \varphi \operatorname{div}(a(x,\sigma)\nabla(y(\nu;\sigma) - y(u;\sigma))\, d\sigma dx \\ &= \int_{\Omega}\int_0^1 \varphi(v-u)\chi_\omega d\sigma dx. \end{aligned}$$

Now, $(1.4.8)$ is equivalent to

$$\int_{\Omega}\int_0^1 (\varphi(v-u)\chi_\omega + Nu)(v-u)\, d\sigma dx = 0 \;\; \forall v \in L^2(\Omega).$$

■

Optimal Averaged control for parabolic distributed systems depending upon a unknown parameter

Consider the following abstract second order parabolic equation

$$\begin{cases} \frac{dy}{dt} + A(x,\sigma)\, y = f + \mathcal{B}(\sigma)\, v & \text{in } Q, \\ y(x,0) = y_0(x) & \text{in } \Omega. \end{cases} \tag{1.4.9}$$

Let $V \subset H$ be Hilbert spaces, $f \in L^2(0,T;V')$, $v \in \mathcal{U}_{ad} \subset L^2(0,T;V)$, where $\mathcal{U}_{ad}$ is a non-empty closed convex set of admissible controls, $\mathcal{B}(\sigma) \in \mathcal{L}(\mathcal{U}_{ad}, L^2(0,T;V'))$ is the control operator

supposed also depending on σ and $y_0 \in V$ is initial state. Then, for every $\sigma \in (0,1)$ the equation $(1.4.9)$ has a unique solution in $y \in L^2(0,T;V)$ (Lions, 1971).
Let $z(x,t) = \int_0^1 y(x,t;\sigma)\, d\sigma \in L^2(0,T;V)$ be the averaged state with respect to σ and z_d a given desired averaged state in $L^2(0,T;V)$. We are interested to following quadratic optimal control problem

$$\inf_{v\in\mathcal{U}_{ad}} J(v) \quad \text{with } J(v) = \|z - z_d\|^2_{L^2(0,T;V)} + N\|v\|^2_{L^2(0,T;V)}. \tag{1.4.10}$$

where $N > 0$. Then,

Theorem 1.8 *The averaged optimal control* u *for* $(1.4.9)$ $(1.4.10)$ *is unique and it's characterized by*

$$\begin{cases} \frac{dy(u)}{dt} + A(x,\nu)\, y = f + \mathcal{B}(\sigma)\, u, \\ -\frac{d\varphi}{dt} + A^*(x,\nu)\,\varphi = \int_0^1 y(x,t;u;\sigma)\, d\sigma - z_d & \text{in } Q, \\ y(u)(x,0) = y_0(x), \varphi(x,T) = 0 & \text{in } \Omega, \end{cases} \tag{1.4.11}$$

with the variational inequality

$$\int_0^T \int_0^1 (\mathcal{B}^*(\sigma)\,\varphi + Nu, v - u)_V\, d\sigma dt \geq 0\,, \forall v \in \mathcal{U}_{ad}. \tag{1.4.12}$$

Proof. Existence and uniqueness task follows by lower semi-continuity, strict convexity, and coercitivity of objective function J.
A first order Euler condition for $(1.4.10)$ gives

$$\int_0^T (z(u) - z_d, z(v) - z(u))_V\, dt + N\int_0^T (u, v-u)_V\, dt \geq 0 \;\forall v \in \mathcal{U}_{ad}, \tag{1.4.13}$$

where $z(v) = \int_0^1 y(v;\sigma)\, d\sigma$, such that $y(v;\sigma)$ is the unique solution of

$$\begin{cases} \frac{dy}{dt} + A(x,\sigma)\, y = \mathcal{B}(\sigma)\, v & \text{in } Q, \\ y(x,0) = 0 & \text{in } \Omega. \end{cases}$$

Let φ be the $\sigma-$dependent adjoint state given by

$$\begin{cases} -\frac{d\varphi}{dt} + A^*(x,\sigma)\,\varphi = z(u) - z_d & \text{in } Q, \\ \varphi(x,T) = 0 & \text{in } \Omega. \end{cases}$$

where $\varphi \in L^2(0,T;V)$ (Lions, 1971).

Now, let's rewrite first order Euler condition $(1.4.13)$ as

$$\begin{aligned}\int_0^T (z(u) - z_d, z(v) - z(u))_V\, dt &= \int_0^T\int_0^1 (z(u) - z_d, y(\nu;\sigma) - y(u;\sigma))_V d\sigma dx \\ &= \int_0^T\int_0^1 \left(-\frac{d\varphi}{dt} + A^*(x,\sigma)\varphi, y(\nu;\sigma) - y(u;\sigma)\right)_V d\sigma dx \\ &= \int_0^T\int_0^1 \left(\varphi, \left(\frac{d}{dt} + A(x,\sigma)\right)(y(\nu;\sigma) - y(u;\sigma))\right)_V d\sigma dx \\ &= \int_0^T\int_0^1 (\varphi, \mathcal{B}(\sigma)(v-u))_V\, d\sigma dx,\end{aligned}$$

and $(1.4.13)$ will be written as

$$\int_0^T\int_0^1 (\mathcal{B}^*(\sigma)\varphi + Nu)(v-u)\, d\sigma dt \geq 0, \forall u \in \mathcal{U}_{ad}.$$

■

Optimal averaged control for hyperbolic distributed systems depending upon a unknown parameter

Consider the following abstract hyperbolic problem

$$\begin{cases} \frac{d^2y}{dt^2} + A(x,\sigma)y = f + \mathcal{B}(\sigma)v & \text{in } Q, \\ y(x,0) = y_0(x),\ \frac{dy}{dt}(x,0) = y_1(x) & \text{in } \Omega. \end{cases} \tag{1.4.14}$$

Let V, H be Hilbert spaces with V is separable and dense in H, $f \in L^2(0,T;H)$, $y_0 \in V$, $y_1 \in H$, $\mathcal{B}(\sigma) \in \mathcal{L}(\mathcal{U}_{ad}, L^2(0,T;H))$ and $\mathcal{U}_{ad} \subset L^2(0,T;H)$. Then, for every $\sigma \in (0,1)$ the equation $(1.4.14)$ has a unique solution in $L^2(0,T;V)$ (Lions & Magenes, 1972). We are interested to the optimal control problem $(1.4.14)$ $(1.4.10)$.

Theorem 1.9 *The averaged optimal control u solution to for $(1.4.14)$ $(1.4.10)$ is unique and it's characterized by*

$$\begin{cases} \frac{d^2y(u)}{dt^2} + A(x,\sigma)y(u) = f + \mathcal{B}(\sigma)u, & \\ \frac{d^2\varphi}{dt^2} + A^*(x,\sigma)\varphi = \int_0^1 y(x,t;u;\sigma)\,d\sigma - z_d & \text{in } Q, \\ y(u)(x,0) = y_0(x), \frac{dy(u)}{dt}(x,0) = y_1(x), & \\ \varphi(T) = 0, \frac{d\varphi}{dt}(T) = 0 & \text{in } \Omega, \end{cases} \tag{1.4.15}$$

with the variational inequality

$$\int_0^T\int_0^1 (\mathcal{B}^*(\sigma)\varphi + Nu, v-u)_H\, d\nu dt \geq 0, \forall v \in \mathcal{U}_{ad}. \tag{1.4.16}$$

Proof. An optimality condition is written as follow :

$$\int_0^T (z(u) - z_d, z(v) - z(u))_V \, dt + N \int_0^T (u, v-u)_V \, dt \geq 0 \; \forall v \in \mathcal{U}_{ad}, \tag{1.4.17}$$

where $z(v) = \int_0^1 y(\nu;\sigma)\, d\sigma$ which is a solution of the system :

$$\begin{cases} \frac{d^2y}{dt^2} + A(x,\sigma)\, y = \mathcal{B}(\sigma)\, v & \text{in } Q, \\ y(x,0) = 0, \;\; \frac{dy}{dt}(x,0) = 0 & \text{in } \Omega. \end{cases}$$

Introduce the $\sigma-$dependent adjoint state given by

$$\begin{cases} \frac{d^2\varphi}{dt^2} + A^*(x,\nu)\, \varphi = z(u) - z_d & \text{in } Q, \\ \varphi(x,T) = 0, \;\; \frac{d\varphi}{dt}(x,T) = 0 & \text{in } \Omega. \end{cases}$$

Now, let's rewrite the first order Euler $(1.4.17)$ condition as

$$\begin{aligned} \int_0^T (z(u) - z_d, z(v) - z(u))_V \, dt &= \int_0^T \int_0^1 (z(u) - z_d, y(\nu;\sigma) - y(u;\sigma))_V \, d\sigma dt \\ &= \int_0^T \int_0^1 \left(\frac{d^2\varphi}{dt^2} + A^*(x,\sigma)\, \varphi, y(\nu;\sigma) - y(u;\sigma) \right)_V d\sigma dt \\ &= \int_0^T \int_0^1 (\varphi, \mathcal{B}(\sigma)(v-u))_H \, d\sigma dt, \end{aligned}$$

so, we get $(1.4.16)$. ■

Chapter 2

Optimal control of some distributed systems of a various kinds and with missing data

In this chapter, we present various of optimal control problems with missing data, with different kinds (second order parabolic equation, fractional diffusion equation, and age structured population dynamics equation) and difficulties. For every problem, we define the no-regret control and low-regret control, then, we characterize them via optimality systems.

2.1 Optimal control of an abstract parabolic equation with missing data

In this section, we study an optimal control problem associated to a second order abstract parabolic equation with partially given initial condition.
Let $\mathcal{Y}$ be Hilbert space of states, $\mathcal{U}$ is a Hilbert space of controls, $\mathcal{U}_{ad}$ is a non empty closed convex subset of admissible controls, F Hilbert space of uncertainties verifies $\mathcal{Y} \subset F \subset \mathcal{Y}'$, G is a subspace of F.
Consider the following abstract parabolic equation :

$$\begin{cases} \frac{\partial y}{\partial t} + \mathcal{A}y = f + \mathcal{B}v, \\ y\left(0\right) = y_0 + g, \end{cases} \tag{2.1.1}$$

where $\mathcal{A} \in \mathcal{L}\left(\mathcal{Y}, \mathcal{Y}'\right)$ is a second order elliptic operator, $f \in L^2\left(0, T; \mathcal{Y}'\right)$, $\mathcal{B} \in \mathcal{L}\left(\mathcal{U}, L^2\left(0, T; \mathcal{Y}'\right)\right)$, y_0 is given in F and $g \in G$.

For every v and g, $(2.1.1)$ has a unique solution $y = y(v, g) \in L^2(0, T; \mathcal{Y})$.
Associate to $(2.1.1)$ the following quadratic cost function :

$$J(v, g) = \|Cy(v, g) - z_d\|^2_{L^2(0,T;\mathcal{Z})} + N\|v\|^2_{L^2(0,T;\mathcal{U})}, \ v \in \mathcal{U}_{ad}, \tag{2.1.2}$$

where $\mathcal{Z}$ is a Hilbert space of observations, $C \in \mathcal{L}(L^2(0, T; \mathcal{Y}), \mathcal{Z})$, z_d is a fixed observation in $\mathcal{Z}$ and $N > 0$.
We are interested with the following optimal control problem

$$\inf_{v \in \mathcal{U}_{ad}} J(v, g) \ \text{ for every } g \in G \text{ s.t. } (2.1.1). \tag{2.1.3}$$

2.1.1 Low-regret control for abstract parabolic equation with missing data

Here, we seek to prove existence and uniqueness for the low-regret control for $(2.1.1)$ and to characterize that control by an approached optimality system. Proceed as in Lemma 1.1 to get

$$J(v, g) - J(0, g) = J(v, 0) - J(0, 0) + 2\int_0^T (y(v, 0) - y(0, 0), C^*C(y(0, g) - y(0, 0)))_{\mathcal{Z}}\, dt. \tag{2.1.4}$$

Introduce $\zeta = \zeta(v)$ solution to the backward adjoint equation

$$\begin{cases} -\frac{\partial \zeta}{\partial t} + \mathcal{A}^*\zeta = C^*C(y(v, 0) - y(0, 0)), \\ \zeta(T) = 0. \end{cases} \tag{2.1.5}$$

Then $(2.1.4)$ becomes

$$J(v, g) - J(0, g) = J(v, 0) - J(0, 0) + 2\langle \zeta(0), g\rangle_{G', G}. \tag{2.1.6}$$

The low-regret control associated to $(2.1.1)$ is a solution of the following standard optimal control problem

$$\inf_{v \in \mathcal{U}_{ad}} \mathcal{J}^\gamma(v) \tag{2.1.7}$$

with

$$\mathcal{J}^\gamma(v) = J(v, 0) - J(0, 0) + \frac{1}{\gamma}\|\zeta(0)\|^2_G, \ \gamma > 0 \tag{2.1.8}$$

where G' is identified to G. Now, we announce the following

Proposition 2.1 *The problem $(2.1.1)$ $(2.1.7)$ $(2.1.8)$ has a unique solution u_γ. Moreover, u_γ converges weakly to the no-regret control u.*

Proof. The proof is identical to the proofs in proposition 1.1 and theorem 1.3. ■

2.1.2 Approximated optimality system for a parabolic equation with missing data

Here, we shall give an approached optimality system characterizing the low-regret control.

Proposition 2.2 *The low-regret control u_γ, solution to* (2.1.7) (2.1.8) *is characterized by the following optimality system*

$$\left\{\begin{array}{c} \frac{\partial y_\gamma}{\partial t}+\mathcal{A}y_\gamma=f+\mathcal{B}u_\gamma, \\ -\frac{\partial \zeta_\gamma}{\partial t}+\mathcal{A}^*\zeta_\gamma=C^*C\left(y_\gamma-y\left(0,0\right)\right), \\ \frac{\partial \rho_\gamma}{\partial t}+\mathcal{A}\rho_\gamma=0, \\ -\frac{\partial p_\gamma}{\partial t}+\mathcal{A}^*p_\gamma=C^*\left(Cy_\gamma-z_d\right)+C^*C\rho_\gamma, \\ y_\gamma\left(0\right)=y_0,\ \zeta_\gamma\left(T\right)=0, \\ \rho_\gamma\left(0\right)=\frac{1}{\gamma}\zeta_\gamma\left(0\right),\ p_\gamma\left(T\right)=0 \\ \left(\mathcal{B}^*p_\gamma+Nu_\gamma,v-u_\gamma\right)_{\mathcal{U}}\geq 0\quad\forall v\in\mathcal{U}_{ad}. \end{array}\right. \tag{2.1.9}$$

Proof. Let u_γ be a solution of (2.1.7) (2.1.8). A first order necessary condition gives for every $v\in\mathcal{U}_{ad}$

$$\begin{array}{l} \int_0^T\left(Cy\left(u_\gamma,0\right)-z_d,C\left(y\left(v-u_\gamma,0\right)-y\left(0,0\right)\right)\right)_{\mathcal{Z}}dt+N\int_0^T\left(u_\gamma,v-u_\gamma\right)_{\mathcal{U}}dt \\ +\frac{1}{\gamma}\int_0^T\left(\zeta\left(u_\gamma\right)\left(0\right),S\left(v-u_\gamma\right)\right)_G dt\geq 0. \end{array} \tag{2.1.10}$$

Let $y_\gamma=y\left(u_\gamma,0\right)$, $\zeta_\gamma=\zeta\left(u_\gamma\right)$, and $\rho_\gamma\in\mathcal{Y}$ be solution of

$$\left\{\begin{array}{l} \frac{\partial \rho_\gamma}{\partial t}+\mathcal{A}\rho_\gamma=0, \\ \rho_\gamma\left(0\right)=\frac{1}{\gamma}\zeta_\gamma\left(0\right). \end{array}\right.$$

Introduce an adjoint state by

$$\left\{\begin{array}{c} -\frac{\partial p_\gamma}{\partial t}+\mathcal{A}^*p_\gamma=C^*\left(Cy_\gamma-z_d\right)+C^*C\rho_\gamma, \\ p_\gamma\left(T\right)=0. \end{array}\right. \tag{2.1.11}$$

In other way, from (2.1.11) we get

$$\begin{array}{rcl} \left(Cy\left(u_\gamma,0\right)-z_d,C\left(y\left(v-u_\gamma,0\right)-y\left(0,0\right)\right)\right)_{\mathcal{Z}} & = & \left(C^*\left(Cy\left(u_\gamma,0\right)-z_d\right),y\left(v-u_\gamma,0\right)-y\left(0,0\right)\right)_{\mathcal{Y}} \\ & = & \left(-\dfrac{\partial p_\gamma}{\partial t}+\mathcal{A}^*p_\gamma-C^*C\rho_\gamma,y\left(v-u_\gamma,0\right)-y\left(0,0\right)\right)_{\mathcal{Y}}. \end{array} \tag{2.1.12}$$

Use (2.1.1) and (2.1.5) to rewrite (2.1.12) as

$$\begin{array}{rcl} \left(Cy\left(u_\gamma,0\right)-z_d,C\left(y\left(v-u_\gamma,0\right)-y\left(0,0\right)\right)\right)_{\mathcal{Z}} & = & \left(-\dfrac{\partial p_\gamma}{\partial t},y\left(v-u_\gamma,0\right)-y\left(0,0\right)\right)_{\mathcal{Y}} \\ & & +\left(p_\gamma,\mathcal{B}\left(v-u_\gamma\right)-\dfrac{\partial y}{\partial t}\left(v-u_\gamma,0\right)+\dfrac{\partial y}{\partial t}\left(0,0\right)\right)_{\mathcal{Y}} \\ & & -\left(\rho_\gamma,-\dfrac{\partial \zeta_\gamma}{\partial t}+\mathcal{A}^*\zeta_\gamma\right)_{\mathcal{Y}}, \end{array}$$

then

$$\left(Cy\left(u_{\gamma},0\right)-z_{d},C\left(y\left(v-u_{\gamma},0\right)-y\left(0,0\right)\right)\right)_{\mathcal{Z}}=\left(\mathcal{B}^{*}p_{\gamma},v-u_{\gamma}\right)_{\mathcal{U}}-\left(\frac{1}{\gamma}\zeta_{\gamma}\left(u_{\gamma}\right)\left(0\right),\zeta_{\gamma}\left(v-u_{\gamma}\right)\left(0\right)\right)_{G}. \tag{2.1.13}$$

Substitute $(2.1.13)$ into $(2.1.10)$, to obtain

$$\int_{0}^{T}\left(\mathcal{B}^{*}p_{\gamma}+Nu_{\gamma},v-u_{\gamma}\right)_{\mathcal{U}}dt\geq 0 \text{ for every } v\in\mathcal{U}_{ad}.$$

■

2.1.3 Abstract optimality system (Optimality system for the no-regret control)

In this subsection, we shall give an optimality system characterizing no-regret control for the abstract parabolic equation $(2.1.1)$. To do this, we introduce $\rho\in\mathcal{L}^{2}\left(0,T;\mathcal{Y}\right)$ given by :

$$\begin{cases}\frac{\partial\rho}{\partial t}+\mathcal{A}\rho=0,\\ \rho\left(0\right)=g,\ g\in G,\end{cases}$$

and $\sigma\in\mathcal{L}^{2}\left(0,T;\mathcal{Y}\right)$ solution to

$$\begin{cases}-\frac{\partial\sigma}{\partial t}+\mathcal{A}^{*}\sigma=C^{*}C\rho,\\ \sigma\left(T\right)=0.\end{cases}$$

Define a continuous operator, $\mathcal{R}:F\rightarrow\mathcal{U}$ by $\mathcal{R}g=\mathcal{B}^{*}\sigma$, and make the following hypothesis :

$$\left\|\mathcal{R}g\right\|_{\mathcal{U}}\geq c\left\|g\right\|_{F},\ c>0,\ \forall g\in G. \tag{2.1.14}$$

Theorem 2.1 *Suppose that* $(2.1.14)$ *holds, then, the no-regret control* u*, solution to* $(2.1.1)$ $(2.1.2)$ *is characterized by the following optimality system*

$$\begin{cases}\frac{\partial y}{\partial t}+\mathcal{A}y=f+\mathcal{B}u,\\ -\frac{\partial\zeta}{\partial t}+\mathcal{A}^{*}\zeta=C^{*}C\left(y-y\left(0,0\right)\right),\\ \frac{\partial\rho}{\partial t}+\mathcal{A}\rho=0,\\ -\frac{\partial p}{\partial t}+\mathcal{A}^{*}p=C^{*}\left(Cy-z_{d}\right)+C^{*}C\rho,\\ y\left(0\right)=y_{0},\zeta\left(T\right)=0,\\ \rho\left(0\right)=\lambda,p\left(T\right)=0,\\ \left(\mathcal{B}^{*}p+Nu,v-u\right)_{\mathcal{U}}\geq 0\ \ \forall v\in\mathcal{U}_{ad},\end{cases}$$

with $\lambda\in\widehat{G}$.

Proof. The proof leads from the approximated optimality system $(2.1.9)$, and some a priori estimates permit to pass to limit when $\gamma\rightarrow 0$ as in Theorem 1.3. ■

2.2 Optimal control of a fractional diffusion equation with missing data

In this section, we consider an optimal control problem of a time-fractional diffusion equation with a missing boundary condition. This problem will be treated via low-regret control and no-regret control notions introduced in the first chapter.

2.2.1 Position of problem

Let Ω be a bounded open set of $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with boundary Γ of class C^2. Let $Q = \Omega \times (0,T)$ where $T > 0$, $\Sigma = \Gamma \times (0,T)$. Consider the following fractional diffusion equation:

$$\begin{cases} {}^{RL}D_t^{\alpha} y - \Delta y = v & \text{in } Q, \\ y = g & \text{on } \Sigma, \\ I^{1-\alpha} y(0) = y_0 & \text{in } \Omega, \end{cases} \tag{2.2.1}$$

where $1/2 < \alpha < 1$, the function $y_0 = y_0(x) \in H^1(\Omega)$, $g = g(x,t)$ is an unknown function in $L^2(\Sigma)$ and $v = v(x,t) \in L^2(Q)$, ${}^{RL}D_t^{\alpha}$, I^{α} denote the Riemann-Liouvillle fractional time derivative and integral resp. both are of order α (see Appendices, Definition 2). The equation $(2.2.1)$ describes a diffusion of a pollutant in the soil with an unknown boundary distribution.
For every $(v, g, y_0) \in L^2(Q) \times L^2(\Sigma) \times H^1(\Omega)$, one can prove by using transposition method that $(2.2.1)$ has a unique solution $y = y(x,t;v,g) \in L^2(Q)$ (Mouphou, 2017).
We are concerned by the following optimal control problem:

$$\inf_{v \in L^2(Q)} J(v,g) \qquad \forall g \in L^2(\Sigma), \tag{2.2.2}$$

where

$$J(v,g) = \|y(v,g) - z_d\|^2_{L^2(Q)} + N\|v\|^2_{L^2(Q)}, \tag{2.2.3}$$

with z_d is a target function in $L^2(\Omega)$ and $N > 0$.
Let's start by the following lemma:

Lemma 2.1 *For every $v \in y_0$ and $g \in L^2(\Sigma)$, we have*

$$J(v,g) - J(0,g) = J(v,0) - J(0,0) + 2\int_0^T \int_\Gamma g \frac{\partial \xi(v)}{\partial \nu} d\Gamma dt, \tag{2.2.4}$$

where $\xi(v) \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ *(Mouphou, 2017) solves the system :*

$$\begin{cases} {}^{RL}D_t^{\alpha}\xi(v) - \Delta\xi(v) = -(y(v,0) - y(0,0)) & \text{in } Q, \\ \xi(v) = 0 & \text{on } \Sigma, \\ \xi(v)(T) = 0 & \text{in } \Omega. \end{cases} \tag{2.2.5}$$

Proof. It's easy to prove that

$$J(v,g) - J(0,g) = J(v,0) - J(0,0) + 2\int_0^T\int_\Omega (y(0,g) - y(0,0))(y(v,0) - y(0,0))\,dxdt,$$

then introduce $\xi(v)$ solution to (2.2.5) and use fractional integration by parts (see Appendices, Theorem 3) to prove that

$$\int_0^T\int_\Omega (y(0,g) - y(0,0))(y(v,0) - y(0,0))\,dxdt = \int_0^T\int_\Gamma g\frac{\partial\xi(v)}{\partial\nu}d\Gamma dt.$$

■

2.2.2 Low-regret control for fractional diffusion equation with missing data

Definition 2.1 *The low-regret control* u_γ *for* $(2.2.1)-(2.2.3)$ *is a solution to*

$$\inf_{v\in L^2(Q)}\sup_{g\in L^2(\Sigma)}\left(J(v,g) - J(0,g) - \gamma\|g\|^2_{L^2(\Sigma)}\right), \gamma > 0. \tag{2.2.6}$$

To reformulate (2.2.6), use (2.2.4) to get

$$\begin{aligned} &\inf_{v\in L^2(Q)}\sup_{g\in L^2(\Sigma)}\left(J(v,g) - J(0,g) - \gamma\|g\|^2_{L^2(\Sigma)}\right) \\ = &\inf_{v\in L^2(Q)}\left(J(v,0) - J(0,0) + \sup_{g\in L^2(\Sigma)}\left(2\int_0^T\int_\Gamma g\frac{\partial\xi(v)}{\partial\nu}d\Gamma dt - \gamma\|g\|^2_{L^2(\Sigma)}\right)\right), \end{aligned}$$

and use Legendre transform, to find that

$$\sup_{g\in L^2(\Sigma)}\left(2\int_0^T\int_\Gamma g\frac{\partial\xi(v)}{\partial\nu}d\Gamma dt - \gamma\|g\|^2_{L^2(\Sigma)}\right) = \frac{1}{\gamma}\left\|\frac{\partial\xi(v)}{\partial\nu}\right\|^2_{L^2(\Sigma)}.$$

Hence, (2.2.6) is equivalent to

$$\inf_{v\in L^2(Q)}\mathcal{J}^\gamma(v) \tag{2.2.7}$$

s.t.

$$\mathcal{J}^\gamma(v) = J(v,0) - J(0,0) + \frac{1}{\gamma}\left\|\frac{\partial\xi(v)}{\partial\nu}\right\|^2_{L^2(\Sigma)}, \gamma > 0. \tag{2.2.8}$$

Proposition 2.3 *The problem* (2.2.1) (2.2.7) (2.2.8) *has a unique solution* u_γ.

Proof. We know that

$$\mathcal{J}^\gamma(v) = J(v,0) - J(0,0) + \frac{1}{\gamma}\left\|\frac{\partial\xi(v)}{\partial\nu}\right\|^2_{L^2(\Sigma)} \geq -J(0,0),$$

then $\inf\limits_{v\in L^2(Q)} \mathcal{J}^\gamma(v)$ exists. Let $(v_n) \subset L^2(Q)$ be a minimizing sequence s.t

$$\mathcal{J}^\gamma(v_n) \underset{n\to\infty}{\to} \inf_{v\in L^2(Q)} \mathcal{J}^\gamma(v).$$

The associated state $y_n = y(v_n, 0)$ satisfies

$$\begin{cases} {}^{RL}D_t^\alpha y_n - \Delta y_n = v_n & \text{in } Q, \\ y_n = 0 & \text{on } \Sigma, \\ I^{1-\alpha} y_n(0) = y_0 & \text{in } \Omega. \end{cases} \tag{2.2.9}$$

Also, there exists $c > 0$, independent of n s.t. (Mophou, Tao & Joseph, 2015)

$$\|y_n\|_{L^2\left(0,T;H_0^1(\Omega)\right)} \leq c\left(\|y_0\|_{H^1(\Omega)} + \|v_n\|_{L^2(Q)}\right), \text{ for every } n \tag{2.2.10}$$

and

$$-J(0,0) \leq J(v_n,0) - J(0,0) + \frac{1}{\gamma}\left\|\frac{\partial\xi(v_n)}{\partial\nu}\right\|^2_{L^2(\Sigma)} \leq c,$$

which gives the following bounds

$$\|v_n\|_{L^2(Q)} \leq c, \quad \left\|\frac{\partial\xi(v_n)}{\partial\nu}\right\|_{L^2(\Sigma)} \leq c\sqrt{\gamma}.$$

Combine with (2.2.9) and (2.2.10) to get

$$\left\|{}^{RL}D_t^\alpha y_n - \Delta y_n\right\|_{L^2(Q)} \leq c, \quad \|y_n\|_{L^2\left(0,T;H_0^1(\Omega)\right)} \leq c.$$

Hence, there exists subsequences still be denoted (v_n) and (y_n) s.t.

$$\begin{aligned} v_n &\rightharpoonup u_\gamma \text{ weakly in } L^2(Q), \\ y_n &\rightharpoonup y_\gamma \text{ weakly in } L^2\left(0,T;H_0^1(\Omega)\right), \\ {}^{RL}D_t^\alpha y_n - \Delta y_n &\rightharpoonup f \text{ weakly in } L^2(Q). \end{aligned}$$

The rest of the proof will be done in three steps, we shall give only a sketch, for more details see (Mouphou, 2015).

First step : We prove that (u_γ, y_γ) satisfies

$$\begin{cases} {}^{RL}D_t^\alpha y_\gamma - \Delta y_\gamma = u_\gamma & \text{in } Q, \\ y_\gamma = 0 & \text{on } \Sigma, \\ I^{1-\alpha} y_\gamma(0) = y_0 & \text{in } \Omega. \end{cases}$$

Second step : We prove that $\xi(v_n)$ converges to $\xi_\gamma = \xi(u_\gamma)$ which verifies the system

$$\begin{cases} {}^{RL}D_t^\alpha \xi_\gamma - \Delta \xi_\gamma = -(y_\gamma - y(0,0)) & \text{in } Q, \\ \xi_\gamma = 0 & \text{on } \Sigma, \\ \xi_\gamma(T) = 0 & \text{in } \Omega. \end{cases}$$

Third step : Because $\mathcal{J}^\gamma$ is lower semicontinous function, we get

$$\liminf_{n\to\infty} \mathcal{J}^\gamma(v_n) \geq \mathcal{J}^\gamma(u_\gamma)$$

then

$$\mathcal{J}^\gamma(u_\gamma) = \inf_{v\in L^2(Q)} \mathcal{J}^\gamma(v).$$

The uniqueness of u_γ follows from strict convexity of $\mathcal{J}^\gamma$. ■

Theorem 2.2 *The low-regret control u_γ, solution to (2.2.1) (2.2.7) (2.2.8) is characterized by the following optimality system*

$$\begin{cases} {}^{RL}D_t^\alpha y_\gamma - \Delta y_\gamma = u_\gamma, \\ {}^{RL}D_t^\alpha \xi_\gamma - \Delta \xi_\gamma = -(y_\gamma - y(0,0)), \\ {}^{RL}D_t^\alpha \rho_\gamma - \Delta \rho_\gamma = 0, \\ {}^{RL}D_t^\alpha p_\gamma - \Delta p_\gamma = y_\gamma - z_d + \frac{1}{\sqrt{\gamma}}\rho_\gamma & \text{in } Q, \\ y_\gamma = 0,\ \xi_\gamma = 0, \\ \rho_\gamma = \frac{1}{\sqrt{\gamma}}\frac{\partial \xi_\gamma}{\partial \nu},\ p_\gamma = 0 & \text{on } \Sigma, \\ I^{1-\alpha} y_\gamma(0) = y_0,\ \xi_\gamma(T) = 0, \\ I^{1-\alpha}\rho_\gamma(0) = 0,\ p_\gamma(T) = 0 & \text{in } \Omega, \\ p_\gamma + N u_\gamma = 0 \ \text{ in } Q, \end{cases} \tag{2.2.11}$$

where $y_\gamma = y(u_\gamma, 0)$, $\rho_\gamma \in L^2(Q)$ and $p_\gamma \in L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega))$.

Proof. As usually, a first order condition gives for every $v \in L^2(Q)$

$$\int_0^T \int_\Omega (y(u_\gamma, 0) - z_d)(y(v,0) - y(u_\gamma, 0))\, dx + N \int_0^T \int_\Omega u_\gamma (v - u_\gamma)\, dt + \frac{1}{\gamma}\int_0^T \int_\Gamma \frac{\partial \xi_\gamma}{\partial \nu}\frac{\partial \xi(v - u_\gamma)}{\partial \nu} d\Gamma dt = 0 \tag{2.2.12}$$

Introduce ρ_γ given by

$$\begin{cases} {}^{RL}D_t^\alpha \rho_\gamma - \Delta \rho_\gamma = 0 & \text{in } Q, \\ \rho_\gamma = \frac{1}{\sqrt{\gamma}} \frac{\partial \xi_\gamma}{\partial \nu} & \text{on } \Sigma, \\ I^{1-\alpha} \rho_\gamma (0) = 0 & \text{in } \Omega, \end{cases}$$

since $\frac{1}{\sqrt{\gamma}} \frac{\partial \xi_\gamma}{\partial \nu} \in L^2(\Sigma)$, by transposition method we prove that $\rho_\gamma \in L^2(Q)$. Then, by Lemma 1 in Appendices we get

$$\int_0^T \int_\Omega (y(v,0) - y(u_\gamma, 0)) \frac{1}{\sqrt{\gamma}} \rho_\gamma = \frac{1}{\gamma} \int_0^T \int_\Gamma \frac{\partial \xi_\gamma}{\partial \nu} \frac{\partial \xi (v - u_\gamma)}{\partial \nu} d\Gamma dt,$$

and $(2.2.12)$ is equivalent to

$$\int_0^T \int_\Omega (y(v,0) - y(u_\gamma, 0)) \left(y(u_\gamma, 0) - z_d + \frac{1}{\sqrt{\gamma}} \rho_\gamma \right) dxdt + N \int_0^T \int_\Omega u_\gamma (v - u_\gamma) \, dt = 0, \, \forall v \in L^2(Q).$$

Moreover, define the adjoint state p_γ, solution of

$$\begin{cases} {}^{RL}D_t^\alpha p_\gamma - \Delta p_\gamma = y(u_\gamma, 0) - z_d + \frac{1}{\sqrt{\gamma}} \rho_\gamma & \text{in } Q, \\ p_\gamma = 0 & \text{on } \Sigma, \\ p_\gamma (T) = 0 & \text{in } \Omega. \end{cases} \tag{2.2.13}$$

We know that $y(u_\gamma, 0) - z_d + \frac{1}{\sqrt{\gamma}} \rho_\gamma \in L^2(Q)$, by Proposition 1 in appendices, $p_\gamma \in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega))$. Multiply $(2.2.13)$ by $y(v,0) - y(u_\gamma, 0)$ and integrate by parts over Q by using, again, Lemma 1 in appendices to rewrite $(2.2.12)$ as

$$\int_0^T \int_\Omega (p_\gamma + N u_\gamma)(v - u_\gamma) \, dxdt = 0, \, \forall v \in L^2(Q).$$

■

Proposition 2.4 *The low regret control u_γ solution to $(2.2.1)\,(2.2.7)\,(2.2.8)$, converges weakly in $L^2(Q)$ to the no-regret control u.*

Proof. u_γ is a low-regret control, then

$$J(u_\gamma, 0) - J(0,0) + \frac{1}{\gamma} \left\| \frac{\partial \xi(u_\gamma)}{\partial \nu} \right\|^2_{L^2(\Sigma)} = \mathcal{J}^\gamma (u_\gamma) \leq \mathcal{J}^\gamma (0) = 0$$

in other words

$$\| y(u_\gamma, 0) - z_d \|^2_{L^2(Q)} + N \| u_\gamma \|^2_{L^2(Q)} + \frac{1}{\gamma} \left\| \frac{\partial \xi(u_\gamma)}{\partial \nu} \right\|^2_{L^2(\Sigma)} \leq \| y(0,0) - z_d \|^2_{L^2(Q)} = \text{Constant}.$$

We deduce that there exists a constant $C > 0$ independent of γ s.t.

$$\begin{aligned}
\|y(u_\gamma, 0)\|_{L^2(Q)} &\leq C, && (2.2.14.a)\\
\|u_\gamma\|_{L^2(Q)} &\leq C, && (2.2.14.b)\\
\left\|\frac{\partial \xi(u_\gamma)}{\partial \nu}\right\|_{L^2(\Sigma)} &\leq C\sqrt{\gamma}, && (2.2.14.c)
\end{aligned}$$

from $(2.2.14.b)$ and $(2.2.11)$ we deduce

$$\left\|{}^{RL}D_t^\alpha y_\gamma - \Delta y_\gamma\right\|_{L^2(Q)} \leq C.$$

Also one can prove (Mophou, Tao & Joseph, 2015) that

$$\|y(u_\gamma, 0)\|_{L^2\left(0,T;H_0^1(\Omega)\right)} \leq C.$$

Then, we can extract a subsequence u_γ, y_γ s.t.

$$\begin{aligned}
u_\gamma &\rightharpoonup u \text{ weakly in } L^2(Q),\\
y_\gamma &\rightharpoonup y \text{ weakly in } L^2\left(0,T;H_0^1(\Omega)\right),\\
{}^{RL}D_t^\alpha y_\gamma - \Delta y_\gamma &\rightharpoonup f \text{ weakly in } L^2(Q).
\end{aligned}$$

Proceed as in the proof of Proposition 2.3, to prove that $y = y(u, 0)$ is a solution of the equation :

$$\begin{cases}
{}^{RL}D_t^\alpha y - \Delta y = u & \text{in } Q,\\
y = 0 & \text{on } \Sigma,\\
I^{1-\alpha} y(0) = y_0 & \text{in } \Omega.
\end{cases} \qquad (2.2.15)$$

and $\xi(u) \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$. By $(2.2.14.c)$

$$\frac{\partial \xi(u_\gamma)}{\partial \nu} \to \frac{\partial \xi(u)}{\partial \nu} = 0 \text{ in } L^2(\Sigma),$$

which leads to $\int\limits_0^T \int_\Gamma g \frac{\partial \xi(u)}{\partial \nu} d\Gamma dt = 0$ i.e., u is a no-regret control. ∎

2.2.3 Optimality system of no-regret control for fractional diffusion equation with missing data

Theorem 2.3 *The no-regret control* $u = \lim\limits_{\gamma \to 0} u_\gamma$ *is characterized by the following optimality system*

$$\begin{cases} {}^{RL}D_t^\alpha y - \Delta y = u, & \\ {}^{RL}D_t^\alpha \xi - \Delta \xi = -\left(y - y\left(0,0\right)\right), & \\ {}^{RL}D_t^\alpha \rho - \Delta \rho = 0, & \\ {}^{RL}D_t^\alpha p - \Delta p = y - z_d + \lambda_2 & \text{in } Q, \\ y = 0,\ \xi = 0, & \\ \rho = \lambda_1,\ p = 0, & \text{on } \Sigma, \\ I^{1-\alpha} y\left(0\right) = y_0,\ \xi\left(T\right) = 0, & \\ I^{1-\alpha} \rho\left(0\right) = 0,\ p\left(T\right) = 0, & \text{in } \Omega, \\ p + Nu = 0 \ \text{ in } Q, & \end{cases} \tag{2.2.16}$$

where $y = y\left(u, 0\right)$, $\rho, p \in L^2(Q)$, *and* $\lambda_1 \in L^2(Q)$,$\lambda_2 \in L^2(\Sigma)$.

Proof. The system that governs y is already given by $(2.2.15)$. One can prove (Dorville, Mophou, and Valmorin, 2011) the bound

$$\left\|\rho_\gamma\right\|_{L^2(Q)} \leq C,$$

where C is a positive constant independent of γ, then

$$\rho_\gamma \rightharpoonup \rho \text{ weakly in } L^2(Q),$$

and from $(2.2.14.c)$ we deduce

$$\frac{1}{\sqrt{\gamma}} \frac{\partial \xi_\gamma}{\partial \nu} \rightharpoonup \lambda_1 \text{ weakly in } L^2(\Sigma).$$

Because $p_\gamma = -N u_\gamma$, we have

$$\left\|p_\gamma\right\|_{L^2(Q)} \leq C.$$

Hence, there exists p s.t.

$$p_\gamma \rightharpoonup p \text{ weakly in } L^2(Q).$$

And with $(2.2.14.a)$, $(2.2.13)$, we get

$$\frac{1}{\sqrt{\gamma}} \rho_\gamma \rightharpoonup \lambda_2 \text{ weakly in } L^2(Q).$$

Then, by passing to limit when $\gamma \to 0$ and proving as in the proof of Proposition 2.3 to get the systems governing ρ and p. ■

The problem of optimal control of a fractional wave equation ($3/2 < \alpha < 2$) with missing initial condition is also treated in (Baleanu, Joseph & Mophou, 2016), where the authors gave a full characterizations for the low-regret control and no-regret control.

2.3 Optimal control for age structured population dynamics with missing data

In this section, we present an optimal control problem for age structured population dynamics with a missing initial population age distribution treated by Jacob and Omrane (2010).

2.3.1 Position of problem

Let's consider a single species population where we are interested in the age factor. Our population lives in a bounded domain Ω in $\mathbb{R}^2$ with a smooth boundary Γ. Let $A > 0$ be the maximum age of individuals in the considered population and $(0, T)$ the time interval with horizon T, $Q = \Omega \times (0, T) \times (0, A)$, $\Omega_T = \Omega \times (0, T)$ and $\Omega_A = \Omega \times (0, A)$. Denote by $p(x, t, a)$ the distribution of the population located at x, at time t and having the age a. The individuals of this population growth following the PDE given by

$$\begin{cases} \frac{\partial p}{\partial t}(x,t,a) + \frac{\partial p}{\partial a}(x,t,a) + \mu(x,t,a)\,p(x,t,a) - k\Delta p(x,t,a) = v(x,t,a) & (x,t,a) \text{ in } Q, \\ \frac{\partial p}{\partial \nu}(x,t,a) = 0 & (x,t,a) \text{ in } \Sigma, \\ p(x,t,0) = \int\limits_0^A \beta(a)\,p(x,t,a)da & (x,t) \text{ in } \Omega_T, \\ p(x,0,a) = p_0(x,a) & (x,a) \text{ in } \Omega_A, \end{cases} \tag{2.3.1}$$

where $\mu \in L^\infty(Q)$, $\mu(x,t,a) \geq 0$ a.e. in Q, $\beta \in L^\infty(0, A)$, $\beta(a) \geq 0$ a.e. in $(0, A)$, $p_0 \in L^2(\Omega_A)$ is the initial distribution supposed unknown with $p_0(x, a) \geq 0$ a.e., in Ω_A and v is a control function in $L^2(Q)$.

The diffusion equation $(2.3.1)$ has a unique solution $p = p(x, t, a; v, p_0)$ (Anita, 2000, p111).

We have to choose a control v s.t. : the population distribution p approaches to a given measurement p_d in $L^2(Q)$.

Then, we want to solve

$$\inf_{v \in L^2(Q)} J(v), \tag{2.3.2}$$

s.t.

$$J(v, p_0) = \|p(v, p_0) - p_d\|^2_{L^2(Q)} + N\|v\|^2_{L^2(Q)}. \tag{2.3.3}$$

2.3.2 No-regret control and low-regret for the age structured population dynamics with missing data

Define the no-regret control for $(2.3.1)-(2.3.3)$ by:

Definition 2.2 *We say that $u \in L^2(Q)$ is a no-regret control for $(2.3.1)-(2.3.3)$ if u solves*

$$\inf_{v \in L^2(Q)} \sup_{p_0 \in L^2(\Omega_A)} \left(J(v, p_0) - J(0, p_0) \right). \tag{2.3.4}$$

As usually, by a simple calculation and by using the Green formula we get

$$J(v, p_0) - J(0, p_0) = J(v, 0) - J(0, 0) + 2 \left(\zeta(v)(x, 0, a), p_0 \right)_{L^2(\Omega_A)}, \tag{2.3.5}$$

where $\zeta(v)$ is a solution of the following backward diffusion equation :

$$\begin{cases} -\frac{\partial \zeta(v)}{\partial t} - \frac{\partial \zeta(v)}{\partial a} + \mu \zeta(v) - k \Delta \zeta(v) = p(v, 0) & \text{in } Q, \\ \frac{\partial \zeta(v)}{\partial \nu}(x, t, a) = 0 & \text{on } \Sigma, \\ \zeta(v)(x, t, A) = 0 & \text{in } \Omega_T, \\ \zeta(v)(x, T, a) = 0 & \text{in } \Omega_A. \end{cases} \tag{2.3.6}$$

To get a characterization of the optimal control, the no-regret control definition has to be relaxed as follows :

Definition 2.3 *We say that $u_\gamma \in L^2(Q)$ is a low-regret control for $(2.3.1)-(2.3.3)$ if u_γ solves*

$$\inf_{v \in L^2(Q)} \sup_{p_0 \in L^2(\Omega_A)} \left(J(v, p_0) - J(0, p_0) - \gamma \|p_0\|^2_{L^2(\Omega_A)} \right), \ \gamma > 0. \tag{2.3.7}$$

From $(2.3.5)$, we know that

$$J(v, p_0) - J(0, p_0) - \gamma \|p_0\|^2_{L^2(\Omega_A)} = J(v, 0) - J(0, 0) + 2 \left(\zeta(v)(x, 0, a), p_0 \right)_{L^2(\Omega_A)} - \gamma \|p_0\|^2_{L^2(\Omega_A)},$$

then

$$\begin{aligned} & \inf_{v \in L^2(Q)} \sup_{p_0 \in L^2(\Omega_A)} \left(J(v, p_0) - J(0, p_0) - \gamma \|p_0\|^2_{L^2(\Omega_A)} \right) \\ = & \inf_{v \in L^2(Q)} \left(J(v, 0) - J(0, 0) + \sup_{p_0 \in L^2(\Omega_A)} \left(2 \left(\zeta(v)(x, 0, a), p_0 \right)_{L^2(\Omega_A)} - \gamma \|p_0\|^2_{L^2(\Omega_A)} \right) \right) \\ = & \inf_{v \in L^2(Q)} \left(J(v, 0) - J(0, 0) + \frac{1}{\gamma} \|\zeta(v)(x, 0, a)\|^2_{L^2(\Omega_A)} \right). \end{aligned}$$

Now, our problem is equivalent to the following standard optimal control problem

$$\inf_{v \in L^2(Q)} \mathcal{J}^\gamma(v), \tag{2.3.8}$$

where

$$\mathcal{J}^\gamma(v) = J(v, 0) - J(0, 0) + \frac{1}{\gamma} \|\zeta(v)(x, 0, a)\|^2_{L^2(\Omega_A)}. \tag{2.3.9}$$

2.3.3 Existence and characterization of low-regret control for age structured population dynamics with missing data

Proposition 2.5 *There is a unique low-regret control u_γ solution to* (2.3.1) (2.3.8) (2.3.9).

Proof. Note that $\mathcal{J}^\gamma(v) \geq -J(0,0)$ for every $v \in L^2(Q)$, then (2.3.8) (2.3.9) has a solution. Let (v_n^γ) be a minimizing sequence i.e., $\lim\limits_{n\to\infty} \mathcal{J}^\gamma(v_n^\gamma) = d_\gamma = \inf\limits_{v\in L^2(Q)} \mathcal{J}^\gamma(v)$, we have

$$-J(0,0) \leq \mathcal{J}^\gamma(v_n^\gamma) = J(v_n^\gamma,0) - J(0,0) + \frac{1}{\gamma}\|\zeta(v_n^\gamma)(x,0,a)\|^2_{L^2(\Omega_A)} \leq d_\gamma + 1. \tag{2.3.10}$$

From (2.3.10) we deduce the bounds

$$\|v_n^\gamma\|_{L^2(Q)} \leq C_\gamma,$$

$$\|p(v_n^\gamma,0)\|_{L^2(Q)} \leq C_\gamma, \tag{2.3.11.b}$$

$$\frac{1}{\sqrt{\gamma}}\|\zeta(v_n^\gamma)(x,0,a)\|_{L^2(\Omega_A)} \leq C_\gamma, \tag{2.3.11.c}$$

where C_γ is a positive constant independent of n. From $(2.3.11.a)$ we deduce that (v_n^γ) is bounded in $L^2(Q)$, then there exists a subsequence still denoted (v_n^γ) converges weakly to u_γ in $L^2(Q)$. Also

$$p(v_n^\gamma,0) \rightharpoonup p_\gamma \text{ weakly in } L^2(Q),$$

and for $\varphi \in \mathcal{D}(Q)$ we have

$$\int\limits_Q p(v_n^\gamma,0)\left(-\frac{\partial\varphi}{\partial t} - \frac{\partial\varphi}{\partial a} + \mu\varphi - k\Delta\varphi\right)dxdtda = \int\limits_Q v_n^\gamma\varphi dxdtda,$$

by passing to limit when $n \to +\infty$ we get

$$\int\limits_Q p(u_\gamma,0)\left(-\frac{\partial\varphi}{\partial t} - \frac{\partial\varphi}{\partial a} + \mu\varphi - k\Delta\varphi\right)dxdtda = \int\limits_Q u_\gamma\varphi dxdtda,$$

Let's prove that $p_\gamma = p(u_\gamma,0)$

$$\frac{\partial p}{\partial t}(v_n^\gamma,0) + \frac{\partial p}{\partial a}(v_n^\gamma,0) + \mu p(v_n^\gamma,0) - k\Delta p(v_n^\gamma,0) \rightharpoonup \frac{\partial p_\gamma}{\partial t} + \frac{\partial p_\gamma}{\partial a} + \mu p_\gamma - k\Delta p_\gamma \text{ in } \mathcal{D}'(Q),$$

then by the uniqueness of limit we deduce that

$$\frac{\partial p_\gamma}{\partial t} + \frac{\partial p_\gamma}{\partial a} + \mu p_\gamma - k\Delta p_\gamma = u_\gamma \text{ in } L^2(Q),$$

then $p_\gamma = p(u_\gamma,0)$. By a similar way, $\zeta(v_n^\gamma)(x,0,a)$ converges weakly to $\zeta(u_\gamma)(x,0,a)$ in $L^2(\Omega_A)$.

Hence,

$$\mathcal{J}^{\gamma}(u_{\gamma}) \leq \inf_{n\in\mathbb{N}} \mathcal{J}^{\gamma}(v_n^{\gamma}) = d_{\gamma},$$

then u_γ is a solution of $(2.3.1)$ $(2.3.8)$ $(2.3.9)$. Uniqueness of u_γ follows from the strict convexity of $\mathcal{J}^{\gamma}$. ■

Now, let's give a characterization of low-regret control for age structured population dynamics with missing data $(2.3.1)$ by the following proposition:

Proposition 2.6 *The low-regret control u_γ solution to $(2.3.1)$ $(2.3.8)$ $(2.3.9)$ is characterized by:*

$$\begin{cases} \frac{\partial p_\gamma}{\partial t} + \frac{\partial p_\gamma}{\partial a} + \mu p_\gamma - k\Delta p_\gamma = u_\gamma, \\ -\frac{\partial \zeta_\gamma}{\partial t} - \frac{\partial \zeta_\gamma}{\partial a} + \mu \zeta_\gamma - k\Delta \zeta_\gamma = p_\gamma, \\ \frac{\partial \rho_\gamma}{\partial t} + \frac{\partial \rho_\gamma}{\partial a} + \mu \rho_\gamma - k\Delta \rho_\gamma = 0, \\ -\frac{\partial \pi_\gamma}{\partial t} - \frac{\partial \pi_\gamma}{\partial a} + \mu \pi_\gamma - k\Delta \pi_\gamma = p_\gamma - p_d + \frac{1}{\sqrt{\gamma}}\rho_\gamma & \text{in } Q, \\ \frac{\partial p_\gamma}{\partial \nu} = \frac{\partial \zeta_\gamma}{\partial \nu} = \frac{\partial \rho_\gamma}{\partial \nu} = \frac{\partial \pi_\gamma}{\partial \nu} = 0 & \text{on } \Sigma, \\ p_\gamma(x,t,0) = \int_0^A \beta(a)\, p_\gamma(x,t,a)da, \ \zeta_\gamma(x,t,A) = 0, \\ \rho_\gamma(x,t,0) = \int_0^A \beta(a)\, \rho_\gamma(x,t,a)da, \ \pi_\gamma(x,t,A) = 0 & \text{in } \Omega_T, \\ p_\gamma(x,0,a) = 0, \ \zeta_\gamma(x,T,a) = 0, \\ \rho_\gamma(x,0,a) = \frac{1}{\sqrt{\gamma}}\zeta_\gamma(x,0,a), \ \pi_\gamma(x,T,a) = 0 & \text{in } \Omega_A, \end{cases} \tag{2.3.12}$$

with

$$u_\gamma = -\frac{1}{N}\pi_\gamma \text{ in } Q, \tag{2.3.13}$$

where $p_\gamma = p(u_\gamma, 0)$, $\zeta_\gamma = \zeta(u_\gamma)$.

Proof. A first order optimality condition gives for every $v \in L^2(Q)$

$$(p(u_\gamma,0) - p_d, p(v-u_\gamma,0))_{L^2(Q)} + N(u_\gamma, v-u_\gamma)_{L^2(Q)} + \frac{1}{\gamma}(\zeta(u_\gamma)(x,0,a), \zeta(v-u_\gamma)(x,0,a))_{L^2(\Omega_A)} = 0. \tag{2.3.14}$$

Let's introduce a new state ρ_γ given by

$$\begin{cases} \frac{\partial \rho_\gamma}{\partial t} + \frac{\partial \rho_\gamma}{\partial a} + \mu \rho_\gamma - k\Delta \rho_\gamma = 0 & \text{in } Q, \\ \frac{\partial \rho_\gamma}{\partial \nu} = 0 & \text{on } \Sigma, \\ \rho_\gamma(x,t,0) = \int_0^A \beta(a)\, \rho_\gamma(x,t,a)da & \text{in } \Omega_T, \\ \rho_\gamma(x,0,a) = \frac{1}{\sqrt{\gamma}}\zeta(u_\gamma)(x,0,a) & \text{in } \Omega_A, \end{cases}$$

by using the Green formula we get

$$\begin{aligned}
& \frac{1}{\gamma}\left(\zeta\left(u_\gamma\right)(x,0,a), \zeta\left(v-u_\gamma\right)(x,0,a)\right)_{L^2(\Omega_A)} \\
= & \left(\rho_\gamma(x,0,a), \frac{1}{\sqrt{\gamma}}\zeta\left(v-u_\gamma\right)(x,0,a)\right)_{L^2(\Omega_A)} \\
= & \int\limits_Q \rho_\gamma\left(-\frac{\partial\zeta\left(v-u_\gamma\right)}{\partial t}-\frac{\partial\zeta\left(v-u_\gamma\right)}{\partial a}+\mu\zeta\left(v-u_\gamma\right)-k\Delta\zeta\left(v-u_\gamma\right)\right)dxdtda \\
= & \left(\rho_\gamma, p\left(v-u_\gamma\right)(x,0,a)\right)_{L^2(\Omega_A)}.
\end{aligned}$$

Also, introduce the adjoint state π_γ by

$$\begin{cases}
-\frac{\partial\pi_\gamma}{\partial t}-\frac{\partial\pi_\gamma}{\partial a}+\mu\pi_\gamma-k\Delta\pi_\gamma=p_\gamma-p_d+\frac{1}{\sqrt{\gamma}}\rho_\gamma & \text{in } Q, \\
\frac{\partial\pi_\gamma}{\partial\nu}=0 & \text{in } \Sigma, \\
\pi_\gamma(x,t,A)=0 & \text{in } \Omega_T, \\
\pi_\gamma(x,T,a)=0 & \text{in } \Omega_A,
\end{cases}$$

again by using the Green formula we have:

$$\int\limits_Q\left(-\frac{\partial\pi_\gamma}{\partial t}-\frac{\partial\pi_\gamma}{\partial a}+\mu\pi_\gamma-k\Delta\pi_\gamma\right)p\left(v-u_\gamma\right)dxdtda=\int\limits_Q\pi_\gamma\left(v-u_\gamma\right)dxdtda.$$

Finally, the optimality condition (2.3.14) is equivalent to

$$\left(\pi_\gamma+Nu_\gamma, v-u_\gamma\right)_{L^2(Q)}=0 \quad \forall v\in L^2(Q).$$

■

2.3.4 Characterization of no-regret control for age structured population dynamics with missing data

The following theorem gives an optimality system characterizing the no-regret control for age structured population dynamics with missing data (2.3.1).

Theorem 2.4 *The no-regret control u solution to $(2.3.1)-(2.3.3)$ is characterized by:*

$$\begin{cases} \frac{\partial p}{\partial t}+\frac{\partial p}{\partial a}+\mu p-k\Delta p=u, & \\ -\frac{\partial \zeta}{\partial t}-\frac{\partial \zeta}{\partial a}+\mu \zeta-k\Delta \zeta=p, & \\ \frac{\partial \rho}{\partial t}+\frac{\partial \rho}{\partial a}+\mu \rho-k\Delta \rho=0, & \\ -\frac{\partial \pi}{\partial t}-\frac{\partial \pi}{\partial a}+\mu \pi-k\Delta \pi=p-p_d+\lambda_2 & \text{in } Q, \\ \frac{\partial p}{\partial \nu}=\frac{\partial \zeta}{\partial \nu}=\frac{\partial \rho}{\partial \nu}=\frac{\partial \pi}{\partial \nu}=0 & \text{on } \Sigma, \\ p(x,t,0)=\int_0^A \beta(a)\, p(x,t,a)da,\ \zeta(x,t,A)=0, & \\ \rho(x,t,0)=\int_0^A \beta(a)\, \rho(x,t,a)da,\ \pi(x,t,A)=0 & \text{in } \Omega_T, \\ p(x,0,a)=0,\ \zeta(x,T,a)=0, & \\ \rho(x,0,a)=\lambda_1,\ \pi(x,T,a)=0 & \text{in } \Omega_A, \end{cases} \tag{2.3.15}$$

with

$$u=-\frac{1}{N}\pi \text{ in } Q,$$

where $u=\lim\limits_{\gamma\to 0} u_\gamma$, $p=\lim\limits_{\gamma\to 0} p_\gamma$, $\zeta=\lim\limits_{\gamma\to 0}\zeta(u_\gamma)$, $\rho=\lim\limits_{\gamma\to 0}\rho_\gamma$, $\pi=\lim\limits_{\gamma\to 0}\pi_\gamma$, $\lambda_1=\lim\limits_{\gamma\to 0}\frac{1}{\sqrt{\gamma}}\zeta_\gamma(x,0,a)$ *and* $\lambda_2=\lim\limits_{\gamma\to 0}\frac{1}{\sqrt{\gamma}}\rho_\gamma$ *with* $\lambda_1\in L^2(\Omega_A)$ *and* $\lambda_2\in L^2(Q)$.

Proof. u_γ is a low-regret control i.e., u_γ solution to $(2.3.8)\,(2.3.9)$, then $\mathcal{J}^\gamma(u_\gamma)\le \mathcal{J}^\gamma(0)$ in other words

$$\|p(u_\gamma,0)-p_d\|^2_{L^2(Q)}+N\|u_\gamma\|^2_{L^2(Q)}+\frac{1}{\gamma}\|\zeta(u_\gamma)(x,0,a)\|^2_{L^2(\Omega_A)}\le \|p_d\|^2_{L^2(Q)}=\text{constant},$$

from which we get the bounds

$$\|u_\gamma\|_{L^2(Q)}\le C, \tag{2.3.17.a}$$

$$\|p(u_\gamma,0)\|_{L^2(Q)}\le C, \tag{2.3.17.a}$$

where C is a constant independent of γ. From $(2.3.17.a)$ we deduce the existence of a subsequence still denoted (u_γ) converges weakly in $L^2(Q)$ to u the no-regret control. Moreover,

$$\frac{\partial p_\gamma}{\partial t}+\frac{\partial p_\gamma}{\partial a}+\mu p_\gamma-k\Delta p_\gamma \rightharpoonup \frac{\partial p}{\partial t}+\frac{\partial p}{\partial a}+\mu p-k\Delta p \text{ in } \mathcal{D}'(Q),$$

by limit uniqueness we get

$$\frac{\partial p}{\partial t}+\frac{\partial p}{\partial a}+\mu p-k\Delta p=u \text{ in } L^2(Q).$$

We also have from (Lions & Magenes, 1972, vol.1, p.44) that

$$\frac{\partial p_\gamma}{\partial \nu}(x,t,a)\rightharpoonup \frac{\partial p}{\partial \nu}(x,t,a) \text{ weakly in } H^{-\frac{1}{2}}(\Gamma).$$

Other equations in $(2.3.15)$ follow by the same way with $\lambda_1=\lim\limits_{\gamma\to 0}\frac{1}{\sqrt{\gamma}}\zeta_\gamma(x,0,a)$ and $\lambda_2=\lim\limits_{\gamma\to 0}\frac{1}{\sqrt{\gamma}}\rho_\gamma$. Finally, because $\pi_\gamma=-Nu_\gamma$ is bounded in $L^2(Q)$ we deduce that $\pi=-Nu$ in $L^2(Q)$. ■

Remark 2.1 *Many studies for optimal control of distributed systems with missing data deserve to be cited like (Aimene, Dorville & Omrane, 2013) where authors studied the problem of optimal control for trees trunk diameter variations in tropical ecology with missing data. In (Mahoui, Moulay and Omrane, 2017) authors treated the case of pointwise control for a diffusion equation with missing data. In this paper, the authors only gave a characterization for the low-regret control.*

Chapter 3

Optimal control of systems governed by hyperbolic PDEs with missing data

In this principal chapter, the first section treats an ill-posed wave equation by a regularization method. The main idea is to approximate an ill-posed equation by a sequence of well-posed equations with missing data, then controlling them by using no-regret control and low-regret control, getting a sequence of optimal controls and passing to limit.
In the second section, we study an optimal control problem for an electromagnetic wave equation with an unknown velocity of propagation and a missing boundary condition, where we introduce the notion of *averaged no-regret control* to solve our problem.

3.1 Optimal control of an ill-posed wave equation via regularization into a well-posed equation with incomplete data

In this section, we characterize the optimal control for an ill-posed wave equation without using the extra hypothesis of Slater (i.e., the set of admissible controls has a non-empty interior). By approaching the ill-posed wave equation by a sequence of parabolic equations with missing data, where we use the notion of no-regret control to obtain a singular optimality system, then we pass to limit and by a corrector of order zero we complete the information about initial conditions.

3.1.1 Description of problem

Consider an open domain $\Omega \subset \mathbb{R}^N$ with smooth boundary Γ, and denote by $Q = \Omega \times (0, T)$ where $T > 0$, and by $\Sigma = \Gamma \times (0, T)$, $v \in L^2(Q)$. It's well known that the following wave equation :

$$\begin{cases} y'' - \triangle y = v & \text{in } Q, \\ y(x,0) = 0 \ ; y'(x,0) = 0 & \text{in } \Omega, \\ y(x,t) = 0 & \text{on } \Sigma, \end{cases}$$

is well-posed with the regularity properties :

$$y \in L^2\left(0,T; H_0^1(\Omega) \cap H^2(\Omega)\right) \ , \ y'' \in L^2\left(0,T; H^{-1}(\Omega)\right)$$

If we substitute the initial condition $y'(0) = 0$ by $y(T) = 0$, the above system will have no solution (ill-posed).

Counter-example: Consider the following one-dimensional wave equation:

$$\begin{cases} \frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = v & \text{in }]0,1[\times]0,T[, \\ y(x,0) = 0, \ y(x,T) = 0 & \text{in }]0,1[, \\ y(0,t) = 0, \ y(1,t) = 0 & \text{in }]0,T[, \end{cases} \tag{3.1.1}$$

where $v \in L^2(0,T; L^2(0,1))$, with

$$v(x,t) = \sqrt{\frac{2}{\pi}} \sum_{n \geq 1} v_n \sin n\pi x$$

and $v_n \in \mathbb{R}$ for every $n \geq 1$. The solution $y(x,t)$ if it exists could be written in the form

$$y(x,t) = \sum_{n \geq 1} y_n(t)\, w_n(x) \tag{3.1.2}$$

where $w_n(x) = \sqrt{\frac{2}{\pi}} \sin n\pi x$. Substitute into $(3.1.1)$ to get the following second order ordinary differential equation for every $n \geq 1$:

$$\begin{cases} \frac{\partial^2 y_n}{\partial t^2} + n^2\pi^2 y_n = v_n, \\ y(0) = 0, \ y(T) = 0. \end{cases}$$

By variation of constants we have:

$$y_n(t) = \frac{2v_n}{n^2\pi^2} \sin \frac{n\pi(t-T)}{2} \frac{\sin \frac{n\pi t}{2}}{\cos \frac{n\pi T}{2}}$$

but $\lim_{n\to\infty} \frac{2v_n}{n^2\pi^2} \sin \frac{n\pi(t-T)}{2} \frac{\sin \frac{n\pi t}{2}}{\cos \frac{n\pi T}{2}}$ does not exist, i.e., the series $(3.1.2)$ diverges, then, the solution does not exist.

Remark 3.1 *The following wave equation has a unique solution for v in some dense subset of $L^2(Q)$.*

$$\begin{cases} y'' - \triangle y = v & \text{in } Q, \\ y(x,0) = 0,\ y(x,T) = 0 & \text{in } \Omega, \\ y(x,t) = 0 & \text{on } \Sigma. \end{cases} \tag{3.1.3}$$

To illustrate this, consider

$$\widetilde{V} = \left\{ w = \sum_{i=1}^{N} \lambda_i w_i \text{ such that: } -\Delta w_i = \lambda_i w_i \text{ in } \Omega, w_i = 0 \text{ on } \partial\Omega \text{ and } w_i \in L^2(\Omega) \right\}.$$

Then, there is $f \in L^2(0,T)$ and $w \in \widetilde{V}$such that:

$$v(x,t) = \left(\sum_{i=1}^{N} \lambda_i w_i(x) \right) f(t),$$

for a given $v \in \widetilde{V} \otimes L^2(0,T)$ (which is dense in $L^2(Q)$). It suffices to take y of the form $y(x,t) = \zeta(t) w(x)$, where $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_N)$ is defined by:

$$\begin{cases} \frac{d^2\zeta_i}{dt^2} + \lambda_i \zeta_i = f(t), \\ \zeta_i(0) = 0 \ ,\ \zeta_i(T) = 0, \end{cases} \quad \text{for every } i \in \{1, ..., N\},$$

which gives ζ, and $(3.1.3)$ has unique solution.■

3.1.2 The optimal control problem

Consider the equation $(3.1.3)$, let $\mathcal{U}_{ad} \subset L^2(Q)$ a non-empty closed convex subset of admissible controls, and consider the following cost function:

$$J(v,y) = \|y(v) - y_d\|^2_{L^2(Q)} + N \|v\|^2_{L^2(Q)} \tag{3.1.4}$$

where $v \in \mathcal{U}_{ad}$, $y_d \in L^2(Q)$ and $N > 0$.

We know that if there exists a pair $(v, y(v)) \in \mathcal{U}_{ad} \times L^2(Q)$ satisfying $(3.1.3)$, then, it is called a control-state feasible pair. Denote by χ_{ad} the set of all admissible feasible pairs. We suppose in what follows that χ_{ad} is non-empty (i.e., there exists at least one control), and we consider the following optimal control problem:

$$\inf_{(v;y) \in \mathcal{U}_{ad} \times L^2(Q)} J(v,y) \tag{3.1.5}$$

which has a unique solution (u, z) that we should characterize.

Lemma 3.1 *The optimal problem* $(3.1.3)-(3.1.5)$ *has one only solution* (u,z).

Proof. *The functional* $J : L^2(Q)\times L^2(Q)\to\mathbb{R}$ *is a lower semi-continuous function, strictly convex, and coercive. Hence, there is a unique admissible pair* (u,z) *solution to* $(3.1.3)-(3.1.5)$. *A classical method to solve this problem is the well-known penalization method, which consists in approximating the pair* (u,z) *by the solution of some penalized problem. More precisely, for some* $\varepsilon>0$, *we define the penalized cost function:*

$$J_\varepsilon(v,y)=J(v,y)+\frac{1}{2\varepsilon}\left\|y''-\triangle y-v\right\|^2_{L^2(Q)}.$$

The optimal pair $(u_\varepsilon,z_\varepsilon)$ *then converges to* (u,z) *when* $\varepsilon\to 0$.

The first order optimality conditions of Euler-Lagrange for $(u_\varepsilon,z_\varepsilon)$ *are the following:*

For z_ε

$$\begin{aligned}\frac{d}{dt}J_\varepsilon\left(u_\varepsilon,z_\varepsilon+t\left(y-z_\varepsilon\right)\right)\Big|_{t=0} &= \left(z_\varepsilon-y_d,y\right)_{L^2(Q)}+\frac{1}{\varepsilon}\left(z''_\varepsilon-\triangle z_\varepsilon-u_\varepsilon,y''-\triangle y\right)_{L^2(Q)} \quad (3.1.6)\\ &= 0 \quad \text{for every } y\in L^2(Q),\end{aligned}$$

and for u_ε

$$\begin{aligned}\frac{d}{dt}J_\varepsilon\left(u_\varepsilon+t\left(v-u_\varepsilon\right),z_\varepsilon\right)\Big|_{t=0} &= N\left(u_\varepsilon,v-u_\varepsilon\right)_{L^2(Q)}+\frac{1}{\varepsilon}\left(z''_\varepsilon-\triangle z_\varepsilon-u_\varepsilon,v-u_\varepsilon\right)_{L^2(Q)}\\ &\geq 0 \quad \text{for every } v\in\mathcal{U}_{ad}. \quad (3.1.7)\end{aligned}$$

■

In the following theorem, we shall use an approach based on the so-called Slater extra hypothesis (Lions, 1985) to obtain a singular optimality system.

Theorem 3.1 *Under hypothesis of Slater*

$$\mathcal{U}_{ad} \text{ has a non-empty interior} \tag{3.1.8}$$

there exists a unique $(u,z)\in\mathcal{U}_{ad}\times L^2(Q)$, *solution to the optimal control problem* $(3.1.3)-(3.1.5)$. *Moreover, this solution is characterized by the following singular optimality system:*

$$\left\{\begin{array}{ccl} z''-\triangle z=u; & p''-\triangle p=z-y_d & \text{in } Q,\\ z(x,0)=0,\ z(x,T)=0; & p(x,0)=0,\ p(x,T)=0 & \text{in } \Omega,\\ z=0; & p=0 & \text{on } \Sigma,\end{array}\right. \tag{3.1.9}$$

and the following variational inequality:

$$\int_0^T\int_\Omega (p+Nu)(v-u)\,dxdt\geq 0 \quad \forall v\in\mathcal{U}_{ad}.$$

Proof. *Again, we introduce the penalized cost function*

$$J_{\varepsilon}(v,y) = J(v,y) + \frac{1}{2\varepsilon}\|y'' - \triangle y - v\|^2_{L^2(Q)}.$$

Let $(u_{\varepsilon}, z_{\varepsilon})$ *be the solution of the optimal control problem*

$$\inf_{(v;y)\in\mathcal{U}_{ad}\times L^2(Q)} J_{\varepsilon}(v,y) \ \text{ such that } \ (v,y) \text{ verifies } (3.1.3).$$

We have

$$z''_{\varepsilon} - \triangle z_{\varepsilon} = u_{\varepsilon} + \sqrt{\varepsilon} f_{\varepsilon}, \ \|f_{\varepsilon}\|_{L^2(Q)} \leq C. \tag{3.1.10}$$

An optimality system is derived by taking

$$p_{\varepsilon} = -\frac{1}{\varepsilon}\left(z''_{\varepsilon} - \triangle z_{\varepsilon} - u_{\varepsilon}\right).$$

Introduce the operator R *given by*

$$\left|\begin{array}{l} D(R) = \{\varphi : \varphi, \varphi'' - \triangle\varphi \in L^2(Q), \varphi(0) = \varphi(T) = 0, \varphi = 0 \text{ on } \Sigma\}, \\ R\varphi = \varphi'' - \triangle\varphi, \end{array}\right.$$

then, from $(3.1.6)$

$$\left(z_{\varepsilon} - y_d, y\right)_{L^2(Q)} = \left(p_{\varepsilon}, Ry\right)_{L^2(Q)} \ , \forall y \in D(R), \tag{3.1.11}$$

and the optimality condition $(3.1.7)$ *is equivalent to*

$$(p_{\varepsilon} + Nu_{\varepsilon}, v - u_{\varepsilon})_{L^2(Q)} \geq 0 \quad \forall v \in \mathcal{U}_{ad}.$$

Then, we will get the result by passing to the limit when $\varepsilon \to 0$*, if we prove that* p_{ε} *is bounded i.e.,*

$$\exists C > 0 \text{ independent of } \varepsilon \text{ s.t. } \|p_{\varepsilon}\|_{L^2(Q)} \leq C.$$

By hypothesis $(3.1.8)$*, we can find* $v_0 \in \mathcal{U}_{ad}$ *and* $r > 0$ *s.t.*

$$\text{if } \|v - v_0\| \leq r \ \text{ then } v \in \mathcal{U}_{ad},$$

then,

$$\text{there exists } y_0 \in D(R) \ \text{ with } Ry_0 = v_0.$$

We have

$$(p_{\varepsilon} + Nu_{\varepsilon}, v - u_{\varepsilon})_{L^2(Q)} = X_{\varepsilon} + (p_{\varepsilon}, v - v_0)_{L^2(Q)},$$

with

$$X_{\varepsilon} = (p_{\varepsilon} + Nu_{\varepsilon}, v_0 - u_{\varepsilon})_{L^2(Q)} + (Nu_{\varepsilon}, v - v_0)_{L^2(Q)},$$

but on taking $y = y_0$ *in* $(3.1.11)$*, this yields*

$$(p_\varepsilon, v_0)_{L^2(Q)} = (z_\varepsilon - y_d, y_0)_{L^2(Q)},$$

and both taking $y = z_\varepsilon$ *and using* $(3.1.10)$ *this gives*

$$(p_\varepsilon, u_\varepsilon)_{L^2(Q)} = (z_\varepsilon - y_d, z_\varepsilon)_{L^2(Q)} + \|f_\varepsilon\|^2_{L^2(Q)},$$

therefore

$$|X_\varepsilon| \leq C,$$

thus

$$(p_\varepsilon, v - v_0)_{L^2(Q)} \geq -C \quad \forall v \in \mathcal{U}_{ad} \quad \textit{with} \quad \|v - v_0\|_{L^2(Q)} \leq r,$$

whence

$$\|p_\varepsilon\|_{L^2(Q)} \leq \frac{C}{r}.$$

■

The previous theorem proof is based on the hypothesis of Slater. Unfortunately, some sets like the convex cone $(L^2(Q))^+ = \{f \in L^2(Q) \text{ s.t. } f \geq 0\}$ which has an empty interior can be used as a set of admissible controls, for this reason, we'll give another approach to solve the optimal control problem $(3.1.3) - (3.1.5)$, it's the regularization approach.

3.1.3 Approximation by a sequence of parabolic equations

Let's approximate the wave equation $(3.1.3)$ by :

$$\begin{cases} y''_\varepsilon - \triangle y_\varepsilon - \varepsilon \Delta y'_\varepsilon = v & \text{in } Q, \\ y_\varepsilon = 0 & \text{on } \Sigma, \\ y_\varepsilon(x,0) = y_\varepsilon(x,T) = 0, y'_\varepsilon(x,0) = g & \text{in } \Omega, \end{cases} \tag{3.1.12}$$

where $g \in L^2(\Omega)$ is unknown, $\varepsilon > 0$. For any data (v, g) there exists a unique solution for the parabolic equation $(3.1.12)$ $y_\varepsilon = y_\varepsilon(v, g)$ (Lions, 1985), with the cost function:

$$J_\varepsilon(v, g) = \|y_\varepsilon(v, g) - y_d\|^2_{L^2(Q)} + N\|v\|^2_{L^2(Q)}. \tag{3.1.13}$$

Hence, we want to solve

$$\inf_{v \in \mathcal{U}_{ad}} J_\varepsilon(v, g) \quad \forall g \in L^2(\Omega),$$

it's an optimal control problem with missing data, to solve her, let's define the no-regret control for the approximated equation $(3.1.12)$ with $(3.1.13)$.

Definition 3.1 *We say that $u \in \mathcal{U}_{ad}$ is a no-regret control for* (3.1.12) (3.1.13) *if u is the solution of:*

$$\inf_{v \in \mathcal{U}_{ad}} \left(\sup_{g \in L^2(\Omega)} \left(J_\varepsilon (v, g) - J_\varepsilon (0, g) \right) \right). \tag{3.1.14}$$

Before continue, let's give the following lemma.

Lemma 3.2 *For every $v \in \mathcal{U}_{ad}$ and $g \in L^2(\Omega)$ we have:*

$$J_\varepsilon (v, g) - J_\varepsilon (0, g) = J_\varepsilon (v, 0) - J_\varepsilon (0, 0) + 2 \left(\xi_\varepsilon (0), g \right)_{L^2(\Omega)}, \tag{3.1.15}$$

where:

$$\begin{cases} \xi''_\varepsilon - \triangle \xi_\varepsilon + \varepsilon \Delta \xi'_\varepsilon = y_\varepsilon (v, 0) & \text{in } Q, \\ \xi_\varepsilon = 0 & \text{on } \Sigma, \\ \xi_\varepsilon (x, T) = 0 \ , \ \xi'_\varepsilon (x, T) = 0 & \text{in } \Omega. \end{cases} \tag{3.1.16}$$

Proof. We have:

$$J_\varepsilon (v, g) - J_\varepsilon (0, g) = J_\varepsilon (v, 0) - J_\varepsilon (0, 0) + 2 \left(y_\varepsilon (v, 0), y_\varepsilon (0, g) \right)_{L^2(Q)},$$

by (3.1.16) and using Green formula we get

$$\left(y_\varepsilon (v, 0), y_\varepsilon (0, g) \right)_{L^2(Q)} = \left(\xi_\varepsilon (0), g \right)_{L^2(\Omega)}.$$

■

Now, let's relax our optimal control problem by defining the low-regret control.

Definition 3.2 *We say that $u \in \mathcal{U}_{ad}$ is a low-regret control for* (3.1.12) (3.1.13) *if u is the solution of:*

$$\inf_{v \in \mathcal{U}_{ad}} \left(\sup_{g \in L^2(\Omega)} \left(J_\varepsilon (v, g) - J_\varepsilon (0, g) - \gamma \|g\|^2_{L^2(\Omega)} \right) \right). \tag{3.1.17}$$

Remark 3.2 *We have:*

$$\begin{aligned} \sup_{g \in L^2(\Omega)} \left(J_\varepsilon (v, g) - J_\varepsilon (0, g) - \gamma \|g\|^2_{L^2(\Omega)} \right) &= J_\varepsilon (v, 0) - J_\varepsilon (0, 0) \\ &\quad + \sup_{g \in L^2(\Omega)} \left(2 \left(\xi_\varepsilon (0), g \right)_{L^2(\Omega)} - \gamma \|g\|^2_{L^2(\Omega)} \right) \\ &= J_\varepsilon (v, 0) - J_\varepsilon (0, 0) + \frac{1}{\gamma} \|\xi_\varepsilon (0)\|^2_{L^2(\Omega)}, \forall v \in \mathcal{U}_{ad}. \end{aligned}$$

Then, (3.1.17) *is equivalent to the following classical optimal control problem :*

$$\inf_{v \in \mathcal{U}_{ad}} \mathcal{J}^\gamma_\varepsilon (v), \tag{3.1.18}$$

with

$$\mathcal{J}^\gamma_\varepsilon (v) = J_\varepsilon (v, 0) - J_\varepsilon (0, 0) + \frac{1}{\gamma} \|\xi_\varepsilon (0)\|^2_{L^2(\Omega)}. \tag{3.1.19}$$

We see that (3.1.18) (3.1.19) is a standard control problem. Then, we apply the classical theory of optimal control to solve her, and to characterize the low-regret control.

Lemma 3.3 *The problem* (3.1.12) (3.1.18) (3.1.19) *has a unique solution* u_ε^γ *called the approximate low-regret control.*

Proof. We have $\mathcal{J}_\varepsilon^\gamma(v) \geq -J_\varepsilon(0,0) = -\|y_d\|^2_{L^2(Q)}$for every $v \in \mathcal{U}_{ad}$, then $d = \inf\limits_{v\in\mathcal{U}_{ad}} \mathcal{J}_\varepsilon^\gamma(v)$ exists. Let (v_n) be a minimizing sequence with $d = \lim\limits_{n\to\infty} \mathcal{J}_\varepsilon^\gamma(v_n)$, then

$$-\|y_d\|^2_{L^2(Q)} \leq \mathcal{J}_\varepsilon^\gamma(v_n) = J_\varepsilon(v_n,0) - J_\varepsilon(0,0) + \frac{1}{\gamma}\|\xi_\varepsilon(0)\|^2_{L^2(\Omega)} \leq d+1,$$

this gives the bounds

$$\|v_n\|_{L^2(Q)} \leq C, \quad \frac{1}{\sqrt{\gamma}}\|\xi_\varepsilon(v_n)(0)\|_{L^2(\Omega)} \leq C, \;\; \|y_\varepsilon(v_n,0) - y_d\|_{L^2(Q)} \leq C,$$

where C is a positive constant independent of n.
Then, there exists u_ε^γ such that $v_n \rightharpoonup u_\varepsilon^\gamma$ weakly in $\mathcal{U}_{ad}$ (closed), also $y_\varepsilon(v_n,0) \rightharpoonup y_\varepsilon(u_\varepsilon^\gamma,0)$ in $L^2(Q)$ because of continuity w.r.t. the data. Since $\mathcal{J}_\varepsilon^\gamma(v)$ is strictly convex u_ε^γ is unique. ■
In the following proposition, we characterize the approximate low-regret control by an optimality system.

Proposition 3.1 *The approximate low-regret control* u_ε^γ *solution to* (3.1.12) (3.1.18) (3.1.19) *is characterized by the unique* $\{u_\varepsilon^\gamma, y_\varepsilon^\gamma, \xi_\varepsilon^\gamma, \rho_\varepsilon^\gamma, p_\varepsilon^\gamma\}$ *given by :*

$$\begin{cases}
y_\varepsilon^{\gamma\prime\prime} - \triangle y_\varepsilon^\gamma - \varepsilon\Delta y_\varepsilon^{\gamma\prime} = u_\varepsilon^\gamma, & \\
\xi_\varepsilon^{\gamma\prime\prime} - \triangle \xi_\varepsilon^\gamma + \varepsilon\Delta \xi_\varepsilon^{\gamma\prime} = y_\varepsilon^\gamma, & \\
\rho_\varepsilon^{\gamma\prime\prime} - \triangle \rho_\varepsilon^\gamma - \varepsilon\Delta \rho_\varepsilon^{\gamma\prime} = 0, & \\
p_\varepsilon^{\gamma\prime\prime} - \triangle p_\varepsilon^\gamma + \varepsilon\Delta p_\varepsilon^{\gamma\prime} = y_\varepsilon^\gamma - y_d + \rho_\varepsilon^\gamma & \textit{in } Q, \\
y_\varepsilon^\gamma = 0, \xi_\varepsilon^\gamma = 0, \rho_\varepsilon^\gamma = 0, p_\varepsilon^\gamma = 0 & \textit{on } \Sigma, \\
y_\varepsilon^\gamma(x,0) = y_\varepsilon^{\gamma\prime}(x,0) = 0, & \\
\xi_\varepsilon^\gamma(x,T) = \xi_\varepsilon^{\gamma\prime}(x,T) = 0, & \\
\rho_\varepsilon^\gamma(x,0) = 0, \rho_\varepsilon^{\gamma\prime}(x,0) = \frac{1}{\gamma}\xi_\varepsilon^\gamma(x,0), & \\
p_\varepsilon^\gamma(x,T) = p_\varepsilon^{\gamma\prime}(x,T) = 0 & \textit{in } \Omega,
\end{cases}$$

and the variational inequality:

$$(p_\varepsilon^\gamma + Nu_\varepsilon^\gamma, v - u_\varepsilon^\gamma)_{L^2(Q)} \geq 0 \;\; \forall v \in \mathcal{U}_{ad}.$$

Proof. A First order necessary condition for (3.1.18)(3.1.19) gives for every $w \in \mathcal{U}_{ad}$:

$$(y_\varepsilon^\gamma - y_d, y_\varepsilon(w,0))_{L^2(Q)} + (Nu_\varepsilon^\gamma, w)_{L^2(Q)} + \left(\frac{1}{\gamma}\xi_\varepsilon^\gamma(0), \xi_\varepsilon(0,w)\right)_{L^2(\Omega)} \geq 0,$$

with $y_\varepsilon^\gamma = y_\varepsilon(u_\varepsilon^\gamma, 0)$, and $\xi_\varepsilon^\gamma = \xi_\varepsilon(u_\varepsilon^\gamma)$. Let $\rho_\varepsilon^\gamma = \rho_\varepsilon(u_\varepsilon^\gamma)$ be a solution to:

$$\begin{cases} \rho_\varepsilon^{\gamma\prime\prime} - \triangle\rho_\varepsilon^\gamma - \varepsilon\Delta\rho_\varepsilon^{\gamma\prime} = 0 & \text{in } Q, \\ \rho_\varepsilon^\gamma = 0 & \text{on } \Sigma, \\ \rho_\varepsilon^\gamma(x,0) = 0, \rho_\varepsilon^{\gamma\prime}(x,0) = \frac{1}{\gamma}\xi_\varepsilon^\gamma(x,0) & \text{in } \Omega. \end{cases}$$

Use Green formula to get:

$$(\rho_\varepsilon^\gamma, y_\varepsilon(w,0))_{L^2(Q)} = (\rho_\varepsilon^{\gamma\prime}(x,0), \xi_\varepsilon(x,0))_{L^2(\Omega)} = \left(\frac{1}{\gamma}\xi_\varepsilon^\gamma(x,0), \xi_\varepsilon(x,0)\right)_{L^2(\Omega)}.$$

Introduce $p_\varepsilon^\gamma = p_\varepsilon(u_\varepsilon^\gamma)$ with:

$$\begin{cases} p_\varepsilon^{\gamma\prime\prime} - \triangle p_\varepsilon^\gamma + \varepsilon\Delta p_\varepsilon^{\gamma\prime} = y_\varepsilon^\gamma - y_d + \rho_\varepsilon^\gamma & \text{in } Q, \\ p_\varepsilon^\gamma = 0 & \text{on } \Sigma, \\ p_\varepsilon^\gamma(x,T) = p_\varepsilon^{\gamma\prime}(x,T) = 0 & \text{in } \Omega, \end{cases}$$

then,

$$(y_\varepsilon^\gamma - y_d + \rho_\varepsilon^\gamma, y_\varepsilon(w,0))_{L^2(Q)} = (p_\varepsilon^\gamma, w)_{L^2(Q)}.$$

Finally, the optimality condition is equivalent to:

$$(p_\varepsilon^\gamma + N u_\varepsilon^\gamma, w)_{L^2(Q)} \geq 0 \text{ for every } w \in \mathcal{U}_{ad}.$$

■

The next step, is the characterization of the approximate no-regret control. Before doing this, we have to get some estimates on $\{u_\varepsilon^\gamma, y_\varepsilon^\gamma, \xi_\varepsilon^\gamma, \rho_\varepsilon^\gamma, p_\varepsilon^\gamma\}$, for this we announce the following proposition.

Proposition 3.2 *There is some $C > 0$ independent of γ s.t. :*

$$\|u_\varepsilon^\gamma\|_{L^2(Q)} \leq C, \ \ \|y_\varepsilon^\gamma\|_{L^2(Q)} \leq C, \ \frac{1}{\sqrt{\gamma}}\|\xi_\varepsilon^\gamma(0)\|_{L^2(\Omega)} \leq C,$$

$$\|y_\varepsilon^{\gamma\prime}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|y_\varepsilon^\gamma\|^2_{L^\infty\left(0,T;H_0^1(\Omega)\right)} \leq C\,,\ \varepsilon\|y_\varepsilon^{\gamma\prime}\|^2_{L^\infty\left(0,T;H_0^1(\Omega)\right)} \leq C\,, \tag{3.1.22}$$

$$\|\xi_\varepsilon^{\gamma\prime}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\xi_\varepsilon^\gamma\|^2_{L^\infty\left(0,T;H_0^1(\Omega)\right)} \leq C\,,\ \varepsilon\|\xi_\varepsilon^{\gamma\prime}\|^2_{L^\infty\left(0,T;H_0^1(\Omega)\right)} \leq C, \tag{3.1.23}$$

$$\|\rho_\varepsilon^{\gamma\prime}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\rho_\varepsilon^\gamma\|^2_{L^\infty\left(0,T;H_0^1(\Omega)\right)} \leq C,\ \varepsilon\|\rho_\varepsilon^{\gamma\prime}\|^2_{L^\infty\left(0,T;H_0^1(\Omega)\right)} \leq C, \tag{3.1.24}$$

$$\|p_\varepsilon^{\gamma\prime}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|p_\varepsilon^\gamma\|^2_{L^\infty\left(0,T;H_0^1(\Omega)\right)} \leq C,\ \varepsilon\|p_\varepsilon^{\gamma\prime}\|^2_{L^\infty\left(0,T;H_0^1(\Omega)\right)} \leq C. \tag{3.1.25}$$

Proof. u_ε^γ is the approximate low-regret control, then

$$\mathcal{J}_\varepsilon^\gamma\left(u_\varepsilon^\gamma\right) \leq \mathcal{J}_\varepsilon^\gamma\left(v\right) \quad \forall v \in \mathcal{U}_{ad}.$$

In particular when $v = 0$

$$J_\varepsilon\left(u_\varepsilon^\gamma, 0\right) - J_\varepsilon\left(0, 0\right) + \frac{1}{\gamma}\left\|\xi_\varepsilon^\gamma\left(0\right)\right\|_{L^2(\Omega)}^2 \leq \frac{1}{\gamma}\left\|\xi_\varepsilon\left(0; 0\right)\right\|_{L^2(\Omega)}^2,$$

but $\xi_\varepsilon\left(0; 0\right) = 0$ in $[0, T] \times \overline{\Omega}$ so

$$\left\|y_\varepsilon\left(u_\varepsilon^\gamma, 0\right) - y_d\right\|_{L^2(Q)}^2 + N\left\|u_\varepsilon^\gamma\right\|_{L^2(Q)}^2 + \frac{1}{\gamma}\left\|\xi_\varepsilon^\gamma\left(0\right)\right\|_{L^2(\Omega)}^2 \leq \left\|y_d\right\|_{L^2(Q)} = \text{constant},$$

then, we obtain $(3.1.21)$.
For $(3.1.22)$, multiply by $y_\varepsilon^{\gamma\prime}$and integrate over $(0, t) \times \Omega$ to get:

$$\frac{1}{2}\left\|y_\varepsilon^{\gamma\prime}\left(t\right)\right\|_{L^2(\Omega)}^2 + \frac{1}{2}\left\|\nabla y_\varepsilon^\gamma\left(t\right)\right\|_{L^2(\Omega)}^2 \leq \int_0^t \left(u_\varepsilon^\gamma, y_\varepsilon^{\gamma\prime}\right) d\sigma$$

$$\Rightarrow \left\|y_\varepsilon^{\gamma\prime}\left(t\right)\right\|_{L^2(\Omega)}^2 + \left\|y_\varepsilon^\gamma\left(t\right)\right\|_{H_0^1(\Omega)}^2 \leq \int_0^t \left[\left\|y_\varepsilon^{\gamma\prime}\left(s\right)\right\|_{L^2(\Omega)}^2 + \left\|u_\varepsilon^\gamma\left(s\right)\right\|_{L^2(Q)}^2\right] ds$$

and by using Gronwall lemma we obtain the first part of $(3.1.22)$. For the second one, we do the same: integrating over $(0, t) \times \Omega$ to obtain

$$2\varepsilon\left\|\nabla y_\varepsilon^{\gamma\prime}\left(t\right)\right\|_{L^2(\Omega)}^2 \leq \left\|u_\varepsilon^\gamma\right\|_{L^2(\Omega)}^2 + \left\|y_\varepsilon^{\gamma\prime}\left(t\right)\right\|_{L^2(\Omega)}^2$$

$$\Rightarrow \varepsilon\left\|y_\varepsilon^{\gamma\prime}\right\|_{L^\infty\left(0,T;H_0^1(\Omega)\right)}^2 \leq C$$

The estimates $(3.1.23)$, $(3.1.24)$ and $(3.1.25)$ follows by the same way. ■
Now, we can announce the following theorem characterizing the approximate no-regret control.

Theorem 3.2 *The approximate no-regret control* $u_\varepsilon = \lim_{\gamma\to 0} u_\varepsilon^\gamma$ *for the approximated equation* $(3.1.16)$ *is characterized by the unique* $\{u_\varepsilon, y_\varepsilon, \xi_\varepsilon, \rho_\varepsilon, p_\varepsilon\}$ *given by:*

$$\begin{cases} y_\varepsilon'' - \triangle y_\varepsilon - \varepsilon\Delta y_\varepsilon' = u_\varepsilon, & \\ \xi_\varepsilon'' - \triangle \xi_\varepsilon + \varepsilon\Delta \xi_\varepsilon' = y_\varepsilon, & \\ \rho_\varepsilon'' - \triangle \rho_\varepsilon - \varepsilon\Delta \rho_\varepsilon' = 0, & \\ p_\varepsilon'' - \triangle p_\varepsilon + \varepsilon\Delta p_\varepsilon' = y_\varepsilon - y_d + \rho_\varepsilon & \text{in } Q, \\ y_\varepsilon = 0,\ \xi_\varepsilon = 0,\ \rho_\varepsilon = 0,\ p_\varepsilon = 0 & \text{on } \Sigma, \\ y_\varepsilon\left(x, 0\right) = y_\varepsilon'\left(x, 0\right) = 0, & \\ \xi_\varepsilon\left(x, T\right) = \xi_\varepsilon'\left(x, T\right) = 0, & \\ \rho_\varepsilon\left(x, 0\right) = 0, \rho_\varepsilon'\left(x, 0\right) = \lambda_\varepsilon\left(x, 0\right), & \\ p_\varepsilon\left(x, T\right) = p_\varepsilon'\left(x, T\right) = 0 & \text{in } \Omega, \end{cases} \tag{3.1.26}$$

with the following limits

$$y_\varepsilon = \lim_{\gamma\to 0} y_\varepsilon^\gamma,\ \xi_\varepsilon = \lim_{\gamma\to 0} \xi_\varepsilon^\gamma,\ \ \rho_\varepsilon = \lim_{\gamma\to 0} \rho_\varepsilon^\gamma,\ \ p_\varepsilon = \lim_{\gamma\to 0} p_\varepsilon^\gamma,$$

and the following variational inequality:

$$(p_\varepsilon + N u_\varepsilon, v - u_\varepsilon)_{L^2(Q)} \geq 0\ \ \forall v \in \mathcal{U}_{ad},$$

where

$$u_\varepsilon, y_\varepsilon, \xi_\varepsilon, \rho_\varepsilon, p_\varepsilon \in L^2(Q),\ \lambda_\varepsilon(x,0) \in L^2(\Omega).$$

Proof. By (3.1.21), $y_\varepsilon^\gamma \rightharpoonup y_\varepsilon$ weakly in $L^2(Q)$, then $y_\varepsilon^\gamma \to y_\varepsilon$ in $\mathcal{D}'(Q)$ (the space of distribution on Q), and

$$y_\varepsilon^{\gamma\prime\prime} - \triangle y_\varepsilon^\gamma - \varepsilon\Delta y_\varepsilon^{\gamma\prime} \to y_\varepsilon'' - \triangle y_\varepsilon - \varepsilon\Delta y_\varepsilon' \text{ in } \mathcal{D}'(Q),$$

with

$$u_\varepsilon^\gamma \rightharpoonup u_\varepsilon \text{ weakly in } L^2(Q).$$

By the uniqueness of the limit

$$y_\varepsilon'' - \triangle y_\varepsilon - \varepsilon\Delta y_\varepsilon' = u_\varepsilon \text{ in } L^2(Q).$$

For ξ_ε system, by (3.1.23) and continuous embedding of L^∞ in L^2 we conclude that

$$\|\xi_\varepsilon^{\gamma\prime}\|^2_{L^2(0,T;L^2(\Omega))} + \|\xi_\varepsilon^\gamma\|^2_{L^2\left(0,T;H_0^1(\Omega)\right)} \leq C,$$

proceed as the last paragraph to get

$$\xi_\varepsilon'' - \triangle\xi_\varepsilon + \varepsilon\Delta\xi_\varepsilon' = y_\varepsilon \text{ in } L^2\left(0,T;L^2(\Omega)\right).$$

Other systems in (3.1.26) follows by the same way using estimates (3.1.24) (3.1.25), except initial condition

$$\rho_\varepsilon^{\gamma\prime}(x,0) = \frac{1}{\gamma}\xi_\varepsilon^\gamma(x,0) \rightharpoonup \lambda_\varepsilon \text{ in } L^2(\Omega),$$

results from (3.1.21). ■

Theorem 3.3 *The no-regret control* $u = \lim_{\varepsilon\to 0} u_\varepsilon$ *for the ill-posed wave equation* (3.1.3) *is character-*

ized by the unique $\{u, y, \xi, \rho, p\}$ *given by:*

$$\begin{cases} y'' - \triangle y = u, \\ \xi'' - \triangle \xi = y, \\ \rho'' - \triangle \rho = 0, \\ p'' - \triangle p = y - y_d + \rho & \text{in } Q, \\ y = 0, \xi = 0, \rho = 0, p = 0 & \text{on } \Sigma, \\ y(x,0) = y'(x,0) = 0, \\ \xi(x,T) = \xi'(x,T) = 0, \\ \rho(x,0) = 0,\ \rho'(x,0) = \lambda, \\ p(x,T) = p'(x,T) = 0 & \text{in } \Omega, \end{cases} \tag{3.1.27}$$

with the following weak limits

$$y = \lim_{\varepsilon \to 0} y_\varepsilon,\ \xi = \lim_{\varepsilon \to 0} \xi_\varepsilon,\ \ \rho = \lim_{\varepsilon \to 0} \rho_\varepsilon,\ \ p = \lim_{\varepsilon \to 0} p_\varepsilon,$$

and the following variational inequality:

$$(p + Nu, v - u)_{L^2(Q)} \geq 0\,, \forall v \in \mathcal{U}_{ad},$$

where

$$u, y, \xi, \rho, p \in L^2(Q) \text{ and } \lambda \in L^2(\Omega).$$

Proof. See (Lions, 1985). ■

3.1.4 Corrector of order 0 (information about $y(x,T)$)

In singular optimality system $(3.1.27)$, that the passage to limit when $\varepsilon \to 0$ gives no information about $y(x,T)$, to complete this information for this we shall use the notion of zero order corrector for parabolic regularization introduced in (Lions, 1973).
Before defining zero order corrector, let's start by making the following regularity hypothesis

$$y, y' \in L^2\left(0, T, H_0^1(\Omega)\right). \tag{3.1.28}$$

Definition 3.3 *We say that* θ_ε *is a zero order corrector if*

$$\left| \begin{array}{c} (\theta''_\varepsilon, \varphi)_{L^2(\Omega)} + \varepsilon (\nabla \theta'_\varepsilon, \nabla \varphi)_{L^2(\Omega)} + (\nabla \theta_\varepsilon, \nabla \varphi)_{L^2(\Omega)} = \langle \varepsilon f_{\varepsilon 1} + \sqrt{\varepsilon} f_{\varepsilon 2}, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}\ \forall \varphi \in H_0^1(\Omega), \\ \theta_\varepsilon + y_\varepsilon \in H_0^1(\Omega), \\ \theta_\varepsilon(0) = 0, \ \ \theta_\varepsilon(T) + y_\varepsilon(T) = 0, \end{array} \right. \tag{3.1.29}$$

where

$$\left| \begin{array}{l} \|f_{\varepsilon 1}\|_{L^2(0,T,H^{-1}(\Omega))} \leq C, \\ \|f_{\varepsilon 2}\|_{L^2(0,T,H^{-1}(\Omega))} \leq C. \end{array} \right.$$

Remark 3.3 *If θ_ε is a zero order corrector then $m\theta_\varepsilon$ is also a zero order corrector, then we'll take the corrector $m\theta_\varepsilon$ with*

$$m = \left| \begin{array}{l} 1 \text{ in the neighborhood of } t = T, \\ 0 \text{ in the neighborhood of } t = 0. \end{array} \right. \tag{3.1.30}$$

Theorem 3.4 *Let θ_ε be a corrector of order 0 defined by* (3.1.29) *and* (3.1.29)*, then*

$$\|y_\varepsilon - (\theta_\varepsilon + y)\|^2_{L^\infty\left(0,T;H_0^1(\Omega)\right)} + \|y'_\varepsilon - (\theta'_\varepsilon + y')\|^2_{L^\infty(0,T;L^2(\Omega))} \leq C\sqrt{\varepsilon}, \tag{3.1.31}$$

$$[y'_\varepsilon - (\theta'_\varepsilon + y')] \rightharpoonup 0 \text{ weakly in } L^2\left(0,T,H_0^1(\Omega)\right). \tag{3.1.32}$$

Proof. Let $w_\varepsilon = y_\varepsilon - (\theta_\varepsilon + y)$ then for every $\varphi \in H_0^1(\Omega)$, we have

$$(w''_\varepsilon, \varphi)_{L^2(\Omega)} + \varepsilon (\nabla w'_\varepsilon, \nabla\varphi)_{L^2(\Omega)} + (\nabla w_\varepsilon, \nabla\varphi)_{L^2(\Omega)} = -\varepsilon (\nabla y'_\varepsilon, \nabla\varphi)_{L^2(\Omega)} - \left\langle \varepsilon f_{\varepsilon 1} + \sqrt{\varepsilon} f_{\varepsilon 2}, \varphi \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}$$

and

$$w_\varepsilon(0) = 0, \quad w'_\varepsilon(0) = 0.$$

Choose $\varphi = w'_\varepsilon$, then

$$\frac{1}{2}\frac{d}{dt}\|w'_\varepsilon\|^2_{L^2(\Omega)} + \varepsilon\|w'_\varepsilon\|^2_{H_0^1(\Omega)} + \frac{1}{2}\frac{d}{dt}\|w_\varepsilon\|^2_{H_0^1(\Omega)} = -\varepsilon (\nabla y'_\varepsilon, \nabla w'_\varepsilon)_{L^2(\Omega)} - \left\langle \varepsilon f_{\varepsilon 1} + \sqrt{\varepsilon} f_{\varepsilon 2}, w'_\varepsilon \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)},$$

by integration over $(0, t)$

$$\begin{aligned} &\|w'_\varepsilon(t)\|^2_{L^2(\Omega)} + \|w_\varepsilon(t)\|^2_{H_0^1(\Omega)} + 2\varepsilon\int_0^t \|w'_\varepsilon(s)\|^2_{H_0^1(\Omega)} ds \\ \leq\ & C\varepsilon\left(\int_0^t \|w'_\varepsilon(s)\|^2_{H_0^1(\Omega)} ds\right)^{\frac{1}{2}} + C\sqrt{\varepsilon}\left(\int_0^t \|w_\varepsilon(s)\|^2_{H_0^1(\Omega)} ds\right)^{\frac{1}{2}}, \forall\varepsilon > 0. \end{aligned}$$

Make $t = T$, then, take the supremum on $(0, T)$ to get

$$\begin{aligned} &\|w'_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))} + \|w_\varepsilon\|^2_{L^\infty\left(0,T;H_0^1(\Omega)\right)} + 2\varepsilon\|w'_\varepsilon\|^2_{L^2\left(0,T;H_0^1(\Omega)\right)} \\ \leq\ & C\sqrt{\varepsilon}\left(\sqrt{\varepsilon}\|w'_\varepsilon\|_{L^2\left(0,T;H_0^1(\Omega)\right)} + \|w_\varepsilon\|_{L^2\left(0,T;H_0^1(\Omega)\right)}\right), \end{aligned}$$

which gives (3.1.31) and

$$\|w'_\varepsilon\|_{L^2\left(0,T;H_0^1(\Omega)\right)} \leq C,$$

we deduce (3.1.32). Also, we have

$$\|w_\varepsilon\|_{L^\infty\left(0,T;H_0^1(\Omega)\right)} \leq C.$$

■

Now, by using zero order corrector with (3.1.31) and (3.1.32) we can complete information about $y(x,T)$ and announce the next theorem:

Theorem 3.5 *The quadruplet $\{u, y, \xi, \rho, p\}$ satisfies by the mean of zero order corrector:*

$$\begin{cases} y'' - \triangle y = u, \\ \xi'' - \triangle \xi = y, \\ \rho'' - \triangle \rho = 0, \\ p'' - \triangle p = y - y_d + \rho & \text{in } Q, \\ y = 0, \xi = 0, \rho = 0, p = 0 & \text{on } \Sigma, \\ y(0) = y(T) = 0, \\ \xi(T) = \xi'(T) = 0, \\ \rho(0) = 0, \rho'(0) = \lambda(0), \\ p(T) = p'(T) = 0 & \text{in } \Omega, \end{cases}$$

and the variational inequality:

$$(p + Nu, v - u)_{L^2(Q)} \geq 0, \ \forall v \in \mathcal{U}_{ad}.$$

3.2 Optimal control of electromagnetic wave displacement with an unknown velocity of propagation and a missing boundary condition

In this section, we consider an electromagnetic wave equation that describes the propagation of electromagnetic waves through a medium (Jackson, 1998) transparent, isotropic and homogeneous. The main problem is that we do not have information about medium permeability and primitivity in many applications. Therefore, the phase velocity that depends on permeability and primitivity is also missing. Moreover, boundary condition value is unknown, which makes us in front of a problem with incomplete information.

Here, our main goal is to act on waves displacement to be closer to a desired displacement observation by working on the source of the waves, in other words, controlling the source of waves. As a motivating example, in biomedical phenomena the X-rays could damage cells, to avoid their harmful effects we have to make the displacement and consequently the energy suitable for the burden of living cells.

Actually, this problem has two kinds of incomplete data: the first is an unknown velocity of propagation parameter, the second is a missing boundary condition, we want to get an optimal control independent of incomplete data. To achieve this goal, we shall use the concepts of averaged control and no-regret control introduced for the first and second kind of incomplete data respectively, to introduce the *averaged no-regret control*.

3.2.1 Position of problem and preliminaries

Consider the following wave equation with a missing parameter σ and with an unknown Dirichlet boundary condition:

$$\begin{cases} \frac{d^2y}{dt^2} - \sigma^2 \Delta y = v & \text{in } Q, \\ y = g & \text{on } \Sigma, \\ y(x,0) = 0,\, \frac{dy}{dt}(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{3.2.1}$$

where Ω is an open bounded domain in $\mathbb{R}^n$, $n = 1, 2$ or 3 with smooth boundary Γ, $t \in [0,T]\,, T > 0$, $Q = \Omega \times]0,T[\,$, $\Sigma = \Gamma \times]0,T[$, σ is the velocity of propagation with $0 < \sigma_1 \leq \sigma \leq \sigma_2 < \infty$ where $\sigma_1$is the minimum speed and σ_2 is the speed of light, $v \in \mathcal{U}_{ad}$ is a distributed control which doesn't depend on σ, $\mathcal{U}_{ad}$ is a non-empty closed convex subset of $L^2(Q)$, g is an unknown function independent of σ and belongs to $L^2(\Sigma)$. Let $y(v,g,\sigma) = y(v,g,\sigma;x,t) \in C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega))$ be the unique weak solution of (3.2.1) (Yamamoto, 1999) which depends continuously on σ. Let $\int_{\sigma_1}^{\sigma_2} y(v,g;\sigma)\, d\sigma$ be the average of the state with respect to σ (Zuazua, 2014).

Consider the quadratic objective function

$$j(v,g,\sigma) = \|y(v,g;\sigma) - y_d\|^2_{L^2(Q)} + N\|v\|^2_{L^2(Q)}\,, \tag{3.2.2}$$

where y_d is a desired state in $L^2(Q)$, $N > 0$. We want to find the optimal control u solution of

$$\inf_{v \in \mathcal{U}_{ad}} j(v,g,\sigma) \text{ for every } g \in L^2(\Sigma) \text{ and every } \sigma \in [\sigma_1, \sigma_2]\,,$$

where $y(v,g,\sigma)$ solves (3.2.1). Naturally, the optimal control u depends on g and σ. One thought to take

$$\inf_{v \in \mathcal{U}_{ad}} \sup_{g \in L^2(\Sigma)} j(v,g,\sigma)\,,$$

but one can get $\sup\limits_{g \in L^2(\Sigma)} j(v,g,\sigma) = +\infty$, to avoid this difficulty another idea was given in (Lions, 1992), it's to look only for controls such that

$$j(v,g,\sigma) \leq j(0,g,\sigma) \;\; \forall g \in L^2(\Sigma) \;\; \forall \sigma \in [\sigma_1, \sigma_2]\,. \tag{3.2.3}$$

Another difficulty arises here, it's the unknown datum σ, the work with cost function $(3.2.2)$ leads to an optimal control which depends on σ, contrariwise we seek to get an optimal control independent of missing data σ and g. To reach our goal, let's take an averaged observation of the state y with respect to the unknown parameter σ (original idea by Zuazua (2014)), instead of the state itself, in other words, we shall substitute the state by its average in the cost function $(3.2.2)$ i.e. define a new cost function by

$$J(v,g) = \left\| \int_{\sigma_1}^{\sigma_2} y(v,g,\sigma)\, d\sigma - z_d \right\|_{L^2(Q)}^2 + N \|v\|_{L^2(Q)}^2, \tag{3.2.4}$$

where z_d is an averaged desired state observation in $L^2(Q)$.

3.2.2 Averaged no-regret control and averaged low-regret control definitions for the electromagnetic wave with missing data

First of all, we define the averaged no-regret control and the averaged low-regret control for the optimal control problem with missing data $(3.2.1)\,(3.2.4)$.

Definition 3.4 *We say that $u \in \mathcal{U}_{ad}$ is an averaged no-regret control for $(3.2.1)\,(3.2.4)$ if u is a solution of*

$$\inf_{v\in\mathcal{U}_{ad}} \sup_{g\in L^2(\Sigma)} \left(J(v,g) - J(0,g)\right), \tag{3.2.5}$$

Let's try to separate the roles of the control v and the missing data g as follows:

Lemma 3.4 *For all $v \in \mathcal{U}_{ad}$ and $g \in L^2(\Sigma)$ we have*

$$J(v,g) - J(0,g) = J(v,0) - J(0,0) + 2(\sigma_1 - \sigma_2) \int_0^T \int_\Gamma t \frac{\partial \xi(v)}{\partial \nu} g \, d\Gamma dt, \tag{3.2.6}$$

where $\xi(v)$ is given by the following backward wave equation

$$\begin{cases} \frac{d^2\xi(v)}{dt^2} - \Delta\xi(v) = \frac{1}{t}\int_{\sigma_1}^{\sigma_2} y(v,0,\sigma)\, d\sigma & \text{in } Q, \\ \xi(v) = 0 & \text{on } \Sigma, \\ \xi(v)(x,T) = 0,\ \frac{d\xi(v)}{dt}(x,T) = 0 & \text{in } \Omega, \end{cases} \tag{3.2.7}$$

which has a unique solution in $C([0,T], H_0^1(\Omega)) \cap C^1([0,T], L^2(\Omega))$ with the hidden regularity propriety

$$\frac{\partial \xi(v)}{\partial \nu} \in L^2(\Sigma). \tag{3.2.8}$$

Proof. As usually, we get

$$J(v,g)-J(0,g)=J(v,0)-J(0,0)+2\int_0^T\int_\Omega\left(\int_{\sigma_1}^{\sigma_2}y(v,0,\sigma)\,d\sigma\right)\left(\int_{\sigma_1}^{\sigma_2}y(0,g,\sigma)\,d\sigma\right)dxdt.$$

Before completing the proof we have to find the system that describes $\int_{\sigma_1}^{\sigma_2}y(0,g,\sigma)\,d\sigma$. We note that $y(0,g,\sigma;x,t)=Y(0,g,1;x,\sigma t)$ where $Y(x,t)$ is a solution of

$$\begin{cases}\frac{d^2Y}{dt^2}-\Delta Y=0 & \text{in } Q,\\ Y(x,\sigma t)=g(x,t) & \text{on } \Sigma,\\ Y(x,0)=0,\ \frac{dY}{dt}(x,0)=0 & \text{in } \Omega,\end{cases}$$

then

$$\int_{\sigma_1}^{\sigma_2}y(0,g,\sigma;x,t)\,d\sigma=\int_{\sigma_1}^{\sigma_2}Y(0,g,1;x,\sigma t)\,d\sigma=\frac{Z(x,\sigma_2 t)-Z(x,\sigma_1 t)}{t}$$

where $Z(x,t)=\int_0^t Y(0,g,1;x,s)\,ds$ verifies

$$\begin{cases}\frac{d^2Z}{dt^2}-\Delta Z=0 & \text{in } Q,\\ Z(x,t)=\int_0^t Y(x,s)\,ds & \text{on } \Sigma,\\ Z(x,0)=0,\ \frac{dZ}{dt}(x,0)=0 & \text{in } \Omega,\end{cases}$$

and now we can write (by using Green formula, Appendices, Theorem 2)

$$\begin{aligned}&\int_0^T\int_\Omega\left(\int_{\sigma_1}^{\sigma_2}y(v,0,\sigma)\,d\sigma\right)\left(\int_{\sigma_1}^{\sigma_2}y(0,g,\sigma)\,d\sigma\right)dxdt\\ =&\int_0^T\int_\Omega\left(\int_{\sigma_1}^{\sigma_2}y(v,0,\sigma)\,d\sigma\right)\frac{Z(x,\sigma_2 t)-Z(x,\sigma_1 t)}{t}dxdt\\ =&\int_0^T\int_\Omega\left(\frac{d^2\xi}{dt^2}-\Delta\xi\right)(Z(x,\sigma_2 t)-Z(x,\sigma_1 t))\,dxdt\\ =&-\int_0^T\int_\Gamma\frac{\partial\xi}{\partial\nu}\int_{\sigma_1 t}^{\sigma_2 t}g\left(x,\frac{s}{\sigma}\right)dsd\Gamma dt\\ =&(\sigma_1-\sigma_2)\int_0^T\int_\Gamma t\frac{\partial\xi}{\partial\nu}gd\Gamma dt.\end{aligned}$$

For the wellposedness of (3.2.7) and regularity propriety (3.2.8): since $\frac{1}{t}\left(\int_{\sigma_1}^{\sigma_2}y(v,0,\sigma)\,d\sigma\right)\in L^2(Q)\subset L^1(0,T;L^2(\Omega))$ (by L' Hopital rule we have $\lim_{t\to 0}\left|\frac{1}{t}\left(\int_{\sigma_1}^{\sigma_2}y(v,0,\sigma)\,d\sigma\right)\right|^2=0$) we refer to (Medeiros et al., 2013, chapter 4) to get the main results. ■

Remark that the no-regret control belongs to

$$K=\left\{v\in L^2(Q)\ \text{such that}\ \int_0^T\int_\Gamma t\frac{\partial\xi(v)}{\partial\nu}gd\Gamma dt=0\quad\forall g\in L^2(\Sigma)\right\}.$$

Unfortunately, the set K is too hard to characterize, so the main problem with no-regret control is the difficulty of its characterization, to avoid her we relax the no-regret definition by making some quadratic perturbation to define the averaged low-regret control as follows:

Definition 3.5 *We say that* $u \in \mathcal{U}_{ad}$ *is an averaged low-regret control for* (3.2.1) (3.2.4) *if* u *is a solution of*

$$\inf_{v\in\mathcal{U}_{ad}} \sup_{g\in L^2(\Sigma)} \left(J(v,g) - J(0,g) - \gamma \|g\|^2_{L^2(\Sigma)}\right) \qquad \text{for every } \gamma > 0.$$

Now, by using (3.2.6) we can write

$$\begin{aligned} &\sup_{g\in L^2(\Sigma)} \left(J(v,g) - J(0,g) - \gamma \|g\|^2_{L^2(\Sigma)}\right) \\ = \;& J(v,0) - J(0,0) + \\ &\sup_{g\in L^2(\Sigma)} \left(2(\sigma_1-\sigma_2)\int_0^T\int_\Gamma t\frac{\partial\xi(v)}{\partial\nu} g\,d\Gamma dt - \gamma\|g\|^2_{L^2(\Sigma)}\right) \\ = \;& J(v,0) - J(0,0) + \frac{(\sigma_2-\sigma_1)}{\gamma}\left\|t\frac{\partial\xi(v)}{\partial\nu}\right\|^2_{L^2(\Sigma)} \quad \text{(by using Legendre transform).} \end{aligned}$$

In the end, we get a new optimal control problem with a new objective function not related to the missing boundary value g and the unknown parameter σ.

$$\inf_{v\in L^2(Q)} \mathcal{J}_\gamma(v) \text{ such that } \mathcal{J}_\gamma(v) = J(v,0) - J(0,0) + \frac{(\sigma_2-\sigma_1)}{\gamma}\left\|t\frac{\partial\xi(v)}{\partial\nu}\right\|^2_{L^2(\Sigma)}. \tag{3.2.9}$$

In another word, we are in front of a standard optimal control problem (3.2.1) (3.2.9) where we can apply the classical optimal control theory and announce the following theorems characterizing the low-regret control.

3.2.3 Averaged low-regret control characterization

Theorem 3.6 *There exists a unique averaged low-regret control* u_γ *solution to* (3.2.1) (3.2.9).

Proof. Observe that $\mathcal{J}_\gamma(v) \geq -J(0,0)$ which means that (3.2.9) has a solution. Let $(v_n) \subset L^2(Q)$ a minimizing sequence such that

$$\mathcal{J}_\gamma(v_n) \underset{n\to+\infty}{\longrightarrow} \inf_{v\in L^2(Q)} \mathcal{J}_\gamma(v) = d_\gamma, \tag{3.2.10}$$

where $y_n = y(v_n, 0; \sigma)$ is solution of

$$\begin{cases} \frac{d^2 y_n}{dt^2} - \sigma^2 \Delta y_n = v_n & \text{in } Q, \\ y_n = 0 & \text{on } \Sigma, \\ y_n(x,0) = 0,\ \frac{dy_n}{dt}(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{3.2.11}$$

We also have

$$-J(0,0) \leq \mathcal{J}_\gamma(v_n) = J(v_n, 0) - J(0,0) + \frac{(\sigma_2 - \sigma_1)}{\gamma} \left\| t \frac{\partial \xi(v_n)}{\partial \nu} \right\|^2_{L^2(\Sigma)} \leq d_\gamma + 1,$$

which gives the bounds

$$\|v_n\|_{L^2(Q)} \leq C_\gamma, \tag{3.2.12a}$$

$$\left\| t \frac{\partial \xi(v_n)}{\partial \nu} \right\|_{L^2(\Sigma)} \leq C_\gamma (\sigma_2 - \sigma_1) \sqrt{\gamma}, \tag{3.2.12b}$$

$$\left\| \int_{\sigma_1}^{\sigma_2} y(v_n, 0; \sigma)\, d\sigma \right\|_{L^2(Q)} \leq C_\gamma, \tag{3.2.12c}$$

where C_γ is a constant independent of n. By continuity with respect to data we also conclude that (Miranda, Medeiros & Louredo, 2013)

$$\left\| \frac{dy_n}{dt} \right\|^2_{L^\infty(0,T;L^2(\Omega))} + \sigma^2 \|y_n\|^2_{L^\infty\left(0,T;H^1_0(\Omega)\right)} \leq C \|v_n\|_{L^2(Q)} \leq C_\gamma. \tag{3.2.12d}$$

We deduce from $(3.2.11)$ and $(3.2.12a)$ that

$$\left\| \frac{d^2 y_n}{dt^2} - \sigma^2 \Delta y_n \right\|_{L^2(Q)} \leq C_\gamma. \tag{3.2.12e}$$

Then there exist subsequences still denoted (v_n), (y_n) and $\int_{\sigma_1}^{\sigma_2} y(v_n, 0; \sigma)\, d\sigma$ such that

$$v_n \rightharpoonup u_\gamma \text{ weakly in } L^2(Q), \tag{3.2.13a}$$

$$y_n \rightharpoonup y_\gamma \text{weakly in } L^\infty\left(0,T; H^1_0(\Omega)\right), \tag{3.2.13b}$$

$$\int_{\sigma_1}^{\sigma_2} y(v_n, 0; \sigma)\, d\sigma \rightharpoonup z_\gamma \text{ weakly in } L^2(Q), \tag{3.2.13c}$$

$$\frac{d^2 y_n}{dt^2} - \sigma^2 \Delta y_n \rightharpoonup f_1 \text{ weakly in } L^2(Q). \tag{3.2.13d}$$

Because of continuous embedding of $L^\infty(0,T;H^1_0(\Omega))$ and $L^\infty(0,T;L^2(\Omega))$ into $L^2(0,T;H^1_0(\Omega))$, $L^2(0,T;L^2(\Omega))$ respectively, we conclude that there exists y_γ such that

$$y_n \rightharpoonup y_\gamma \text{ weakly in } L^2\left(0,T; H^1_0(\Omega)\right) \tag{3.2.14a}$$

then

$$\frac{dy_n}{dt} \rightharpoonup \frac{dy_\gamma}{dt} \text{ weakly in } \mathcal{D}'(Q),$$

and we deduce that

$$\frac{dy_n}{dt} \rightharpoonup \frac{dy_\gamma}{dt} \text{ weakly in } L^2(Q), \tag{3.2.14b}$$

The main part of proof will be done in the following three steps.
First step: We prove that $y_\gamma = y(u_\gamma, 0)$ as follows:
Let $\mathcal{D}(Q)$ be the of test functions on Q (functions that belong to C^∞ with compact support) with dual $\mathcal{D}'(Q)$. Multiply by $\varphi \in \mathcal{D}(Q)$ and use $(3.2.13a)$ and $(3.2.14a)$ to get

$$\frac{d^2 y_n}{dt^2} - \sigma^2 \Delta y_n \rightharpoonup \frac{d^2 y_\gamma}{dt^2} - \sigma^2 \Delta y_\gamma \text{ in } \mathcal{D}'(Q).$$

Then, from $(3.2.13d)$ and limit uniqueness we have

$$\frac{d^2 y_\gamma}{dt^2} - \sigma^2 \Delta y_\gamma = f_1.$$

Hence,

$$\frac{d^2 y_n}{dt^2} - \sigma^2 \Delta y_n \rightharpoonup \frac{d^2 y_\gamma}{dt^2} - \sigma^2 \Delta y_\gamma \text{ in } L^2(Q).$$

By $(3.2.11)$, $(3.2.13a)$ and uniqueness of the limit

$$\frac{d^2 y_\gamma}{dt^2}(x,t) - \sigma^2 \Delta y_\gamma(x,t) = u_\gamma(x,t), \quad (x,t) \in Q.$$

From $(3.2.14a)$ and $(3.2.14b)$ we know that $y_\gamma(x,0)$, $\frac{dy_\gamma}{dt}(x,0) \in L^2(\Omega)$. In view of initial conditions in $(3.2.11)$ we deduce that

$$y_\gamma(x,0) = 0, \frac{dy_\gamma}{dt}(x,0) = 0 \text{ in } \Omega.$$

Multiply $(3.2.11_1)$ by $\varphi \in \mathcal{D}(Q)$, where φ is chosen such that $\varphi(x,T) = \frac{d\varphi}{dt}(x,T) = 0$ in Ω, $\varphi = 0$ on Σ, and integrate by parts to get

$$\int_Q \left(\frac{d^2\varphi}{dt^2}(x,t) - \sigma^2 \Delta\varphi(x,t) \right) y_n(x,t)\, dxdt = \int_Q v_n(x,t)\varphi(x,t)\, dxdt,$$

pass to the limit to find

$$\int_Q \left(\frac{d^2\varphi}{dt^2}(x,t) - \sigma^2 \Delta\varphi(x,t) \right) y_\gamma(x,t)\, dxdt = \int_Q u_\gamma(x,t)\varphi(x,t)\, dxdt,$$

integrate by parts again to get

$$\int_\Sigma y_\gamma \frac{\partial\varphi}{\partial\nu} d\Sigma = 0,$$

which leads to $y_\gamma = 0$ on Σ.
Let's prove that also $\int_{\sigma_1}^{\sigma_2} y(u_\gamma, 0, \sigma)\, d\sigma = z_\gamma$. We know that the operator $y(v_n, 0, \sigma) \to \int_{\sigma_1}^{\sigma_2} y(v_n, 0, \sigma)\, d\sigma$ is bounded from $L^2(Q)$ to $L^2(Q)$ then from $(3.2.14a)$ we deduce that

$$\int_{\sigma_1}^{\sigma_2} y(v_n, 0; \sigma)\, d\sigma \rightharpoonup \int_{\sigma_1}^{\sigma_2} y(u_\gamma, 0; \sigma)\, d\sigma \text{ weakly in } L^2(Q) \tag{3.2.15}$$

From $(3.2.13c)$ and by limit uniqueness we get $\int_{\sigma_1}^{\sigma_2} y(u_\gamma, 0; \sigma)\, d\sigma = z_\gamma$.
Second step: We prove that $\xi_\gamma = \xi(u_\gamma)$. We know that $\xi(v_n)$ is a solution of

$$\begin{cases} t\left(\frac{d^2\xi(v_n)}{dt^2} - \Delta\xi(v_n)\right) = \int_{\sigma_1}^{\sigma_2} y(v_n, 0; \sigma)\, d\sigma & \text{in } Q, \\ \xi(v_n) = 0 & \text{on } \Sigma, \\ \xi(v_n)(x,T) = 0,\, \frac{d\xi(v_n)}{dt}(x,T) = 0 & \text{in } \Omega, \end{cases} \tag{3.2.16}$$

Multiply $(3.2.16_1)$ by $\frac{d\xi(v_n)}{dt}$ and apply Green formula to find

$$\frac{1}{2}\int_0^T \int_\Omega t\frac{d}{dt}\left[\left|\frac{d\xi(v_n)}{dt}\right|^2 + |\nabla\xi(v_n)|^2\right] dxdt = \int_0^T \int_\Omega \int_{\sigma_1}^{\sigma_2} y(v_n, 0; \sigma)\, d\sigma \frac{d\xi(v_n)}{dt} dxdt,$$

integrate by parts with respect of time variable, and use Cauchy-Schwartz inequality to obtain

$$\frac{1}{2}\int_0^T \int_\Omega \left[\left|\frac{d\xi(v_n)}{dt}\right|^2 + |\nabla\xi(v_n)|^2\right] dxdt \leq \frac{1}{2}\int_0^T \int_\Omega \left[\left|\frac{d\xi(v_n)}{dt}\right|^2 + \left|\int_{\sigma_1}^{\sigma_2} y(v_n, 0; \sigma)\, d\sigma\right|^2\right] dxdt,$$

by linking with $(3.2.12c)$ we get

$$\int_0^T \int_\Omega |\nabla\xi(v_n)|^2\, dxdt \leq \int_0^T \int_\Omega \left|\int_{\sigma_1}^{\sigma_2} y(v_n, 0; \sigma)\, d\sigma\right|^2 dxdt \leq C_\gamma,$$

this means that

$$\|\xi(v_n)\|^2_{L^2\left(0,T,H_0^1(\Omega)\right)} \leq C_\gamma,$$

then there exists a subsequence still be denoted $\xi(v_n)$ such that

$$\xi(v_n) \rightharpoonup \xi_\gamma \text{ weakly in } L^2\left(0, T, H_0^1(\Omega)\right),$$

which gives

$$t\left(\frac{d^2\xi(v_n)}{dt^2} - \Delta\xi(v_n)\right) \rightharpoonup t\left(\frac{d^2\xi_\gamma}{dt^2} - \Delta\xi_\gamma\right) \text{ in } \mathcal{D}'(Q).$$

From $(3.2.13c)$ we deduce

$$t\left(\frac{d^2\xi(v_n)}{dt^2} - \Delta\xi(v_n)\right) \rightharpoonup f_2 \text{ weakly in } L^2(Q).$$

By limit uniqueness we get

$$t\left(\frac{d^2\xi_\gamma}{dt^2} - \Delta\xi_\gamma\right) = f_2 \in L^2(Q).$$

By passing to limit in $(3.2.16_1)$ and using $(3.2.13c)$ we deduce that

$$\frac{d^2\xi_\gamma}{dt^2} - \Delta\xi_\gamma = \frac{1}{t}\int_{\sigma_1}^{\sigma_2} y(u_\gamma, 0; \sigma)\, d\sigma.$$

The boundary and initial conditions follows by a similar reasoning to the first step. Summarize all the above by saying that $\xi(u_\gamma)$ verifies:

$$\begin{cases} \frac{d^2\xi(u_\gamma)}{dt^2} - \Delta\xi(u_\gamma) = \frac{1}{t}\int_{\sigma_1}^{\sigma_2} y(u_\gamma, 0; \sigma)\, d\sigma & \text{in } Q, \\ \quad \xi(u_\gamma) = 0 & \text{on } \Sigma, \\ \xi(u_\gamma)(x,T) = 0, \frac{d\xi}{dt}(u_\gamma)(x,T) = 0 & \text{in } \Omega. \end{cases}$$

Third step: Since $v \to \mathcal{J}_\gamma(v)$ is a lower semi-continous function, we have

$$\mathcal{J}_\gamma(u_\gamma) \leq \liminf_{n\to+\infty} \mathcal{J}_\gamma(v_n),$$

and according to (3.2.10) we get

$$\mathcal{J}_\gamma(u_\gamma) = \inf_{v\in L^2(Q)} \mathcal{J}_\gamma(v).$$

Uniqueness of u_γ follows because $\mathcal{J}_\gamma$ is strictly convex. ■

Theorem 3.7 *The averaged low-regret control u_γ for (3.2.1) (3.2.9) is characterized by :*

$$\begin{cases} \frac{d^2 y_\gamma}{dt^2} - \sigma^2\Delta y_\gamma = u_\gamma, \\ \frac{d^2\xi_\gamma}{dt^2} - \Delta\xi_\gamma = \frac{1}{t}\int_{\sigma_1}^{\sigma_2} y(u_\gamma, 0; \sigma)\, d\sigma, \\ \frac{d^2\rho_\gamma}{dt^2} - \Delta\rho_\gamma = 0, \\ \frac{d^2 p_\gamma}{dt^2} - \sigma^2\Delta p_\gamma = \int_{\sigma_1}^{\sigma_2} y(u_\gamma, 0; \sigma)\, d\sigma - z_d + (\sigma_2-\sigma_1)\frac{\rho_\gamma}{t} & \text{in } Q, \\ y_\gamma = 0, \xi_\gamma = 0,\ \rho_\gamma = -\frac{t^2}{\gamma}\frac{\partial\xi_\gamma}{\partial\nu},\ p_\gamma = 0 & \text{on } \Sigma, \\ y_\gamma(x,0) = 0, \frac{dy_\gamma}{dt}(x,0) = 0, \\ \xi_\gamma(x,T) = 0, \frac{d\xi_\gamma}{dt}(x,T) = 0, \\ \rho_\gamma(x,0) = 0, \frac{d\rho_\gamma}{dt}(x,0) = 0, \\ p_\gamma(x,T) = 0, \frac{dp_\gamma}{dt}(x,T) = 0 & \text{in } \Omega, \end{cases} \tag{3.2.17}$$

where $y_\gamma = y(u_\gamma, 0; \sigma) \in L^\infty(0,T; H_0^1(\Omega))$, $\xi_\gamma = \xi(u_\gamma) \in L^\infty(0,T; H_0^1(\Omega))$, $\rho_\gamma \in L^\infty(0,T; L^2(\Omega))$ and $p_\gamma = p_\gamma(\sigma) \in L^\infty(0,T; H_0^1(\Omega))$.
With the variational inequality:

$$\left(\int_{\sigma_1}^{\sigma_2} p_\gamma(\sigma)\, d\sigma + N u_\gamma, v - u_\gamma\right)_{L^2(Q)} \geq 0 \quad \textit{for every } v \in \mathcal{U}_{ad}. \tag{3.2.18}$$

Proof. A first order condition for (3.2.9) gives

$$\left(\int_{\sigma_1}^{\sigma_2} y(u_\gamma, 0; \sigma)\, d\sigma - z_d, \int_{\sigma_1}^{\sigma_2} y(v-u_\gamma, 0; \sigma)\, d\sigma\right)_{L^2(Q)} + N(u_\gamma, v-u_\gamma)_{L^2(Q)}$$
$$+ \frac{(\sigma_2-\sigma_1)}{\gamma}\left(t\frac{\partial\xi}{\partial\nu}(u_\gamma), t\frac{\partial\xi}{\partial\nu}(v-u_\gamma)\right)_{L^2(\Sigma)} \geq 0 \ \ \forall v \in \mathcal{U}_{ad}. \tag{3.2.19}$$

Introduce a new state ρ_γ given by

$$\begin{cases} \frac{d^2\rho_\gamma}{dt^2} - \Delta\rho_\gamma = 0 & \text{in } Q, \\ \rho_\gamma = -\frac{t^2}{\gamma}\frac{\partial\xi_\gamma}{\partial\nu} & \text{on } \Sigma, \\ \rho_\gamma(x,0) = 0, \frac{d\rho_\gamma}{dt}(x,0) = 0 & \text{in } \Omega, \end{cases}$$

where $\xi_\gamma = \xi(u_\gamma)$. As $\frac{t^2}{\gamma}\frac{\partial\xi_\gamma}{\partial\nu} \in L^2(\Sigma)$ (by regularity propriety (3.2.8)) the latter problem has a unique solution defined by transposition method in $L^\infty(0,T;L^2(\Omega))$ (Lions, 1971). Then, we have

$$\begin{aligned} \left(\rho_\gamma, \frac{1}{t}\int_{\sigma_1}^{\sigma_2} y(v-u_\gamma,0;\sigma)\,d\sigma\right)_{L^2(Q)} &= \int_0^T\int_\Omega \rho_\gamma\left(\frac{d^2\xi}{dt^2}(v-u_\gamma) - \Delta\xi(v-u_\gamma)\right)dxdt \\ &= \int_0^T\int_\Omega \left(\frac{d^2\rho_\gamma}{dt^2} - \Delta\rho_\gamma\right)\xi(v-u_\gamma)\,dxdt \\ &\quad -\int_0^T\int_\Gamma \rho_\gamma(s,t)\frac{\partial\xi}{\partial\nu}(v-u_\gamma)(s,t)\,d\Gamma dt \\ &= \frac{1}{\gamma}\left(t\frac{\partial\xi}{\partial\nu}(u_\gamma), t\frac{\partial\xi}{\partial\nu}(v-u_\gamma)\right)_{L^2(\Sigma)}. \end{aligned}$$

Define another state $p_\gamma = p_\gamma(\sigma)$ by

$$\begin{cases} \frac{d^2p_\gamma}{dt^2} - \sigma^2\Delta p_\gamma = \int_{\sigma_1}^{\sigma_2} y(u_\gamma,0;\sigma)\,d\sigma - z_d + (\sigma_2-\sigma_1)\frac{\rho_\gamma}{t} & \text{in } Q, \\ p_\gamma = 0 & \text{on } \Sigma, \\ p_\gamma(x,T) = 0, \frac{dp_\gamma}{dt}(x,T) = 0 & \text{in } \Omega. \end{cases}$$

It remains to be checking the validity of the variational inequality (3.2.18)

$$\begin{aligned} &\left(\int_{\sigma_1}^{\sigma_2} y(u_\gamma,0;\sigma)\,d\sigma - z_d + (\sigma_2-\sigma_1)\frac{\rho_\gamma}{t}, \int_{\sigma_1}^{\sigma_2} y(v-u_\gamma,0;\sigma)\,d\sigma\right)_{L^2(Q)} \\ &= \int_{\sigma_1}^{\sigma_2}\int_0^T\int_\Omega \left(\frac{d^2p_\gamma}{dt^2} - \sigma^2\Delta p_\gamma\right) y(v-u_\gamma,0,\sigma)\,dxdtd\sigma \\ &= \int_{\sigma_1}^{\sigma_2}\int_0^T\int_\Omega p_\gamma\left(\frac{d^2y}{dt^2}(v-u_\gamma,0,\sigma) - \sigma^2\Delta y(v-u_\gamma,0,\sigma)\right)_{L^2(Q)} dxdtd\sigma \\ &= \left(\int_{\sigma_1}^{\sigma_2} p_\gamma d\sigma, v-u_\gamma\right)_{L^2(Q)}. \end{aligned}$$

Finally, the optimality condition (3.2.19) is equivalent to

$$\left(\int_{\sigma_1}^{\sigma_2} p_\gamma(\sigma)\,d\sigma + Nu_\gamma, v-u_\gamma\right)_{L^2(Q)} \geq 0 \text{ for every } v \in \mathcal{U}_{ad}.$$

■

3.2.4 Averaged no-regret control characterization

Since the averaged no-regret control u is expected to be a limit of averaged low-regret control sequence u_γ when $\gamma \to 0$, we have to get some a priori estimates for the mentioned states in (3.2.17) before getting the averaged no-regret control characterization.

Theorem 3.8 *There exists some $C > 0$ independent of γ such that*

$$\left\| \int_{\sigma_1}^{\sigma_2} y\left(u_\gamma, 0; \sigma\right) d\sigma \right\|_{L^2(Q)} \leq C, \tag{3.2.20a}$$

$$\left\| u_\gamma \right\|_{L^2(Q)} \leq C, \tag{3.2.20b}$$

$$\left\| t \frac{\partial \xi_\gamma}{\partial \nu} \right\|_{L^2(\Sigma)} \leq C\sqrt{\gamma}, \tag{3.2.20c}$$

$$\left\| \frac{dy_\gamma}{dt} \right\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \quad \left\| y_\gamma \right\|_{L^\infty\left(0,T;H_0^1(\Omega)\right)} \leq C, \tag{3.2.21}$$

$$\left\| \frac{d\xi_\gamma}{dt} \right\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \quad \left\| \xi_\gamma \right\|_{L^\infty\left(0,T;H_0^1(\Omega)\right)} \leq C, \tag{3.2.22}$$

$$\left\| \frac{d\rho_\gamma}{dt} \right\|_{L^\infty(0,T;H^{-1}(\Omega))} \leq C, \quad \left\| \rho_\gamma \right\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \tag{3.2.23}$$

$$\left\| \frac{dp_\gamma}{dt} \right\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \quad \left\| p_\gamma \right\|_{L^\infty\left(0,T;H_0^1(\Omega)\right)} \leq C. \tag{3.2.24}$$

Proof. u_γ is the minimum of $\mathcal{J}_\gamma$ then $\mathcal{J}_\gamma\left(u_\gamma\right) \leq \mathcal{J}_\gamma(0) = 0$ which leads to

$$\left\| \int_{\sigma_1}^{\sigma_2} y\left(u_\gamma, 0; \sigma\right) d\sigma - z_d \right\|_{L^2(Q)}^2 + N \left\| u_\gamma \right\|_{L^2(Q)}^2 + \frac{\left(\sigma_2 - \sigma_1\right)}{\gamma} \left\| t \frac{\partial \xi_\gamma}{\partial \nu} \right\|_{L^2(\Sigma)}^2 \leq J(0,0) = \left\| z_d \right\|_{L^2(Q)}^2$$

this gives (3.2.20).
Multiply both sides of the first equation in (3.2.17) by y_γ and integrate over $(0,t)$ to get

$$\left\| \frac{dy_\gamma}{dt}(t) \right\|_{L^2(\Omega)}^2 + 2\sigma^2 \left\| \nabla y_\gamma(t) \right\|_{(L^2(\Omega))^n}^2 \leq \int_0^t \left(\left\| u_\gamma(s) \right\|_{L^2(\Omega)}^2 + \left\| \frac{dy_\gamma}{dt}(s) \right\|_{L^2(\Omega)}^2 \right) ds,$$

then, use Gronwall lemma to obtain

$$\sup_{t \in (0,T)} \left\| \frac{dy_\gamma}{dt} \right\|_{L^2(\Omega)} \leq C, \ \sup_{t \in (0,T)} \left\| \nabla y_\gamma \right\|_{(L^2(\Omega))^n} \leq C(\sigma).$$

Because $\frac{1}{t} \int_{\sigma_1}^{\sigma_2} y\left(u_\gamma, 0, \sigma\right) d\sigma \in L^2(Q)$ and by the same way we get (3.2.22).

According to $(3.2.20c)$, it's easy to prove by contradiction that

$$\frac{1}{\gamma}\left\| t\frac{\partial \xi_\gamma}{\partial \nu}\right\|^2_{L^2(\Sigma)} \leq C \Rightarrow \frac{1}{\gamma}\left\| t\frac{\partial \xi_\gamma}{\partial \nu}\right\|_{L^2(\Sigma)} \leq C \quad \text{for every } \gamma < 1,$$

this means that $\frac{t^2}{\gamma}\frac{\partial \xi_\gamma}{\partial \nu} \in L^2(\Sigma)$ which allows to use theorem 4.3 in (Apolaya, 1994) to prove $(3.2.23)$.

For $(3.2.24)$, remark that $\int_{\sigma_1}^{\sigma_2} y(u_\gamma, 0)\, d\sigma - z_d + (\sigma_2 - \sigma_1)\frac{\rho_\gamma}{t} \in L^2(Q)$ to obtain the estimates by the same way of $(3.2.21)$ and $(3.2.22)$. ■

Proposition 3.3 *The low-regret control sequences u_γ converges to the no-regret control u solution of* $(3.2.1)$ $(3.2.4)$.

Proof. In view of $(3.2.17_1)$ and $(3.2.20b)$ we deduce

$$\left\| \frac{d^2 y_\gamma}{dt^2} - \sigma^2 \Delta y_\gamma \right\|_{L^2(Q)} \leq C. \tag{3.2.25}$$

From $(3.2.20b)$ we get

$$u_\gamma \rightharpoonup u \text{ weakly in } L^2(Q). \tag{3.2.26a}$$

The a priori estimates $(3.2.21_2)$ implies that

$$y_\gamma \rightharpoonup y \text{ weakly in } L^\infty\left(0, T; H_0^1(\Omega)\right). \tag{3.2.26b}$$

By proceeding as in the proof of theorem 3.6 and by using $(3.2.26a)$, $(3.2.26c)$ we prove that $y = y(u, 0; \sigma)$ verifies

$$\begin{cases} \frac{d^2 y}{dt^2} - \sigma^2 \Delta y = u & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = 0, \frac{dy}{dt}(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{3.2.26}$$

and from the estimates in $(3.2.22)$ we have

$$\xi_\gamma \rightharpoonup \xi \text{ weakly in } L^\infty\left(0, T; H_0^1(\Omega)\right)$$

By a similar way $\xi = \xi(x, t; u) \in L^\infty(0, T; H_0^1(\Omega))$ is a solution of

$$\begin{cases} \frac{d^2 \xi(u)}{dt^2} - \Delta \xi(u) = \frac{1}{t}\int_{\sigma_1}^{\sigma_2} y(u, 0; \sigma)\, d\sigma & \text{in } Q, \\ \xi(u) = 0 & \text{on } \Sigma, \\ \xi(u)(x,T) = 0, \frac{d\xi(u)}{dt}(x,T) = 0 & \text{in } \Omega, \end{cases} \tag{3.2.27}$$

In view of $(3.2.20c)$ we have

$$t\frac{\partial \xi(u_\gamma)}{\partial \nu} \to t\frac{\partial \xi(u)}{\partial \nu} = 0 \text{ strongly in } L^2(\Sigma)$$

which implies that $\int_0^T \int_\Gamma t\frac{\partial \xi(u)}{\partial \nu} g\, d\Gamma\, dt = 0$ for every $g \in L^2(\Sigma)$, this means that u is a no-regret control. ■

Theorem 3.9 *The averaged no-regret control* u *solution of* (3.2.1) (3.2.4) *is characterized by the following system:*

$$\begin{cases} \frac{d^2y}{dt^2} - \sigma^2 \Delta y = u, \\ \frac{d^2\xi}{dt^2} - \Delta \xi = \frac{1}{t}\int_{\sigma_1}^{\sigma_2} y\left(u,0;\sigma\right) d\sigma, \\ \frac{d^2\rho}{dt^2} - \Delta \rho = 0, \\ \frac{d^2p}{dt^2} - \sigma^2 \Delta p = \int_{\sigma_1}^{\sigma_2} y\left(u,0;\sigma\right) d\sigma - z_d + \left(\sigma_2-\sigma_1\right)\frac{\rho}{t} & \text{in } Q, \\ y=0, \xi=0, \ \rho=\lambda \ , \ p=0 & \text{on } \Sigma, \\ y\left(x,0\right)=0, \frac{dy}{dt}\left(x,0\right)=0, \\ \xi\left(x,T\right)=0, \frac{d\xi}{dt}\left(x,T\right)=0, \\ \rho\left(x,0\right)=0, \frac{d\rho}{dt}\left(x,0\right)=0, \\ p\left(x,T\right)=0, \frac{dp}{dt}\left(x,T\right)=0 & \text{in } \Omega, \end{cases} \tag{3.2.28}$$

and the variational inequality:

$$\left(\int_{\sigma_1}^{\sigma_2} p\left(\sigma\right) d\sigma + Nu, v-u\right)_{L^2(Q)} \geq 0 \text{ for every } v \in \mathcal{U}_{ad} \tag{3.2.29}$$

with the following limits:

$u = \lim_{\gamma\to 0} u_\gamma$, $y = \lim_{\gamma\to 0} y\left(u_\gamma,0\right)$, $\xi = \lim_{\gamma\to 0}\xi_\gamma$, $\rho = \lim_{\gamma\to 0}\rho_\gamma$, $p = p\left(\sigma\right) = \lim_{\gamma\to 0} p_\gamma$ *and* $\lambda = \lim_{\gamma\to 0} -\frac{t^2}{\gamma}\frac{\partial\xi_\gamma}{\partial\nu}$.

Proof. We have already the systems that govern the states y, ξ in (3.2.26) and (3.2.27) resp., it remains to find the systems that govern ρ and p.
From $(3.2.23_2)$ we deduce the existence of ρ such that

$$\rho_\gamma \rightharpoonup \rho \text{ weakly in } L^\infty\left(0,T;L^2\left(\Omega\right)\right),$$

by continuity of embedding from $L^\infty\left(0,T;L^2\left(\Omega\right)\right)$ to $L^2\left(Q\right)$ we get

$$\rho_\gamma \rightharpoonup \rho \text{ weakly in } L^2\left(Q\right), \tag{3.2.30}$$

and from $(3.2.20c)$ we deduce that there exist $\lambda \in L^2\left(\Sigma\right)$ such that

$$-\frac{t^2}{\gamma}\frac{\partial\xi_\gamma}{\partial\nu} \rightharpoonup \lambda \text{ weakly in } L^2\left(\Sigma\right). \tag{3.2.31}$$

By passing to limit in $(3.2.17_3)$ and using (3.2.30), (3.2.31), we prove as in the first step of proof of theorem 3.6 that ρ satisfies

$$\begin{cases} \frac{d^2\rho}{dt^2} - \Delta\rho = 0 & \text{in } Q, \\ \rho = \lambda & \text{on } \Sigma, \\ \rho\left(x,0\right)=0, \frac{d\rho}{dt}\left(x,0\right)=0 & \text{in } \Omega. \end{cases}$$

From $(3.2.24_2)$ there exist p

$$p_\gamma \rightharpoonup p \text{ weakly in } L^\infty\left(0,T;H_0^1(\Omega)\right).$$

Again by continuity of embedding from $L^\infty(0,T;H_0^1(\Omega))$ to $L^2(0,T;H_0^1(\Omega))$ we get

$$p_\gamma \rightharpoonup p \text{ weakly in } L^2\left(0,T;H_0^1(\Omega)\right)$$

because $t\int_{\sigma_1}^{\sigma_2} y(u,0;\sigma)\,d\sigma - tz_d + (\sigma_2-\sigma_1)\,\rho_\gamma \in L^2(Q)$ and by reasoning by the same way of the second step in the proof of theorem 4 to find

$$\begin{cases} \frac{d^2p}{dt^2} - \sigma^2\Delta p = \int_{\sigma_1}^{\sigma_2} y(u,0;\sigma)\,d\sigma - z_d + (\sigma_2-\sigma_1)\,\frac{\rho}{t} & \text{in } Q, \\ p = 0 & \text{on } \Sigma, \\ p(x,T) = 0,\ \frac{dp}{dt}(x,T) = 0 & \text{in } \Omega. \end{cases}$$

■

Conclusion & perspectives

Our work has led us to prove existence, uniqueness and to characterize the optimal control for an electromagnetic wave equation modeling a biomedical phenomenon (X-rays penetration in living cells), where we controlled her displacement to be compatible with living bodies with unknown physical proprieties.

We have coupled the notions of no-regret control and averaged control to introduce a novel notion in control theory to solve this problem, it's the averaged no-regret control. We have avoided the missing velocity of propagation parameter by controlling the average of the state with respect to this parameter, we averted the missing boundary condition by taking the no-regret control. Then, we get an optimality system characterizing the averaged no-regret control.

As well as, we have proved existence, uniqueness and we have characterized the optimal control for an ill-posed wave equation by regularization into a well-posed equation with missing data, where we used the concepts of no-regret control and low-regret control to control the well-posed one and consequently the ill-posed one.

We note that in both studied problems, the optimal control has characterized by an optimality system which has a more complex structure comparing with optimal control of classical distributed systems (systems with complete data) i.e., it contains four systems governing states characterizing the optimum, on contrary, classical systems in her characterization need only two states. Consequently, this structure makes numerical treatment of optimal control problems with missing data more difficult than classical problems.

In the future, the notion of averaged no-regret control could be applied to control other distributed systems depending on an unknown parameter and with incomplete data, where we can get more interesting results.

To further our research, we plan to study more complicated and general cases as abstract equations containing an operator depending upon an uncertainty parameter and with missing data, for example, missing source, boundary conditions, initial conditions...

Further work needs to be done, like numerical simulations of the main problem to test the efficiency of our method.

Appendices

Definition 1 *Let $J : U \subset X \to Y$ be an operator with Banach spaces X, Y and $U \neq \emptyset$ open. J is called directionally differentiable at $x \in U$ if the limit*

$$dJ(x,h) = \lim_{t \to 0^+} \frac{J(x+th) - J(x)}{t} \in Y$$

exists for all $h \in X$. J is called Gâteaux differentiable at $x \in U$ if J is directionally differentiable at x and the directional derivative $J'(x) : h \in X \to dJ(x,h) \in Y$ is bounded and linear, i.e., $J(x) \in \mathcal{L}(X,Y)$.

Theorem 1 *Let X be a Banach space and $U \subset X$ be nonempty and convex. Furthermore, let $J : V \to R$ be defined on an open neighborhood of U. Let u be a local solution of*

$$\inf_{v \in U} J(v),$$

at which J is Gâteaux-differentiable. Then the following optimality condition holds:

$$\langle J'(u), v - u \rangle_{X',X} \geq 0 \ \ \forall v \in U.$$

If J is convex on U, the last condition is necessary and sufficient for global optimality.

Theorem 2 (Green formulas) *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, and ν be the outward unit normal vector on $\Gamma = \partial\Omega$. Then we have*

For $u \in H^1(\Omega)$ and $v \in H^2(\Omega)$ we have the half Green formula

$$\int_\Omega u \Delta v dx = -\int_\Omega \nabla u \nabla v dx + \int_\Gamma u \frac{\partial v}{\partial \nu} d\Gamma.$$

For $u, v \in H^2(\Omega)$ we have the full Green formula

$$\int_\Omega (u\Delta v - v\Delta u)\, dx = \int_\Gamma \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) d\Gamma.$$

Definition 2 (Riemann-Liouville fractional derivative) (Samko, Kilbas, and Marichev, 1993) *Let $f : \mathbb{R}_+ \to \mathbb{R}$ and $\alpha \in (0,1)$, the left Riemann-Liouville fractional derivative of f of order α is defined by*

$${}^{RL}D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \ \ t > 0,$$

where $\Gamma(\alpha)$ is the Euler gamma function.

Definition 3 (Samko, Kilbas & Marichev, 1993) *Let $f : \mathbb{R}_+ \to \mathbb{R}$ and $\alpha > 0$. Then*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \ \ t > 0,$$

is the Riemann-Liouville fractional integral of f of order α.

Lemma 1 (Mophou, 2011) Let $0 < \alpha < 1$, Then for every $\varphi, y \in C^{\infty}\left(\overline{Q}\right)$, we have

$$\begin{aligned}\int_0^T \int_\Omega \left({}^{RL}D_t^\alpha y(x,t) - \Delta y(x,t)\right)\varphi(x,t)\,dxdt &= \int_\Omega \left[\varphi(x,T)\, I^{1-\alpha}(x,T) - \varphi\left(x,0^+\right) I^{1-\alpha}\left(x,0^+\right)\right] dxdt \\ &+ \int_0^T \int_\Gamma \left[y(x,t)\frac{\partial \varphi}{\partial \nu} - \varphi(x,t)\frac{\partial y}{\partial \nu}(x,t)\right] d\Gamma dt \\ &+ \int_0^T \int_\Omega y(x,t)\left({}^{RL}D_t^\alpha \varphi(x,t) - \Delta\varphi(x,t)\right) dxdt.\end{aligned}$$

where Q, Ω and Γ are described in subsection 2.2.1.

Proposition 1 (Mophou, 2015) Let $\psi \in L^2(Q)$. Then equation

$$\begin{cases} {}^{RL}D_t^\alpha \varphi - \Delta\varphi = \psi & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(T) = 0 & \text{in } \Omega, \end{cases}$$

has a unique solution $\varphi \in L^2((0,T); H^2(\Omega) \cap H_0^1(\Omega))$. Moreover, there exists a constant $C > 0$ such that

$$\|\varphi\|_{L^2((0,T);H^2(\Omega))} + \left\|{}^{RL}D_t^\alpha \varphi\right\|_{L^2(Q)} \leq C\,\|\varphi\|_{L^2(Q)},$$

where Q, Ω, Γ and Σ are described in subsection 2.2.1.

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