A New Class of Rectangular Distribution Properties and Application

Salma Omar Bleed
Al-asmarya University

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Salma Omar Bleed
College of Science, Statistics Department, Al-asmarya University, Libya
E-mail: SalmaBleed@yahoo.com

Abstract

In this article, we generalize the Rectangular distribution using the quadratic rank transmutation map studied by Shaw and Buckley (2007) to develop a transmuted Rectangular distribution. The expectation, variance, moments, reliability function, hazard rate function, Cumulative Hazard function, the moment generating function, and the characteristic function of the new distribution are provided. In addition, the model parameters are estimated by the maximum likelihood method. Finally, an application to generated data sets is illustrated.

Keywords: Rectangular distribution, Uniform distribution, Moments, Transmutation Map, Maximum Likelihood Estimation, Reliability Function.

1- Introduction and Motivation

In probability theory and statistics, the continuous uniform distribution or Rectangular distribution is a family of symmetric probability distributions, such that for each member of the family all intervals of the same length on the distribution's support are equally probable. A uniform random variable is equally likely to take any value between its lower limit (a) and its upper limit (b). The distribution is often abbreviated $U(a,b)$, [Park and Anil, 2009] . For a uniform random variable between a and b, the function $f_X(x)$ is constant between a and b. Since the probability that the random variable falls between a and b must be 1, the function is given by

$$f_X(x) = \begin{cases} 0, & if \quad x < a \\ \dfrac{1}{b-a}, & if \quad a \le x \le b \\ 0, & if \quad x > b \end{cases}$$

With the distribution function (cdf)

$$F(x)=\begin{cases} 0 & x\le a \\ \dfrac{x-a}{b-a} & a\le x\le b \\ 1 & b\le x \end{cases}$$

A random variable *X* that is uniformly distributed between *a* and *b* has:

Expected Value, median	$=\dfrac{(a+b)}{2}$
Variance	$=\dfrac{(b-a)^2}{12}$
Standard Deviation	$=\dfrac{(b-a)}{\sqrt{12}}$
mode	Any value between a and b
MGF	$=\dfrac{\left(e^{tb}-e^{ta}\right)}{t\,(b-a)}$
Raw Moments	$=\dfrac{\sum_{r=0}^{k} a^r b^{k-r}}{k+1}$
Order statistics	$=\dfrac{k}{n+1}$

Notice that, if X has a standard uniform distribution, then by the inverse transform sampling method, $Y=-\lambda^{-1}\, ln(X)$ has an exponential distribution with (rate) parameter λ. If X has a standard uniform distribution, then $Y = X^n$ has a beta distribution with parameters $(1/n, 1)$. As such, If X has a standard uniform distribution, then $Y = X$ is also a special case of the beta distribution with parameters *(1,1)*. The Irwin–Hall distribution is the sum of *n* i.i.d. *U(0,1)* distributions. The sum of two independent, equally distributed uniform distributions yields a symmetric triangular distribution. The distance between two i.i.d. uniform random variables also has a triangular distribution, although not symmetric, form. For more details, see Johnson et al. (1995).

In this article, transmutation map approach suggested by Shaw and Buckley (2007) to define a new model, which generalizes the Rectangular model is used. It is called the generalized distribution as the transmuted Rectangular distribution and denoted by ***TRD***. According to the Quadratic Rank Transmutation Map (QRTM), approach the cumulative distribution function (cdf) satisfy the relationship

$$F(x) = (1+\lambda)F_1(x) - \lambda F_1^2(x) \rightarrow (1)$$

where $F_1(x)$ is the cumulative distribution function (*cdf*) of the base distribution, which on differentiation yields, $f(x)$, such that

$$f(x) = (1+\lambda)f_1(x) - 2\lambda f_1(x)F_1(x) \rightarrow (2)$$

If $\lambda = 0$ then the distribution of the base random variable is obtained. In the rest of this paper, mathematical formulations of the transmuted Rectangular distribution and some of its properties are provided. For more information about the quadratic rank transmutation map, see Shaw and Buckley (2009).

2- Transmuted Rectangular Distribution

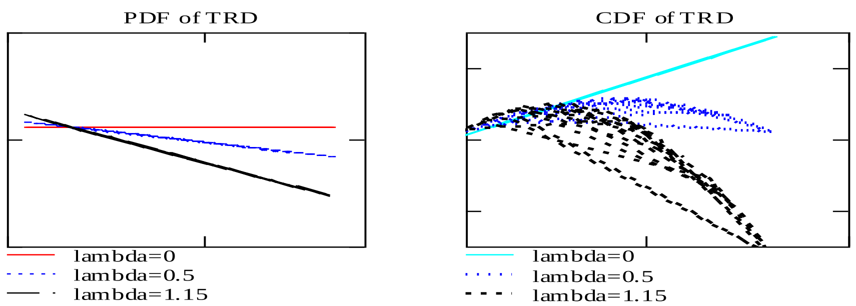
By using Eq.(1) and Eq.(2), the *cdf* of transmuted Rectangular distribution (**TRD**) has the following form

$$F(x) = (1+\lambda)F_1(x) - \lambda F_1^2(x) = \frac{(1+\lambda)(x-a)}{(b-a)} - \frac{\lambda(x-a)^2}{(b-a)^2}, \quad a \le x \le b \rightarrow (3)$$

Where λ is the transmuted parameter. The corresponding probability density function of Eq.(3) is given as follows

$$f(x) = (1+\lambda)f_1(x) - \lambda f_1(x)F_1(x) = \frac{(1+\lambda)}{(b-a)} - \frac{2\lambda(x-a)}{(b-a)^2}, \quad a \le x \le b \rightarrow (4)$$

It is observed that, the transmuted Rectangular distribution is an extended model to analyze data from complex situations, as shown in the following plots which are the *pdf* and *cdf* of ***TRD*** distribution for selected values of the parameters.



3- Statistical Properties

In this section, some statistical properties of the new generalization are provided.

3-1 The Reliability or the Survivor function: There is a relation between the *cdf* and the reliability function, i.e., $RF + F(x) = 1$. Therefore, the reliability function of the transmuted Rectangular distribution (***RF*~TRD~**) also known as the survivor function and is defined as:

$$RF_{TRD} = 1 - F(x) = 1 - \frac{(1+\lambda)(x-a)}{(b-a)} + \frac{\lambda(x-a)^2}{(b-a)^2}, \quad a \le x \le b \rightarrow (5)$$

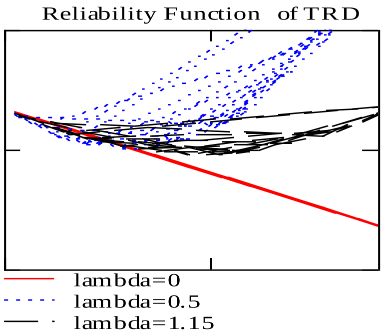
3-2 The Hazard function : There is a relation between the *pdf*, reliability and hazard function, i.e., $HF(x) = \frac{f(x)}{RF(x)}$. Therefore, the hazard function of the transmuted Rectangular distribution (***HF*~TRD~**) is defined as:

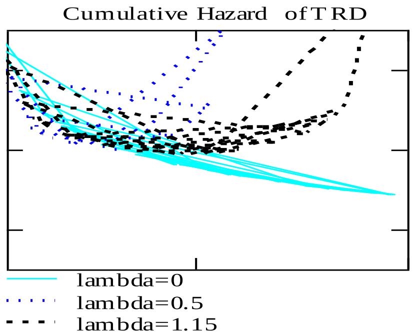
$$HF_{TRD} = \left(\frac{(1+\lambda)}{(b-a)} - \frac{2\lambda(x-a)}{(b-a)^2}\right)\left[1 - \frac{(1+\lambda)(x-a)}{(b-a)} + \frac{\lambda(x-a)^2}{(b-a)^2}\right]^{-1}, \quad a \le x \le b \rightarrow (6)$$

3-3 The Cumulative Hazard function : There is a relation between the *cdf* and the cumulative hazard function, i.e., $CHF(x) = -\ln[F(x)]$. Therefore, the cumulative hazard function of the transmuted Rectangular distribution (***CHF*~TRD~**) is defined as :

$$CHF_{TRD} = -\ln\left[1 - \frac{(1+\lambda)(x-a)}{(b-a)} + \frac{\lambda(x-a)^2}{(b-a)^2}\right], \quad \rightarrow (7)$$

It is observed that, the transmuted Rectangular distribution has increasing patterns reliability and cumulative hazard function, as shown in the following plots.





3-4 Random number generation and parameter estimation

To generate random numbers when the parameters a and b are known, we can use the method of inversion from the transmuted Rectangular distribution as

$$x_{r.v} = \left[u\left(b-a\right) + (1+\lambda)a - \frac{\lambda a^2}{\left(b-a\right)} \right] . \left[(1+\lambda) - \frac{\left(2a + x_{r.v}\right)\lambda}{\left(b-a\right)} \right]^{-1}, \quad \rightarrow (8)$$

Eq.(8) doesn't have a closed form solution, so u will be generated as uniform random variables from $U(0,1)$, and then solve for $x_{r.v}$ in order to generate random numbers from ***TRD*** distribution.

3-5 Moments

In applications, it is necessary and important in any statistical analysis to derive moments, so in this subsection, the r^{th} moment for (*TRD*) distribution are derived.

Theorem 1: If X has *TRD* then the r^{th} moment for (*TRD*) distribution are

$$\mu_r' = \frac{(1+\lambda)\left(b^{r+1} - a^{r+1}\right)}{\left(b-a\right)\left(r+1\right)} + \frac{2\lambda\left(b^{r+2} - a^{r+2}\right)}{\left(b-a\right)^2\left(r+1\right)\left(r+2\right)} - \frac{2\lambda b^{r+1}}{\left(b-a\right)\left(r+1\right)}, \quad \rightarrow (9)$$

Proof:

From Eq. (2), The r^{th} ordinary moment of the (*TRD*) distribution is given by

$$\mu_r' = \int_a^b x^r f(x)dx = \int_a^b x^r \left[\frac{(1+\lambda)}{(b-a)} - \frac{2\lambda(x-a)}{(b-a)^2}\right]dx$$

$$= \frac{(1+\lambda)}{(b-a)}\int_a^b x^r dx - \frac{2\lambda}{(b-a)^2}\int_a^b x^r(x-a)dx =$$

$$= \frac{(1+\lambda)\left(b^{r+1}-a^{r+1}\right)}{(b-a)(r+1)} + \frac{2\lambda\left(b^{r+2}-a^{r+2}\right)}{(b-a)^2(r+1)(r+2)} - \frac{2\lambda b^{r+1}}{(b-a)(r+1)}$$

Notices that, if $\lambda = 0$ then $\mu_r' = \frac{\left(b^{r+1}-a^{r+1}\right)}{(b-a)(r+1)}$, which are the r^{th} ordinary moment of the Rectangular distribution.

3-6 Moment Generating Function

The moment generating function is important especially if it is existing. Then in this section, the moment generating function of (*TRD*) distribution is derived**.**

Theorem 2: If X has ***TRD*** then the moment generating function of (***TRD***) distribution is

$$MGF = \frac{(1+\lambda)\left(e^{bt}-e^{at}\right)}{t(b-a)} + \frac{2\lambda\left(e^{bt}-e^{at}\right)}{t^2(b-a)^2} - \frac{2\lambda e^{bt}}{t(b-a)}, \quad \rightarrow (10)$$

Proof:

From Eq. (2), The moment generating function of the (***TRD***) distribution is given by

$$MGF = E(e^{tx}) = \int_a^b e^{tx} f(x)dx = \frac{(1+\lambda)}{(b-a)}\int_a^b e^{tx}dx - \frac{2\lambda}{(b-a)^2}\int_a^b e^{tx}(x-a)dx =$$

$$= \frac{(1+\lambda)\left(e^{bt}-e^{at}\right)}{t(b-a)} + \frac{2\lambda\left(e^{bt}-e^{at}\right)}{t^2(b-a)^2} - \frac{2\lambda e^{bt}}{t(b-a)}$$

Notice that, if $\lambda = 0$ then we have

$$MGF = \frac{\left(e^{bt}-e^{at}\right)}{t(b-a)},$$

which is the moment generating function of the Rectangular distribution. The r^{th} moment of X can be obtained by using the MGF (10), which on differentiation yields r^{th} moment. The moment generating function of the (***TRD***) be expressed as

$$MGF = A + t\left[\frac{(b+a)}{2} - \frac{\lambda(b-a)}{6}\right] + \frac{t^2}{2!}\left[\frac{(b^2+ba+a^2)}{3} - \frac{\lambda(b^2+a^2)(b+a)}{6(b-a)}\right]$$
$$+\frac{t^3}{3!}\left[\frac{3(1+\lambda)(b^4-a^4)}{2(b-a)} + \frac{3\lambda(b^5-a^5)}{5(b-a)^2} - \frac{3\lambda b^4}{(b-a)}\right] + \dots \rightarrow (11)$$

Where, $A = \left[(1+\lambda) + \frac{2\lambda}{t(b-a)} - \frac{2\lambda(b+t^{-1})}{(b-a)} + \frac{(b^2-a^2)}{2}\right]$. From Eq.(11) notice that, the r^{th} moment is the coefficient of $\frac{t^r}{r!}$, therefore

$$\mu_1' = \left[\frac{(b+a)}{2} - \frac{\lambda(b-a)}{6}\right],$$
$$\mu_2' = \left[\frac{(b^2+ba+a^2)}{3} - \frac{\lambda(b^2+a^2)(b+a)}{6(b-a)}\right],$$
$$\mu_3' = \left[\frac{3(1+\lambda)(b^4-a^4)}{2(b-a)} + \frac{3\lambda(b^5-a^5)}{5(b-a)^2} - \frac{3\lambda b^4}{(b-a)}\right]$$

Therefore, the mean and the variance of the (***TRD***) distribution are

$$Mean = \mu_1' = \left[\frac{(b+a)}{2} - \frac{\lambda(b-a)}{6}\right],$$
$$Variance = \mu_2' - \mu_1'^2 = \left[\frac{(b^2+ba+a^2)}{3} - \frac{\lambda(b^2+a^2)(b+a)}{6(b-a)}\right] - \left[\frac{(b+a)}{2} - \frac{\lambda(b-a)}{6}\right]^2$$
$$= \frac{(b-a)^2}{12} - \frac{\lambda(b^3+a^3)}{3(b-a)} - \frac{\lambda^2(b-a)^2}{36}$$

Notice that, if $\lambda = 0$ then the mean, variance, and the 1^{th} three moments of the Rectangular distribution are

$$Mean=\left[\frac{(b+a)}{2}\right],\qquad Variance=\frac{(b-a)^2}{12}$$

$$\mu_1'=\left[\frac{(b+a)}{2}\right],\ \mu_2'=\left[\frac{(b^2+ba+a^2)}{3}\right],\ \mu_3'=\left[\frac{3(b^4-a^4)}{2(b-a)}\right]$$

In addition, we can obtain the characteristic function of the ***TRD*** distribution.

Theorem 3: If X has ***TRD*** with the moment generating function MGF(11), then the characteristic function QF of the ***TRD*** distribution is

$$QF=E(e^{itx})=A+it\left[\frac{(b+a)}{2}-\frac{\lambda(b-a)}{6}\right]+\frac{t^2}{2!}\left[\frac{(b^2+ba+a^2)}{3}-\frac{\lambda(b^2+a^2)(b+a)}{6(b-a)}\right]+\ldots\rightarrow(12)$$

Proof:

From Eq.(2), and MGF(11) of the ***TRD*** the characteristic function QF of the ***TRD*** is given by

$$QF=E(e^{itx})=m_x(it)$$

Notice that, if $\lambda=0$ then $QF=E(e^{itx})=A+it\left[\frac{(b+a)}{2}\right]+\frac{t^2}{2!}\left[\frac{(b^2+ba+a^2)}{3}-\right]+\ldots$ which is the characteristic function of the Rectangular distribution.

4- Maximum Likelihood Estimates

The maximum likelihood estimates, MLEs, of the parameters that are inherent within the transmuted Rectangular probability distribution function is given by the following: Let $x_1, x_2, \cdots, x_n$ be a sample of size n from a transmuted Rectangular distribution, then the likelihood function is given by

$$L=\frac{(1+\lambda)^n}{(b-a)^n}-\frac{2\lambda}{(b-a)^2}\prod_{i=1}^{n}(x_i-a)\quad\rightarrow(13)$$

The log-likelihood function of Eq.(13) is given by

$$\log L=n\log(1+\lambda)-n\log(b-a)-\frac{2\lambda}{(b-a)^2}\sum_{i=1}^{n}\log(x_i-a)\quad\rightarrow(14)$$

The log-likelihood can be maximized by differentiating Eq.(14) to obtain the maximum likelihood estimate (*MLE*) of the unknown parameter λ. The *MLE* of the unknown parameter λ has closed form and is given by

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{(1+\lambda)} - \frac{2}{(b-a)^2}\sum_{i=1}^{n}\log(x_i - a)$$

Therefore,

$$\hat{\lambda} = n\left[\frac{2}{(b-a)^2}\sum_{i=1}^{n}\log(x_i - a)\right]\text{-1}$$

The variance of the unknown parameter λ is given by

$$I(\lambda) = -\frac{\partial^2 \log L}{\partial \lambda^2} = \frac{n}{(1+\lambda)^2}$$

5- Application of Transmuted Rectangular distribution

The estimators and the corresponding summary statistics are obtained by the proposed model using MathCAD program. For a given samples with different choices of a, b, and λ we obtain the maximum likelihood estimators (MLEs), the mean squared error (MSE), relative bias (RAB) and the confidence interval for λ, Table 1 summarizes the results. Estimate the true parameter λ , well with relatively small MSEs and RAB. We also notice that the coverage probabilities of the asymptotic confidence interval are close to the nominal level. These results indicate that the proposed model and the asymptotic approximation work well under the situation.

Table 1: summarizes the results of the estimates for λ

$b = 0.4, a = 0.001, \lambda = 1.990$					
parameter	N	*MLE*	*MSE*	*ARBias*	*C.I*
λ	30	1.9793	1.2E-4	0.0054	1.5354-2.0090
$b = 0.4, a = 0.001, \lambda = 1.993$					
parameter	N	*MLE*	*MSE*	*ARBias*	*C.I*
λ	30	1.98190	1.2E-4	0.0056	1.6345-2.0110
$b = 0.4, a = 0.001, \lambda = 2.0$					
parameter	N	*MLE*	*MSE*	*ARBias*	*C.I*
λ	50	2.00000	1.5E-7	0.0001929	1.8790-2.2431
$b = 0.4, a = 0.001, \lambda = 2.0$					
parameter	N	*MLE*	*MSE*	*ARBias*	*C.I*
λ	70	1.9880	3.0E-5	0.0027	1.7780-1.9987
$b = 0.4, a = 0.001, \lambda = 1.993$					

$parameter$	N	MLE	MSE	$ARBias$	$C.I$
λ	90	2.00000	1.6E-8	0.000063	1.8971-2.1324
$b = 0.4, a = 0.001, \lambda = 1.992$					
$parameter$	N	MLE	MSE	$ARBias$	$C.I$
λ	180	1.99200	5.1E-5	0.0036	1.8670-2.0912

6- Conclusion

In this article, a new model called the transmuted Rectangular distribution was proposed, which extends the Rectangular, or the uniform distribution in the analysis of data with real support. Expansions for the expectation, variance, moments, reliability function, hazard rate function, the moment generating function, the characteristic function, the MLE of the unknown parameter with its variance were derived. An application of the transmuted Rectangular distribution to generated data show that the new distribution can be used quite effectively to provide better fits than the Rectangular distribution, and that an obvious reason for generalizing a standard distribution of the Rectangular distribution, so it provides greater flexibility in modeling real data.

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الخلاصـــــــــة

في هذه المقالة تم اقتراح نموذج جديد يسمى بـ transmuted Rectangular distribution و الذي يعتبر تعميم للتوزيع الاحصائي المعروف بالتوزيع المنتظم او Rectangular, or the uniform distribution و ذلك بأستخدام quadratic rank transmutation map لـ Shaw and Buckley (2007). تم اشتقاق المتوسط الحسابي، التباين، دالة الموثوقية، دالة المخاطرة، العزوم، الدالة المولدة للعزوم، و الدالة المميزة للتوزيع الجديد transmuted Rectangular distribution. اضافة لذلك تم تقدير معالم التوزيع المجهولة بأستخدام طريقة الامكان الاكبر و ايجاد التباين.

ومن ثم أجراء تجارب محاكاة على حجوم عينات مختلفة ($n = 30, 50$ و 70 و 90 و 180) لتوضيح الجانب التطبيقي لهذا التوزيع الجديد transmuted Rectangular distribution، و لقد بينت النتائج أن التوزيع الجديد transmuted Rectangular distribution اكثر مرونة من التوزيع المنتظم Rectangular, or the uniform distribution و يمكن أن يستعمل عمليا ليعطي توفيقات افضل من التوزيع النتظم.