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## Regional Boundary Gradient Closed Loop Control System and $\Gamma^*$ AGFO-Observer

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**Abstract:** The main objective of this paper is to study the problem of regional boundary asymptotic gradient full order observer ( $\Gamma^*$ AGFO-observer) in link with the structures of sensors and actuators. Important results have been obtained related to diverse kinds of measurements and controls, of domains and boundary conditions. It has been shown that the structures of measurements and feedback controls allow the existence of  $\Gamma^*$ AGFO-observer in order to achieve regional boundary gradient closed loop control system (RBGCLC-system). It has also been found that there is a dynamical system which does not represent the observer in the usual sense, but it could be interpreted as a regional observer.

**Keywords:**  $\Gamma^*$ AGFO-observer, closed loop, feedback control, RBGCLC-system, sensor.

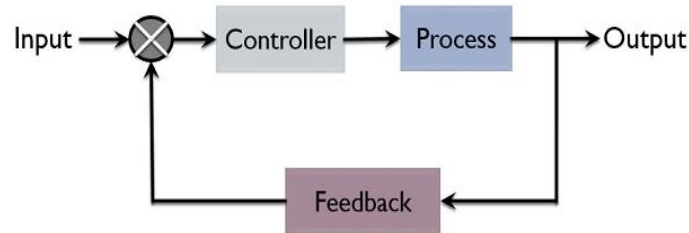
### 1. Introduction

The important of a control system is played a good roll to simplify some system, and controls other systems, as the human civilization is being modernized day by day the demand for automation is increasing accordingly. Automation highly requires control of devices to achieve desired results [1-2].

The main purpose of a control system is that there should be a clear mathematical relationship between input and output of the system [3-4]. In the case where the input and output of the system can be described via by a linear suitably, the system is called a linear control system [5]. One of the most important notion of control systems is closed loop control system [6-7]. Control system in which the output has an effect on the input measure in such a way that the input measure will regulate itself based on the output generated is said to be control system of closed loop type. A scientific example of a closed-loop control system is missile launched and auto tracked by radar, An air conditioner and cooling system in car or in house control, estimation and optimization of energy efficient buildings [8-10].

Then we can define the main characteristics of regional closed-loop control problem as being to reduce errors by automatically adjusting the systems input, to improve stability of an unstable system [11-13]. Thus, the closed-loop systems are considered as fully automatic control system because it is designed in a way that the achieved output is automatically compared with the reference input to have the required output maybe realize via an associated regional observer and strategic sensor [11-15](figure 1).





**Fig. 1:** Input, output, system and feedback.

The purpose of this paper is to extend the previous results in [17-19] demonstrate that the structures of measurements and control allows the existence of *RBAGFO*-Observer in order to achieve *RBGCLC*-system.

The outline of this paper is structured as follow. Section 2 concerns the class of considered system and formulation problem. Section 3, devotes to the  *$\Gamma^*$ AGFO*-Observer building problem. Section 4, is linked to extend the regional closed loop control system to the regional boundary gradient case. Final section, presents various results related to different types of measurements and controls have been accomplished in order to guarantee the stability of considered system.

## 2. Considered system and formulation problem

Consider a distributed parameter system defined with the following forms:

- ◊  $\Omega$  is an open bounded subset of  $R^n$  with smooth boundary  $\partial\Omega$ .
- ◊  $\Gamma$  is a sub-region of  $\partial\Omega$  with positive measure.
- ◊ Symbolize  $\Pi = \Omega \times ]0, \infty[$ ; and  $\Xi = \partial\Omega \times ]0, \infty[$ .
- ◊ Space  $\mathbb{W} = H^1(\Omega)$ ;  $\mathbb{U} = L^2(0, T, \mathcal{R}^p)$ ; and  $\mathbb{Y} = L^2(0, T, \mathcal{R}^q)$  are designed in this paper as separable Hilbert spaces and represented as state space, input space and measurement space with  $p$  and  $q$  are the sensors; actuator numbers [3-5].
- ◊  $\mathcal{A} = \sum_{i,j=1}^n \frac{\partial}{\partial w_j} (a_{ij} \frac{\partial}{\partial w_j})$  with  $a_{ij} \in D(\bar{\mathcal{A}})$  (domain of  $\bar{\mathcal{A}}$ ) is a linear differential operator of second order, which produces a semi-group  $(S_{\mathcal{A}}(t))_{t \geq 0}$  of strongly continuous kind on  $H^1(\Omega)$ . Thus,  $\mathcal{A}$  is of self-adjoint kind and compact resolvent [15-16].
- ◊ The reflected system is given by

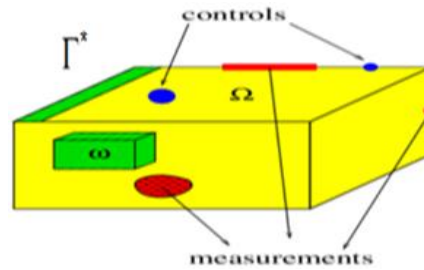
$$\begin{cases} \frac{\partial w}{\partial t}(\zeta, t) = \mathcal{A}w(\zeta, t) + Bu(t) & \Pi \\ w(\zeta, 0) = w_0(\zeta) & \Omega \\ \frac{\partial w}{\partial \nu}(\mu, t) = 0 & \Xi \end{cases} \quad (1)$$

where  $\zeta \in \Omega, \mu \in \partial\Omega, t \in [0, T]$  and  $(\zeta, t) \in \Pi, (\mu, t) \in \Xi, (\zeta, 0) \in \Omega$

- ◊ The information can be attained by utilizing various sensor locations [3, 5]. Therefore, corresponding measurement to system (1) is specified by

$$y(., t) = Cw(., t) \quad \Pi \quad (2)$$

where the operators  $B \in L(\mathcal{R}^p, \mathbb{W})$  and  $C \in L(H^1(\Omega), \mathcal{R}^q)$  are depended on the structure of controls and information [3] (figure 2).



**Fig. 2:** Regions  $\Omega$ ,  $\Gamma^*$ , and sensor and control locations.

◊ Under the given assumption above, the system (1) has a unique solution given by the following form [17].

$$w(\zeta, t) = S_{\mathcal{A}}(t)w_0(\xi) + \int_0^t S_{\mathcal{A}}(t-s)Bu(s)ds \quad \Pi \tag{3}$$

◊ The problem is how to create a regional boundary closed loop control system (RBCL-system) via an appropriate observer to the gradient of state on a given  $\Gamma^*$ .

For deriving an regional boundary asymptotic gradient estimator of  $Tw(\zeta)$  on  $\Gamma$ , we consider the following points:

- Now, we consider the operator  $K$  given by the form

$$K: \mathbb{W} \rightarrow \mathbb{Y}$$

$$w \rightarrow CS_{\mathcal{A}}(\cdot)w$$

where  $K$  is bounded linear operator as in [4-5]. Thus, the adjoint operator  $K^*$  of  $K$  is defined by  $K^*: \mathbb{Y} \rightarrow \mathbb{w}$ , and represented by the form

$$K^*: L^2(0, T, \mathcal{R}^q) \rightarrow H^1(\Omega)$$

$$y^* \rightarrow \int_0^t S_{\mathcal{A}}^*(s)C^*y^*(s)ds$$

- The transformation gradient  $\nabla$  is given by

$$\begin{cases} \nabla: H^1(\Omega) \rightarrow (H^1(\Omega))^n \\ w \rightarrow \nabla_w = \left( \frac{\partial w}{\partial \zeta_1}, \dots, \frac{\partial w}{\partial \zeta_n} \right) \end{cases}$$

with  $\nabla^*$  the adjoint of  $\nabla$  assumed by

$$\begin{cases} \nabla^*: (H^1(\Omega))^n \rightarrow H^1(\Omega) \\ w \rightarrow \nabla_w^* = v \end{cases}$$

where  $v$  is a solution of the Dirichlet problem

$$\begin{cases} \Delta_v = -div(z) & \Omega \\ v = 0 & \partial\Omega \end{cases}$$

- Trace operator is described by [20]

$$\gamma_0: H^1(\Omega) \rightarrow H^{1/2}(\partial\Omega)$$

which is linear, subjective and continuous [21-22]. Thus, the extension of the trace operator of order zero which is denoted by  $\gamma$  defined as

$$\gamma: (H^1(\Omega))^n \rightarrow (H^{1/2}(\partial\Omega))^n$$

with  $\gamma_0^*$  and  $\gamma^*$  represent the adjoints.

- Intended for a region  $\Gamma^*$  of  $\partial\Omega$ , contemplate  $\tilde{\chi}_{\Gamma^*}$  via the form

$$\begin{cases} \tilde{\chi}_{\Gamma^*}: H^{1/2}(\partial\Omega) \rightarrow H^{1/2}(\Gamma^*) \\ w \rightarrow \chi_{\Gamma^*} w = w|_{\Gamma^*} \end{cases}$$

with  $w|_{\Gamma^*}$  is the restriction of the state  $x$  to  $\Gamma^*$ , and

$$\chi_{\Gamma^*}: (H^1(\partial\Omega))^n \rightarrow (H^{1/2}(\Gamma^*))^n$$

where the adjoints are respectively given by  $\tilde{\chi}_{\Gamma^*}^*$  and  $\chi_{\Gamma^*}^*$ .

- Finally, consider the operator  $\chi_{\Gamma^*}^* \gamma \nabla K^*$  from  $\mathbb{Y}$  into  $(H^{1/2}(\Gamma^*))^n$  and the adjoint of this operator given by  $K \nabla^* \gamma^* \chi_{\Gamma^*}^*$ .

- We first recall a sensor is defined by any pair  $(D, f)$  such that  $D$  be a closed subset of  $\Omega$ , and which is characterized the sensor spatial supports and  $f \in L^2(D)$  denotes the measurements sensing distributions in  $D$ .

- It may be zone, if  $D \subset \Omega$  and  $f \in L^2(D)$ . In this case, the operator  $C$  is bounded [18] and the output function (2) may be given by the form

$$y(t) = \int_D f(\zeta) w(\zeta, t) d\zeta = C w(\zeta, t) \quad \Pi \quad (4)$$

- It may be pointwise, if  $D = \{b\}$  with  $b \in \Omega$  and  $f = \delta(\cdot - b)$ , where  $\delta$  is the mass of Dirac, which is concentrated in  $b$ . Then, operator  $C$  is un bounded and the measurement information (2) may be specified by the following

$$y(t) = \int_{\Omega} w(\zeta, t) \delta_b(\zeta - b) d\zeta \quad \Pi \quad (5)$$

In this section, we present some definitions and descriptions of regional boundary gradient observability, detectability and strategic sensor, which is derived of [21-23]. Thus, deliberate the system

$$\begin{cases} \frac{\partial w}{\partial t}(\zeta, t) = \mathcal{A} w(\zeta, t) & \Pi \\ w(\zeta, 0) = w_0(\zeta) & \Omega \\ \frac{\partial w}{\partial \vartheta}(\mu, t) = 0 & \Xi \end{cases} \quad (6)$$

The solution of (7) is given by the following form

$$w(\zeta, t) = S_{\mathcal{A}}(t) w_0(\xi) \quad \text{for all } t \in [0, T] \quad (7)$$

- The systems (6)-(7) are said to be exactly regionally boundary gradient observable ( $\mathbb{E} \Gamma^* G$ -observable) on  $\Gamma^*$  if

$$\text{Im } \chi_{\Gamma^*} \nabla K^* = (H^{1/2}(\Gamma^*))^n$$

- The systems (6)-(7) are said to be weakly regionally boundary gradient observable ( $\mathbb{W}\Gamma^*G$ -observable) on  $\Gamma^*$  if

$$\overline{\text{Im } \chi_{\Gamma^*} \nabla K^*} = (H^{1/2}(\Gamma^*))^n$$

It is equivalent to say that the systems (1)-(2) are  $\mathbb{W}\Gamma^*G$ -observable if

$$\text{Ker } H^* = \text{ker } K \quad \nabla^* \chi_{\Gamma^*} = \{0\}$$

- If the systems (6)-(7) are  $\mathbb{W}\Gamma^*G$ -observable, then  $w_0(\zeta, 0)$  is given by

$$w_0 = (K^*K)^{-1}K^*y = K^\dagger y, \quad (8)$$

where  $K^\dagger$  is the pseudo-inverse of the operator  $K$  [15-16].

- A sensor  $(D, f)$  is regional boundary gradient strategic ( $\Gamma^*G$ -strategic) on  $\Gamma^*$  if the observed system is  $\mathbb{W}\Gamma^*G$ -observable.

- The measurements can be obtained by the use of zone or pointwise sensors, which may be located in  $\Omega$  or  $\partial\Omega$  [21].

- The semi-group  $(S_{\mathcal{A}}(t))_{t \geq 0}$  is regionally boundary asymptotically gradient stable on  $(H^{1/2}(\Gamma^*))^n$  ( $\Gamma^*AG$ -stable), then for all  $w_0 \in H^1(\Omega)$ , the solution of autonomous system associated to system (1) coverage to zero when  $t$  tend to  $\infty$ .

- The system (6) is said to be  $\Gamma^*AG$ -stable if the operator  $A$  generates a semi-group which is  $\Gamma^*AG$ -stable on the space  $(H^{1/2}(\Gamma^*))^n$ .

- A system is said to be  $\Gamma^*AG$ -stable if and only if there exists some positive constants  $M_{\Gamma^*}$ ,  $\alpha_{\Gamma^*}$ , such that

$$\|\chi_{\Gamma^*} \gamma \nabla S_{\mathcal{A}}(\cdot)\|_{L((H^{1/2}(\Gamma^*))^n, H^1(\Omega))} \leq M_{\Gamma^*} e^{\alpha_{\Gamma^*} t}, \quad \forall t \geq 0. \quad (9)$$

- If the semi-group  $(S_{\mathcal{A}}(t))_{t \geq 0}$  is  $\Gamma^*AG$ -stable, then for all  $w_0 \in H^1(\Omega)$ , the solution of autonomous system (6) associated to system (1) satisfies

$$\begin{aligned} \|\chi_{\Gamma^*} \gamma \nabla w(\cdot, t)\|_{(H^{1/2}(\Gamma^*))^n} &= \|\chi_{\Gamma^*} \gamma \nabla S_{\mathcal{A}}(t) w_0\|_{(H^{1/2}(\Gamma^*))^n} \\ &\leq M_{\Gamma^*} e^{\alpha_{\Gamma^*} t} \|w_0\|_{(H^{1/2}(\Gamma^*))^n} \end{aligned}$$

and then, we have

$$\lim_{t \rightarrow \infty} \|\chi_{\Gamma^*} \gamma \nabla w(t)\|_{(H^{1/2}(\Gamma^*))^n} = 0.$$

- The system (1)-(2) is said to be regionally boundary asymptotically gradient detectable ( $\Gamma^*AG$ -detectable) on  $\Gamma^*$ , if there exists an operator  $H_{\Gamma^*AG}: \mathcal{R}^q \rightarrow (H^{1/2}(\Gamma^*))^n$ , such that the operator  $(\mathcal{A} - H_{\Gamma^*AG}C)$  generates a strongly continuous semi-group  $(S_{H_{\Gamma^*AG}}(t))_{t \geq 0}$ , which is  $\Gamma^*AG$ -stable.

- The dynamical system associated to the considered systems (1)-(2) is offered via

$$\begin{cases} \frac{\partial v}{\partial t}(\zeta, t) = F_{\Gamma^*AG} v(\zeta, t) + G_{\Gamma^*AG} u(t) + H_{\Gamma^*AG} y(t) & \Pi \\ v(\zeta, 0) = v(\zeta) & \Omega \\ \frac{\partial v}{\partial \theta}(\mu, t) = 0 & \Xi \end{cases} \quad (10)$$

where  $F_{\Gamma^*AG}$  generates a  $SCS$ -group  $(S_{F_{\Gamma^*AG}}(t))_{t \geq 0}$  which is  $F_{\Gamma^*AG}$ -stable on  $\mathbb{V}$  and  $G_{\Gamma^*AG} \in L(\mathbb{W}, \mathbb{V})$ ,  $H_{\Gamma^*AG} \in L(\mathbb{Y}, \mathbb{W})$ . The system (10) can be designed  $\Gamma^*AG$ -estimator for  $T_{\Gamma^*AG} w(\zeta, t) = \chi_{\Gamma^*} \nabla T w(\xi, t)$ , where  $T: \mathbb{W} \rightarrow \mathbb{V}$  with

$$T_{\Gamma^*AG} w(\xi, t) = v(\xi, t)$$

- Require that the process provided by  $v(\cdot, t) \in \mathbb{V}$  specific by

$$\begin{cases} \frac{\partial v}{\partial t}(\zeta, t) = Av(\zeta, t) + Bu(t) - H_{\Gamma^*AG} C(w(\zeta, t) - v(\zeta, t)) & \Pi \\ v(\zeta, 0) = v_0(\zeta) & \Omega \\ \frac{\partial v}{\partial \theta}(\mu, t) = 0 & \Xi \end{cases} \quad (11)$$

In this status  $F_{\Gamma^*AG}$  in system (10) [16-18] is given by  $F_{\Gamma^*AG} = A - H_{\Gamma^*AG}C$  where  $T_{\Gamma^*AG} = I_{\Gamma^*AGFO}$  the operator of identity type. Therefore  $A - H_{\Gamma^*AG}C$  is generator of a SCS group  $(S_{A-H_{\Gamma^*AG}C}(t))_{t \geq 0}$  on space of Hilbert in separable case  $\forall$  so that  $\Gamma^*$ GA-stable.

Hence,  $\exists M_{A-H_{\Gamma^*AG}C}, \alpha_{A-H_{\Gamma^*AG}C} > 0$  such that

$$\|S_{A-H_{\Gamma^*AG}C}(\cdot)\| \leq M_{A-H_{\Gamma^*AG}C} e^{-\alpha_{A-H_{\Gamma^*AG}C} t}, \forall t \geq 0.$$

From the solution of (1) and (11), we get

$$v(\zeta, t) = S_{A-H_{\Gamma^*AG}C}(t)v_0(\zeta) + \left[ \int_0^t S_{A-H_{\Gamma^*AG}C}(t-\tau) Bu(\tau) H_{\Gamma^*AG}y(\tau) \right] d\tau$$

• The system (11) defines  $\Gamma^*$ AGFO-estimator such that

$$v(\zeta, t) = \chi_{\Gamma^*} \nabla T_{\Gamma^*AGFO} w(\xi, t) = I_{\Gamma^*AGFO} w(\zeta, t) \in (H^{1/2}(\Gamma^*))^n$$

with  $w(\zeta, t)$  form the to (1)-(2), for

$$\lim_{t \rightarrow \infty} \|v(\cdot, t) - \chi_{\Gamma^*} \nabla T_{\Gamma^*AGFO} w(\zeta, t)\|_{(H^{1/2}(\Gamma^*))^n} = 0,$$

and  $\chi_{\Gamma^*} \nabla \chi_{\Gamma^*} \nabla I_{\Gamma^*AGFO}$  maps  $D(A)$  into  $D(A - H_{\Gamma^*AG}C)$  where  $v(\xi, t)$  form a solution (11).

• The process (11) form  $\Gamma^*$ AGFO-observer to (1)-(2) such that the next outcome holds:

1- If there is

$$M_{\Gamma^*AGFO} \in L(R, (H^{1/2}(\Gamma^*))^n) \text{ and } N_{\Gamma^*AGFO} \in L((H^{1/2}(\Gamma^*))^n)$$

such that

$$M_{\Gamma^*AGFO}C + N_{\Gamma^*AGFO} = I_{\Gamma^*AGFO}.$$

2-  $A - F_{\Gamma^*AGFO} = H_{\Gamma^*AGFO}C$  and  $G_{\Gamma^*AGFO} = B$ .

3- The system (11) defines  $\Gamma^*$ AGFO-estimator for  $x(\xi, t)$ .

4- The purpose of  $\Gamma^*$ AGFO-observer is to supply an estimation to investigated system. This estimation is specified via

$$\hat{w}(t) = M_{\Gamma^*AGFO}y(t) + N_{\Gamma^*AGFO} v(t).$$

• The system (1)-(2) form  $\Gamma^*$ AGFO -observable, the process be  $\Gamma^*$ AGFO -observer.

• If a system is  $\Gamma^*$ AGFO -observable then, the corresponding sensor is  $\Gamma^*$ AGFO- strategic sensor.

### 3. $\Gamma^*$ AGFO-Observer Building

As well known in [16-18], we interested to extend the characterization consequences that devote the process  $\Gamma^*$ AGFO-observer and  $\Gamma^*$ AGFO-detectability which is described a sufficient condition for  $\Gamma^*$ AGFO-observer in the next main sequel.

#### **Theorem 3.1:**

System (1)-(2) form  $\Gamma^*$ AGFO-detectable, then, the process (11) is  $\Gamma^*$ AGFO-observer, i.e.

$$\lim_{t \rightarrow \infty} \|w(\cdot, t) - T_{\Gamma^*AGFO} v(\cdot, t)\|_{\frac{1}{H^2(\Gamma^*)}} = \lim_{t \rightarrow \infty} \|w(\cdot, t) - v(\cdot, t)\|_{H^{1/2}(\Gamma^*)} = 0 \quad (12)$$

**Proof :** From the assumptions of section 2, the system (1) can be decomposed by the projections  $P$  and  $I - P$  on two parts, unstable and stable [14].

The state vector may be given by where  $w_1(\xi, t)$  is the state component of the unstable part of the system (1), may be written in the form

$$\begin{cases} \frac{\partial w_1}{\partial t}(\zeta, t) = Aw_1(\zeta, t) + Bu(t) & \Pi \\ w_1(\zeta, t) = w_{0_1}(\zeta) & \Omega \\ \frac{\partial w_1}{\partial \vartheta}(\mu, t) = 0 & \Xi \end{cases} \quad (13)$$

with  $w_2(\xi, t)$  represents stable part of (1) specific by

$$\begin{cases} \frac{\partial w_2}{\partial t}(\zeta, t) = Aw_{21}(\zeta, t) + Bu(t) & \Pi \\ w_2(\zeta, t) = w_{0_2}(\zeta) & \Omega \\ \frac{\partial w_2}{\partial \vartheta}(\mu, t) = 0 & \Xi \end{cases} \quad (14)$$

Put

$$e(\zeta, t) = w(\zeta, t) - v(\zeta, t)$$

where  $v(\zeta, t)$  is the solution of the system (11). By deriving the above equation and substituting equations (1) and (11), we obtain

$$\begin{aligned} \frac{\partial e}{\partial t}(\zeta, t) &= \frac{\partial w}{\partial t}(\zeta, t) - \frac{\partial v}{\partial t}(\zeta, t) \\ &= Aw(\zeta, t) - Av(\zeta, t) - H_{\Gamma^*AGFO}C(w(\cdot, t) - v(\zeta, t)) \\ &= (A - H_{\Gamma^*AGFO}C)e(\zeta, t) \end{aligned}$$

System (4.1)-(4.2) form  $\Gamma^*$ AG-detectable, so  $H_{\Gamma^*AGFO} \in L(R^q, H^{1/2}(\Gamma^*))$ , is so that the transformation  $(A - H_{\Gamma^*AGFO}C)$ , described a generator of SCS group  $(S_{H_{\Gamma^*AGFO}}(t))_{t \geq 0}$  on the space  $H^{1/2}(\Gamma^*)$ , that means  $\exists M_{\Gamma^*AGFO}, \alpha_{\Gamma^*AGFO} > 0$ , which is satisfied the following inequality

$$\|\chi_{\Gamma^*} \gamma \nabla S_A(\cdot)\|_{H^{1/2}(\Gamma^*)} \leq M_{\Gamma^*AGFO} e^{-\alpha_{\Gamma^*AGFO} t}$$

Finally, we have

$$\begin{aligned} \|e(\cdot, t)\|_{H^{1/2}(\Gamma^*)} &\leq \|\chi_{\Gamma^*} \gamma \nabla S_{H_{\Gamma^*AGFO}}(\cdot)\|_{H^{1/2}(\Gamma^*)} \|e_0(\cdot)\|_{H^{1/2}(\Gamma^*)} \\ &\leq M_{\Gamma^*AGFO} e^{-\alpha_{\Gamma^*AGFO} t} \|e_0(\cdot)\|_{H^{1/2}(\Gamma^*)} \end{aligned}$$

and

$$e_0(\zeta, t) = w(\zeta, t) - v(\zeta, t)$$



therefore

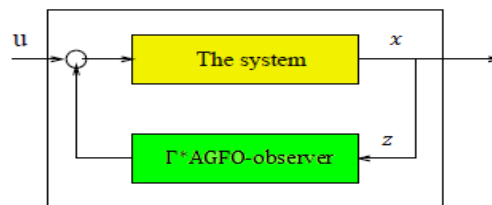
$$\lim_{t \rightarrow \infty} \|e(\cdot, t)\|_{H^{1/2}(\Gamma^*)} = 0.$$

Consequently, the process (11) form a  $\Gamma^*$ AGFO-observer to (1)-(2).■

**Corollary 3.2:** If the system is  $\Gamma^*$ AG-detectable, therefore it is can be built  $\Gamma^*$ AGFO-observer to the same system.

#### 4. Regional closed loop control system and $\Gamma^*$ AGFO-observer

This section devotes to extend the regional closed loop control system to the regional boundary case. The reconstruction of  $\Gamma^*$ AGFO-observer in distributed parameter system gives an estimator of  $Tw(\zeta, t)$  or  $w(\zeta, t)$ , it is important to consider the effect induced by



**Fig. 3** Regional boundary closed loop control system

using this  $\Gamma^*$ AGFO-estimator instead of a feedback control (figure 3). Thus, the problem that naturally arises is how to design a regional boundary closed-loop control system by using only partial information about the state  $w(\zeta, t)$  through the output function (2). In addition, we use the measurements (partial information) to estimate the full state in a region  $\Gamma^*$  by constructing a  $\Gamma^*$ AGFO-observer and to apply the feedback control law on the estimated state  $T_{\Gamma^*AGFO}w(\zeta, t)$  (figure 6). For this purpose consider now the system (1) which is excited by regional boundary feedback control

$$u(t) = -D_{\Gamma^*}w(\cdot, t), \tag{15}$$

where  $D_{\Gamma^*} = \chi_{\Gamma^*} \gamma \nabla D$  is a bounded linear operator defined by

$$\begin{aligned} -D_{\Gamma^*}: (H^{1/2}(\Gamma^*))^n &\rightarrow \mathbb{U} \\ h &\rightarrow uh \end{aligned}$$

**Theorem 4.1:** If the whole system (1)-(2), (11), (12) and (15) is given the matrix form

$$\begin{bmatrix} \frac{\partial w}{\partial t} \\ \frac{\partial e}{\partial t} \end{bmatrix} = \mathcal{A}_{\Gamma^*} \begin{bmatrix} w \\ e \end{bmatrix} \tag{16}$$

where the operator  $A_{\Gamma^*}$  is defined by

$$\mathcal{A}_{\Gamma^*} = \begin{bmatrix} \mathcal{A} - BD_{\Gamma^*} & -BD_{\Gamma^*} \\ 0 & \mathcal{A} - H_{\Gamma^*GAFO}C \end{bmatrix}$$

then, the spectrum of system (16) is the reunion of the spectrum of regional boundary closed-loop control system (20) and the spectrum of  $\Gamma^*$ AGFO-observer (21) and then, achieve the stability of system (1).

**Proof:** Consider again the system (1)-(2), (12), (16) and (23) augmented with the related dynamical system (11) represented as  $\Gamma^*$ AFO-observer by the following equations

$$\begin{cases} \frac{\partial w}{\partial t}(\zeta, t) = \mathcal{A}w(\zeta, t) + Bu(t) & \Pi \\ w(\zeta, 0) = w_0(\zeta) & \bar{\Omega} \\ \frac{\partial w}{\partial \vartheta}(\mu, t) = 0 & \Xi \\ y(\cdot, t) = Cw(\cdot, t) & \Pi \\ u(t) = -D_{\Gamma^*}w(\cdot, t) & \Pi \\ \lim_{t \rightarrow \infty} \|w(\cdot, t) - v(\cdot, t)\|_{H^{1/2}(\Gamma^*)} = 0 & \Pi \\ \frac{\partial \hat{v}}{\partial t}(\zeta, t) = \mathcal{A}v(\zeta, t) + Bu(t) - H_{\Gamma^*GAFO}C(w(\zeta, t) - v(\zeta, t)) & \Pi \\ v(\zeta, 0) = v_0(\zeta) & \Omega \\ \frac{\partial v}{\partial \vartheta}(\mu, t) = 0 & \Xi \end{cases} \quad (17)$$

Substituting equations (2) and (15) in (17), we have the form

$$\begin{cases} \frac{\partial w}{\partial t}(\zeta, t) = \mathcal{A}w(\zeta, t) - BD_{\Gamma^*}w(\cdot, t); & \Pi \\ w(\zeta, 0) = w_0(\zeta) & \Omega \\ \frac{\partial w}{\partial \vartheta}(\mu, t) = 0 & \Xi \\ \frac{\partial v}{\partial t}(\zeta, t) = \mathcal{A}v(\zeta, t) - BD_{\Gamma^*}w(\zeta, t) - H_{\Gamma^*GAFO}C(w(\zeta, t) - v(\zeta, t)) & \Pi \\ v(\zeta, 0) = v_0(\zeta) & \Omega \\ \frac{\partial v}{\partial \vartheta}(\mu, t) = 0 & \Xi \end{cases} \quad (18)$$

Using theorem 3.8 in equation (18), we obtain

$$\begin{cases} \frac{\partial w}{\partial t}(\zeta, t) = (\mathcal{A} - BD_{\Gamma^*})w(\zeta, t) - BD_{\Gamma^*}e(\zeta, t) & \Pi \\ w(\zeta, 0) = w_0(\zeta) & \Omega \\ \frac{\partial w}{\partial \vartheta}(\mu, t) = 0 & \Xi \\ \dot{e}(\zeta, t) = (\mathcal{A} - H_{\Gamma^*GAFO}C)e(\zeta, t) & \Pi \\ e(\zeta, 0) = e_0(\zeta) & \Omega \\ \frac{\partial e}{\partial \vartheta}(\mu, t) = 0 & \Xi \end{cases} \quad (19)$$

where  $e(\zeta, t) = w(\zeta, t) - v(\zeta, t)$ . Now, equation (19) allow to consider the following regional boundary control closed loop system

$$\begin{cases} \frac{\partial w}{\partial t}(\zeta, t) = (\mathcal{A} - BD_{\Gamma^*})w(\zeta, t) - BD_{\Gamma^*}e(\zeta, t) & \Pi \\ w(\zeta, 0) = w_0(\zeta) & \Omega \\ w(\mu, t) = 0 & \Xi \end{cases} \quad (20)$$

From the proof of theorem 3.8, we can get the system

$$\begin{cases} \frac{\partial e}{\partial t}(\zeta, t) &= (\mathcal{A} - H_{\Gamma^* \text{GAFO}} C)e(\zeta, t) & \Pi \\ e(\zeta, 0) &= e_0(\zeta) & \Omega \\ \frac{\partial e}{\partial \vartheta}(\mu, t) &= 0 & \Xi \end{cases} \quad (21)$$

Thus, the combining systems (20)-(21) can be written by the matrix form

$$\begin{bmatrix} \frac{\partial w}{\partial t} \\ \frac{\partial e}{\partial t} \\ \frac{\partial e}{\partial t} \end{bmatrix} = \mathcal{A}_{\Gamma^*} \begin{bmatrix} w \\ e \end{bmatrix} \quad (22)$$

where the operator  $\mathcal{A}_{\Gamma^*}$  is defined by

$$\mathcal{A}_{\Gamma^*} = \begin{bmatrix} \mathcal{A} - BD_{\Gamma^*} & -BD_{\Gamma^*} \\ 0 & \mathcal{A} - H_{\Gamma^* \text{GAFO}} C \end{bmatrix} \quad (23)$$

Consider the previous results as  $\mathcal{A}, B$ , and  $D_{\Gamma^*}$  are bounded linear operators, then, by using perturbation theory linear operator, we can deduce that the operator  $\mathcal{A}_{\Gamma^*}$  generates a strongly continuous semi-group. Therefore, the resolvent  $\rho(\mathcal{A}_{\Gamma^*})$  is non-empty and can be expressed by

$$\rho(\mathcal{A}_{\Gamma^*}) = \rho(\mathcal{A} - BD_{\Gamma^*}) \cap \rho(BD_{\Gamma^*})$$

Finally, we have

$$\sigma(\mathcal{A}_{\Gamma^*}) = \sigma(\mathcal{A} - BD_{\Gamma^*}) \cup \sigma(BD_{\Gamma^*})$$

where  $\sigma(\mathcal{A}_{\Gamma^*})$  denotes the spectrum of  $\mathcal{A}_{\Gamma^*}$ . Hence the semi group of  $\mathcal{A}_{\Gamma^*}$

$$(S_{\mathcal{A}_{\Gamma^*}}(t))_{t \geq 0} = \begin{bmatrix} S_{\mathcal{A} - BD_{\Gamma^*}}(t) & S_{BD_{\Gamma^*}}(t) \\ 0 & S_{\mathcal{A} - H_{\Gamma^* \text{GAFO}} C}(t) \end{bmatrix}$$

is stable on the space

$$(H^{1/2}(\Gamma^*))^n \oplus (H^{1/2}(\Gamma^*))^n \quad [5],$$

such that

$$\left\| \begin{bmatrix} w(\cdot, t) \\ e(\cdot, t) \end{bmatrix} \right\|_{(H^{1/2}(\Gamma^*))^n \oplus (H^{1/2}(\Gamma^*))^n} \leq M_{\mathcal{A}_{\Gamma^*}} e^{-\alpha_{\mathcal{A}_{\Gamma^*}} t} \left\| \begin{bmatrix} w_0(\cdot) \\ e_0(\cdot) \end{bmatrix} \right\|_{(H^1(\Omega))^n \oplus (H^1(\Omega))^n}$$

Consequently, we have

$$\|w(\cdot, t)\|_{(H^{1/2}(\Gamma^*))^n} \leq \bar{M}_{\mathcal{A}} e^{-\bar{\alpha}_{\mathcal{A}} t} \|w_0\|_{(H^1(\Omega))^n}$$

where  $\bar{M}_{\mathcal{A}}, \bar{\alpha}_{\mathcal{A}} > 0$  and therefore the system (1) is  $\Gamma^*$ GAFO-stable. ■

## 5. Application to sensors and controls structures

In this section we consider the distributed diffusion systems defined on  $\Omega = ]0, 1[ \times ]0, 1[$ . Various results related to different types of measurements and controls have been explored; domains and boundary conditions.

### 5.1 Case of a zone sensor

**5.1.1 Rectangular domain.** In view of example 3.9 and theorem 3.8 the system (1) augmented with output function (2) with internal zone sensor and control is described by

$$\begin{cases} \frac{\partial w}{\partial t}(\zeta_1, \zeta_2, t) = \Delta w(\zeta_1, \zeta_2, t) + \chi_{\bar{D}}g(\zeta_1, \zeta_2)u(t) & \Pi \\ w(\zeta_1, \zeta_2, t) = w_0(\zeta_1, \zeta_2) & \Omega \\ \frac{\partial w}{\partial \vartheta}(\mu_1, \mu_1, t) = 0 & \Xi \\ y(t) = \chi_D f(\zeta_1, \zeta_2)w(\zeta_1, \zeta_2, t) & \Pi \end{cases} \quad (24)$$

where

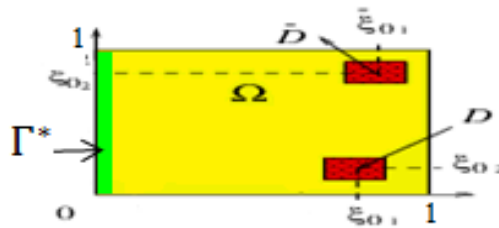
$$D = [\zeta_{0_1} - l_1, \zeta_{0_1} + l_1] \times [\zeta_{0_2} - l_2, \zeta_{0_2} + l_2] \subset \Omega$$

and

$$\bar{D} = [\bar{\zeta}_{0_1} - l_1, \bar{\zeta}_{0_2} - l_2] \times [\bar{\zeta}_{0_2} - l_2, \bar{\zeta}_{0_2} + l_2] \subset \Omega$$

are the locations of the zone sensor (actuator) (figure 4). Since the zone sensor is couple  $(D, f)$  of  $D$  and  $f$ , then

$$y(t) = Cw(\zeta, t) = \chi_D f(\zeta_1, \zeta_2)w(\zeta_1, \zeta_2, t) \quad (25)$$



**Fig.4:**  $D$  and  $\bar{D}$  internal zone sensor and control locations.

In this case, the considered region  $\Gamma^*$  is defined by

$$\Gamma^* = \{0\} \times ]0,1[ \subset ]0,1[ \times ]0,1[.$$

and the operator  $Bu(t)$  in system (24) is given by

$$Bu(t) = \chi_{\bar{D}}g(\zeta_1, \zeta_2)u(t) \quad (26)$$

and by regional boundary feedback control in (15) is defined by

$$Bu(t) = -\chi_{\bar{D}}g(\zeta_1, \zeta_2)D_{\Gamma^*}w(\cdot, t)$$

and then,

$$BD_{\Gamma^*}\chi_{\bar{D}}g = \chi_{\bar{D}}gD_{\Gamma^*} \quad (27)$$

Under the condition of example 3.9 the corresponding dynamical system (24) represented by

$$\begin{cases} \frac{\partial v}{\partial t}(\zeta_1, \zeta_2, t) = \Delta v(\zeta_1, \zeta_2, t) + \chi_{\bar{D}}g(\zeta_1, \zeta_2)u(t) \\ -H_{\Gamma^*AGFO}\chi_D f(w(\zeta_1, \zeta_2, t) - v(\zeta_1, \zeta_2, t)) & \Pi \\ v(\zeta_1, \zeta_2, 0) = v_0(\zeta_1, \zeta_2) & \Omega \\ \frac{\partial v}{\partial \vartheta}(\mu_1, \mu_1, t) = 0 & \Xi \end{cases} \quad (28)$$

and  $\Gamma^*$ AGFO-observer,

**Proposition 4.2:** If the whole system (24), (25), (28) and (12) is given the matrix form

$$\begin{bmatrix} \frac{\partial w}{\partial t} \\ \frac{\partial e}{\partial t} \end{bmatrix} = \Delta_{D_{\Gamma^*} \chi_{\bar{D}} g} \begin{bmatrix} w \\ e \end{bmatrix} \quad (29)$$

where the operator  $A_{D_{\Gamma^*} \chi_{\bar{D}} g}$  is defined by

$$\Delta_{D_{\Gamma^*} \chi_{\bar{D}} g} = \begin{bmatrix} A - \chi_{\bar{D}} g D_{\Gamma^*} \chi_{\bar{D}} g & -\chi_{\bar{D}} g D_{\Gamma^*} \chi_{\bar{D}} g \\ 0 & A - H_{\Gamma^*} \text{GAGFO} \chi_{\bar{D}} f \end{bmatrix}$$

then, the spectrum of system (24) is the reunion of the spectrum of regional boundary closed-loop control system (33) and the spectrum of  $\Gamma^*$ AGFO-observer (34) and then, achieve the stability of system (28).

**Proof:** The system (24), (25) and (12) augmented with the related dynamical system (36) described as  $\Gamma^*$ AFO-observer by the form

$$\left\{ \begin{array}{ll} \frac{\partial w}{\partial t}(\zeta, t) = \Delta w(\zeta, t) + \chi_{\bar{D}} g(\zeta_1, \zeta_2) u(t) & \Pi \\ w(\zeta, 0) = w_0(\zeta) & \Omega \\ \frac{\partial w}{\partial \vartheta}(\mu, t) = 0 & \Xi \\ y(\cdot, t) = \chi_{\bar{D}} f w(\cdot, t) & \Pi \\ u(t) = -D_{\Gamma^*} \chi_{\bar{D}} g w(\cdot, t) & \Pi \\ \lim_{t \rightarrow \infty} \|w(\cdot, t) - v(\cdot, t)\|_{H^{1/2}(\Gamma^*)} = 0 & \Pi \\ \frac{\partial v}{\partial t}(\zeta, t) = \Delta v(\zeta, t) + \chi_{\bar{D}} g(\zeta_1, \zeta_2) u(t) & \\ -H_{\Gamma^*} \text{GAGFO} \chi_{\bar{D}} f(w(\zeta, t) - v(\zeta, t)) & \Pi \\ v(\zeta, 0) = v_0(\zeta) & \Omega \\ \frac{\partial v}{\partial \vartheta}(\mu, t) = 0 & \Xi \end{array} \right. \quad (30)$$

inserting equations (24) and (25) in (30), we have the form

$$\left\{ \begin{array}{ll} \frac{\partial w}{\partial t}(\zeta, t) = \Delta w(\zeta, t) - B D_{\Gamma^*} \chi_{\bar{D}} g w(\cdot, t); & \Pi \\ w(\zeta, 0) = w_0(\zeta) & \Omega \\ \frac{\partial w}{\partial \vartheta}(\mu, t) = 0 & \Xi \\ \frac{\partial v}{\partial t}(\zeta, t) = \Delta v(\zeta, t) - B D_{\Gamma^*} \chi_{\bar{D}} g w(\zeta, t) & \\ -H_{\Gamma^*} \text{GAGFO} \chi_{\bar{D}} f(w(\zeta, t) - v(\zeta, t)) & \Pi \\ v(\zeta, 0) = v_0(\zeta) & \Omega \\ \frac{\partial v}{\partial \vartheta}(\mu, t) = 0 & \Xi \end{array} \right. \quad (31)$$

From theorem 3.8 and equation (31), we can get

$$\begin{cases} \frac{\partial w}{\partial t}(\zeta, t) = (\Delta - BD_{\Gamma^* \chi_{\bar{D}g}})w(\zeta, t) - BD_{\Gamma^* \chi_{\bar{D}g}}e(\zeta, t) & \Pi \\ w(\zeta, 0) = w_0(\zeta) & \Omega \\ \frac{\partial w}{\partial \vartheta}(\mu, t) = 0 & \Xi \\ \dot{e}(\zeta, t) = (\Delta - H_{\Gamma^* \text{GAFO}} \chi_{Df})e(\zeta, t) & \Pi \\ e(\zeta, 0) = e_0(\zeta) & \Omega \\ \frac{\partial e}{\partial \vartheta}(\mu, t) = 0 & \Xi \end{cases} \quad (32)$$

where  $e(\zeta, t) = w(\zeta, t) - v(\zeta, t)$ . Now, equation (38) permit to defend the following regional boundary control closed loop system

$$\begin{cases} \frac{\partial w}{\partial t}(\zeta, t) = (\Delta - BD_{\Gamma^* \chi_{\bar{D}g}})w(\zeta, t) - BD_{\Gamma^* \chi_{\bar{D}g}}e(\zeta, t) & \Pi \\ w(\zeta, 0) = w_0(\zeta) & \Omega \\ \frac{\partial w}{\partial \vartheta}(\mu, t) = & \Xi \end{cases} \quad (33)$$

From the proof of theorem 3.8, we have

$$\begin{cases} \frac{\partial e}{\partial t}(\zeta, t) = (\Delta - H_{\Gamma^* \text{GAFO}} \chi_{Df})e(\zeta, t) & \Pi \\ e(\zeta, 0) = e_0(\zeta) & \Omega \\ \frac{\partial e}{\partial \vartheta}(\mu, t) = 0 & \Xi \end{cases} \quad (34)$$

Thus, the combining systems (33)-(34) can be written by the matrix form

$$\begin{bmatrix} \frac{\partial w}{\partial t} \\ \frac{\partial e}{\partial t} \end{bmatrix} = \Delta_{D_{\Gamma^* \chi_{\bar{D}g}}} \begin{bmatrix} w \\ e \end{bmatrix} \quad (35)$$

where the operator  $D_{\Gamma^* \chi_{\bar{D}g}}$  is defined by

$$\Delta_{D_{\Gamma^* \chi_{\bar{D}g}}} = \begin{bmatrix} \Delta - BD_{\Gamma^* \chi_{\bar{D}g}} & -BD_{\Gamma^* \chi_{\bar{D}g}} \\ 0 & \Delta - H_{\Gamma^* \text{GAFO}} \chi_{Df} \end{bmatrix} \quad (36)$$

Consider the previous results as  $\Delta, B$ , and  $D_{\Gamma^* \chi_{\bar{D}g}}$  are bounded linear operators, then, by using perturbation theory linear operator, we can deduce that the operator  $\Delta_{D_{\Gamma^* \chi_{\bar{D}g}}}$  generates a strongly continuous semi-group. Therefore, the resolvent  $\rho(\Delta_{D_{\Gamma^* \chi_{\bar{D}g}}})$  is non-empty and can be expressed by

$$\rho(\Delta_{D_{\Gamma^* \chi_{\bar{D}g}}}) = \rho(\Delta - BD_{D_{\Gamma^* \chi_{\bar{D}g}}}) \cap \rho(BD_{D_{\Gamma^* \chi_{\bar{D}g}}})$$

Finally, we have

$$\sigma(\Delta_{D_{\Gamma^* \chi_{\bar{D}g}}}) = \sigma(\Delta - BD_{\chi_{\bar{D}g}}) \cup \sigma(BD_{D_{\Gamma^* \chi_{\bar{D}g}}})$$

where  $\sigma(\Delta_{D_{\Gamma^* \chi_{\bar{D}g}}})$  denotes the spectrum of  $\Delta_{D_{\Gamma^* \chi_{\bar{D}g}}}$ . Hence the semi group of  $\Delta_{D_{\Gamma^* \chi_{\bar{D}g}}}$

$$(S_{\Delta_{D_{\Gamma^* \chi_{\bar{D}g}}}}(t))_{t \geq 0} = \begin{bmatrix} S_{\Delta - BD_{\Gamma^* \chi_{\bar{D}g}}}(t) & S_{BD_{\Gamma^* \chi_{\bar{D}g}}}(t) \\ 0 & S_{\Delta - H_{\Gamma^* \text{GAFO}}}(t) \end{bmatrix}$$

is stable on the space

$$(H^{1/2}(\Gamma^*))^n \oplus (H^{1/2}(\Gamma^*))^n \quad [30],$$

such that

$$\|w(\cdot, t)\|_{(H^{1/2}(\Gamma^*))^n \oplus (H^{1/2}(\Gamma^*))^n} \leq M_{\Delta_{D_{\Gamma^*} \chi_{\bar{D}g}}} e^{-\alpha_{\Delta_{\Gamma^*} t}} \|w_0(\cdot)\|_{(H^1(\Omega))^n \oplus (H^1(\Omega))^n}$$

Consequently, we have

$$\|w(\cdot, t)\|_{(H^{1/2}(\Gamma^*))^n} \leq \bar{M}_{\Delta_{D_{\Gamma^*} \chi_{\bar{D}g}}} e^{-\bar{\alpha}_{\Delta_{D_{\Gamma^*} \chi_{\bar{D}g}} t}} \|w_0\|_{(H^1(\Omega))^n}$$

where  $\bar{M}_{\Delta_{D_{\Gamma^*} \chi_{\bar{D}g}}, \bar{\alpha}_{\Delta_{D_{\Gamma^*} \chi_{\bar{D}g}}} > 0$  and therefore the system (24) is  $\Gamma^*$ GAFO-stable. ■

### 5.1.2 Circular domain

**Remark 5.3:** The previous results can be extend to the case of circular domain with the following system

$$\begin{cases} \frac{\partial w}{\partial t}(r, \theta, t) = \Delta w(r, \theta, t) + w(r, \theta, t) & \Pi \\ w(r, \theta, 0) = w(r, \theta) & \bar{\Omega} \\ \frac{\partial w}{\partial \vartheta}(a, \theta, t) = 0 & \Xi \end{cases} \quad (37)$$

augmented output function is defined by

$$y(t) = \int_D \frac{\partial v}{\partial \vartheta}(a, \theta, t) f(a, \theta) dr d\theta \quad (38)$$

where  $\Omega = D ]0, a[, r = a > 0, \theta \in [0, 2\pi]$ , for internal zone and pointwise sensor with another output function [23].

### 5.2 Case of a pointwise sensor

In this subsection, we consider the following cases:

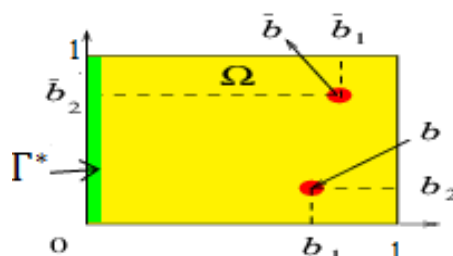
#### 5.2.1 The domain $\Omega = ]0, 1[ \times ]0, 1[$

By the same way in subsection 5.1, we can develop system (30) with internal pointwise sensor in the following equations

$$\begin{cases} \frac{\partial w}{\partial t}(\zeta_1, \zeta_2, t) = \Delta w(\zeta_1, \zeta_2, t) + \delta_{\bar{b}}(\zeta_1, \zeta_2)u(t) & \Pi \\ w(\zeta_1, \zeta_2, t) = w_0(\zeta_1, \zeta_2) & \Omega \\ \frac{\partial w}{\partial \vartheta}(\mu_1, \mu_1, t) = 0 & \Xi \\ y(t) = \delta_b(b_1, b_2)w(\zeta_1, \zeta_2, t) & \Pi \end{cases} \quad (39)$$

where  $\delta_{\bar{b}}(\xi_1, \xi_2) = \delta(\xi_1 - \bar{b}_1, \xi_2 - \bar{b}_1)$  and  $\bar{b} = (b_{\bar{b}_1}, b_{\bar{b}_2}) \in \Omega$  is location of the internal pointwise control  $(b, \delta_b)$  (figure 5). Since the zone sensor is couple  $(b, \delta_b)$  of  $b$  and  $\delta_b$ , then

$$y(t) = Cw(\zeta_1, \zeta_2, t) = \delta_b(b_1, b_2)w(\zeta_1, \zeta_2, t) \quad (40)$$



**Fig 5:**  $b$  and  $\bar{b}$  pointwise sensor and control locations.

and the operator  $Bu(t)$  in system (39) is given by

$$Bu(t) = \delta_{\bar{b}}(b_1, b_2)u(t) \quad (41)$$

and by regional boundary feedback control in (23) is defined by

$$Bu(t) = -\delta_{\bar{b}}(b_1, b_2)D_{\Gamma^*}w(\cdot, t)$$

and then,

$$BD_{\Gamma^*}\delta_{\bar{b}} = \delta_{\bar{b}}D_{\Gamma^*} \quad (42)$$

Under the condition of example 3.9 the corresponding dynamical system (24) represented by

$$\begin{cases} \frac{\partial v}{\partial t}(\zeta_1, \zeta_2, t) = \Delta v(\zeta_1, \zeta_2, t) + \delta_{\bar{b}}(b_1, b_2)u(t) \\ -H_{\Gamma^*AGFO}\delta_b(b_1, b_2)(w(\zeta_1, \zeta_2, t) - v(\zeta_1, \zeta_2, t)) & \Pi \\ v(\zeta_1, \zeta_2, 0) = v_0(\zeta_1, \zeta_2) & \Omega \\ \frac{\partial v}{\partial \vartheta}(\mu_1, \mu_1, t) = 0 & \Xi \end{cases} \quad (43)$$

and  $\Gamma^*$ AGFO-observer,

**Proposition 4.2:** If the overall system (39), (40), (43) and (12) is given the matrix form

$$\begin{bmatrix} \frac{\partial w}{\partial t} \\ \frac{\partial e}{\partial t} \end{bmatrix} = \Delta_{D_{\Gamma^*}\delta_{\bar{b}}} \begin{bmatrix} w \\ e \end{bmatrix} \quad (44)$$

where the operator  $\Delta_{D_{\Gamma^*}\delta_{\bar{b}}}$  is defined by

$$\Delta_{D_{\Gamma^*}\delta_{\bar{b}}} = \begin{bmatrix} \Delta - \delta_{\bar{b}}D_{\Gamma^*}\delta_{\bar{b}} & -\delta_{\bar{b}}D_{\Gamma^*}\delta_{\bar{b}} \\ 0 & \Delta - H_{\Gamma^*GAFO}\delta_b \end{bmatrix}$$

then, the spectrum of system (44) is the reunion of the spectrum of regional boundary closed-loop control system (48) and the spectrum of  $\Gamma^*$ AGFO-observer (49) and then, achieve the stability of system (39).

**Proof:** The system (33), (34) and (12) augmented with the related dynamical system (28) described as  $\Gamma^*$ AFO-observer by the form



$$\left\{ \begin{array}{l} \frac{\partial w}{\partial t}(\zeta, t) = \Delta w(\zeta, t) + \delta_{\bar{b}}(b_1, b_2)u(t) \quad \Pi \\ w(\zeta, 0) = w_0(\zeta) \quad \Omega \\ \frac{\partial w}{\partial \vartheta}(\mu, t) = 0 \quad \Xi \\ y(\cdot, t) = \delta_b w(\cdot, t) \quad \Pi \\ u(t) = -D_{\Gamma^*} \delta_{\bar{b}} w(\cdot, t) \quad \Pi \\ \lim_{t \rightarrow \infty} \|w(\cdot, t) - v(\cdot, t)\|_{H^{1/2}(\Gamma^*)} = 0 \quad \Pi \\ \frac{\partial v}{\partial t}(\zeta, t) = \Delta v(\zeta, t) + \delta_{\bar{b}}(b_1, b_2)u(t) \\ -H_{\Gamma^* \text{GAFO}} \delta_b (w(\zeta, t) - v(\zeta, t)) \quad \Pi \\ v(\zeta, 0) = v_0(\zeta) \quad \Omega \\ \frac{\partial v}{\partial \vartheta}(\mu, t) = 0 \quad \Xi \end{array} \right. \quad (45)$$

From the equations (397), (40) and (45), we have the form

$$\left\{ \begin{array}{l} \frac{\partial w}{\partial t}(\zeta, t) = \Delta w(\zeta, t) - BD_{\Gamma^*} \delta_{\bar{b}} w(\cdot, t); \quad \Pi \\ w(\zeta, 0) = w_0(\zeta) \quad \Omega \\ \frac{\partial w}{\partial \vartheta}(\mu, t) = 0 \quad \Xi \\ \frac{\partial v}{\partial t}(\zeta, t) = \Delta v(\zeta, t) - BD_{\Gamma^*} \delta_{\bar{b}} w(\zeta, t) \\ -H_{\Gamma^* \text{GAFO}} \delta_b (w(\zeta, t) - v(\zeta, t)) \quad \Pi \\ v(\zeta, 0) = v_0(\zeta) \quad \Omega \\ \frac{\partial v}{\partial \vartheta}(\mu, t) = 0 \quad \Xi \end{array} \right. \quad (46)$$

From theorem 3.8 and equation (46), we can get

$$\left\{ \begin{array}{l} \frac{\partial w}{\partial t}(\zeta, t) = (\Delta - BD_{\Gamma^*} \delta_{\bar{b}})w(\zeta, t) - BD_{\Gamma^*} \delta_{\bar{b}} e(\zeta, t) \quad \Pi \\ w(\zeta, 0) = w_0(\zeta) \quad \Omega \\ \frac{\partial w}{\partial \vartheta}(\mu, t) = 0 \quad \Xi \\ \dot{e}(\zeta, t) = (\Delta - H_{\Gamma^* \text{GAFO}} \delta_b)e(\zeta, t) \quad \Pi \\ e(\zeta, 0) = e_0(\zeta) \quad \Omega \\ \frac{\partial e}{\partial \vartheta}(\mu, t) = 0 \quad \Xi \end{array} \right. \quad (47)$$

where  $e(\zeta, t) = w(\zeta, t) - v(\zeta, t)$ . Now, equation (47) permit to defend the following regional boundary control closed loop system

$$\left\{ \begin{array}{l} \frac{\partial w}{\partial t}(\zeta, t) = (\Delta - BD_{\Gamma^*} \delta_{\bar{b}})w(\zeta, t) - BD_{\Gamma^*} \delta_{\bar{b}} e(\zeta, t) \quad \Pi \\ w(\zeta, 0) = w_0(\zeta) \quad \Omega \\ \frac{\partial v}{\partial \vartheta}(\mu, t) = 0 \quad \Xi \end{array} \right. \quad (48)$$

From the proof of theorem 3.8, we have

$$\left\{ \begin{array}{l} \frac{\partial e}{\partial t}(\zeta, t) = (\Delta - H_{\Gamma^* \text{GAFO}} \delta_b)e(\zeta, t) \quad \Pi \\ e(\zeta, 0) = e_0(\zeta) \quad \Omega \\ \frac{\partial e}{\partial \vartheta}(\mu, t) = 0 \quad \Xi \end{array} \right. \quad (49)$$

Thus, the combining systems (48)-(49) can be written by the matrix form

$$\begin{bmatrix} \frac{\partial w}{\partial t} \\ \frac{\partial e}{\partial t} \end{bmatrix} = \Delta_{\Gamma^* \delta} \begin{bmatrix} w \\ e \end{bmatrix} \quad (50)$$

where the operator  $D_{\Gamma^*} \chi_{\bar{D}} g^*$  is defined by

$$\Delta_{\Gamma^* \delta} = \begin{bmatrix} \Delta - BD_{\Gamma^* \delta_{\bar{b}}} & -BD_{\Gamma^* \delta_{\bar{b}}} \\ 0 & \Delta - H_{\Gamma^* \text{GAFO}} \delta_b \end{bmatrix} \quad (51)$$

Consider the previous results as  $\Delta, B$ , and  $D_{\Gamma^* \delta_{\bar{b}}}$  are bounded linear operators, then, by using perturbation theory linear operator, we can deduce that the operator  $\Delta_{\Gamma^* \delta}$  generates a strongly continuous semi-group. Therefore, the resolvent  $\rho(\Delta_{\Gamma^* \delta})$  is non-empty and can be expressed by

$$\rho(\Delta_{\Gamma^* \delta}) = \rho(\Delta - BD_{\Gamma^* \delta_{\bar{b}}}) \cap \rho(BD_{\Gamma^* \delta_{\bar{b}}})$$

Finally, we have

$$\sigma(\Delta_{\Gamma^* \delta}) = \sigma(\Delta - BD_{\Gamma^* \delta_{\bar{b}}}) \cup \sigma(BD_{\Gamma^* \delta_{\bar{b}}})$$

where  $\sigma(\Delta_{\Gamma^* \delta})$  denotes the spectrum of  $\Delta_{\Gamma^* \delta}$ . Hence the semi group of  $\Delta_{\Gamma^* \delta}$

$$(S_{\Delta_{\Gamma^* \delta}}(t))_{t \geq 0} = \begin{bmatrix} S_{\Delta - BD_{\Gamma^* \delta_{\bar{b}}}}(t) & S_{BD_{\Gamma^* \delta_{\bar{b}}}}(t) \\ 0 & S_{\Delta - H_{\Gamma^* \text{GAFO}}} \delta_b(t) \end{bmatrix}$$

is stable on the space

$$(H^{1/2}(\Gamma^*))^n \oplus (H^{1/2}(\Gamma^*))^n \quad [30],$$

such that

$$\left\| \begin{bmatrix} w(\cdot, t) \\ e(\cdot, t) \end{bmatrix} \right\|_{(H^{1/2}(\Gamma^*))^n \oplus (H^{1/2}(\Gamma^*))^n} \leq M_{\Delta_{\Gamma^* \delta}} e^{-\alpha_{\Delta_{\Gamma^* \delta}} t} \cdot \left\| \begin{bmatrix} w_0(\cdot) \\ e_0(\cdot) \end{bmatrix} \right\|_{(H^1(\Omega))^n \oplus (H^1(\Omega))^n}$$

Consequently, we have

$$\|w(\cdot, t)\|_{(H^{1/2}(\Gamma^*))^n} \leq \bar{M}_{\Delta_{\Gamma^* \delta}} e^{-\bar{\alpha}_{\Delta_{\Gamma^* \delta}} t} \|w_0\|_{(H^1(\Omega))^n}$$

where  $\bar{M}_{\Delta_{\Gamma^* \delta}}, \bar{\alpha}_{\Delta_{\Gamma^* \delta}} > 0$  and therefore the system (24) is  $\Gamma^*$ GAFO-stable. ■

**5.2.1 The domain**  $\Omega = D ]0, a[, r = a > 0, \theta \in [0, 2\pi]$ ,

**Remark 5.3:** The previous results can be extend to the case of circular domain with system (15) augmented to the output function

$$y(t) = \int_D \frac{\partial v}{\partial v}(a, \theta, t) f(a, \theta) dr d\theta \quad (52)$$

where  $\Omega = D ]0, a[, r = a > 0, \theta \in [0, 2\pi]$ , for internal zone and pointwise sensor with another output function [23].

## 6. Conclusion

Various characterizations have been established using the associated gradient sensors structures and feedback control for the RBAGFO-observer. New results have been obtained linked to divert kind of the information: controls; domains; conditions of boundary. It has been shown that the structure of sensors and feedback controls pledge the existence of RBAGFO-observer which is enable to characterize regional

boundary closed loop control system. Numerous applications have illustrated and demonstrated to diffusion distributed parameter systems in different situations. May be interested to extend these results to the case of hyperbolic systems as in [25].

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