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Equivalent reliability polynomials modeling EAS and their geometries

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Abstract. In this paper we shall introduce two equivalent techniques in order to evaluate reliability analysis of electrical aircrafts systems (EAS): (i) graph theory technique, and (ii) simplifying diffeomorphism technique. Geometric modeling of reliability models is based on algebraic hypersurfaces, whose intrinsic properties are able to select those models which are relevant for applications. The basic idea is to cover the reliability hypersurfaces by exponentially decay curves. Most of the calculations made in this paper have used Maple and Matlab software.

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1 Introduction

Network reliability analysis receives great attention for the planning, effectiveness, and protection of many real world systems, such as computers, communications, electrical circuits, or power networks [7, 8]. The units of a network are subject to random failures, as more companies and institutions become dependent upon networked computing applications. Failure of a single unit may directly affect the operation of a network, so the probability of each unit of a network is a very important while considering the reliability of a network. Hence the reliability consideration is an important factor in networked computing. There are many exact techniques for computation of complex system or network reliability. The bridge structure or network model is a directed stochastic graph $G = (V, E)$, where V is the vertex (node) set, and E is the set of directed edges (arcs). The edges represent units that can fail with known probabilities. In real problems, these probabilities are usually evaluated from statistical data [10]-[14]. In the exact methods there are two types for the computation of the network reliability. The first type deals with the enumeration of all the minimum *cuts* or *paths*. The probabilistic computations uses the *sum of disjoint product method* because this enumeration provide non-disjoint events. Many researcher works about this type of technique have been presented in the papers. In the second type, the algorithms are based on applying equivalent transformation techniques in terms of reliability, similar to the transformation theorems for electrical systems applied to determine the equivalent impedance between two nodes [2]. In the first process we reduce the size of the graph by removing some structures. For that we use *delta star reductions* [1]. In this way, we will be able to compute the reliability in linear time and the reduction will result in a single edge.

2 Mean time to failure

We will present first the concepts in network topology and in graph theory ([7]- [14]), which are needed to calculate the network reliability.

Definition 2.1. *A graph $G = (V, E)$, where V is the set of vertices (or nodes) and E the set of edges (or arcs), is called a network.*

Definition 2.2. *The average*

$$MTTF = \int_0^\infty R(t)dt,$$

is called mean time to failure (MTTF).

($MTTF$) is a basic measure of reliability for non-reparable systems. It represents the average failure free operating time, during a particular measurement period under stated conditions.

This number do not depend on the reliability polynomials, but only on their pullbacks. Two different reliability polynomials can have the same pullback $R(t)$. Consequently, we can introduce a theory of systems with assigned pullback. For example, giving $R(t) = ae^{-\frac{t^2}{2}}$, $a < \frac{2}{\sqrt{\pi}}$ we find $(MTTF) = a\frac{\sqrt{\pi}}{2} < 1$.

Synthesis problem *Choose the structure and the components of a system, to realize a given pullback.*

3 Bridge structure

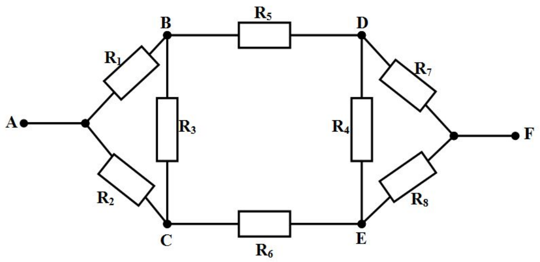

FIG. 1. A bridge network

In some engineering systems, units may be connected in a bridge configuration as shown in Fig. 1 which represent a three-phase electrical generator, part of the airplane power system, powered by a three-phase electric motor [1].

The system in Fig. 1 can be transform into its equivalent series and parallel form by using detla-star technique [2].

We describe our system as a directed network consisting of nodes and arcs (paths), as illustrated in Fig. 1, the first node is considered as the *source* (node A in the figure), and a last node is considered as a *sink* (node F). Each component of the network is identified as a path passing from one node to another. The arcs are numbered for identification. A *failure of a component* is equivalent to an arc being removed or cut out from the network. The system is *successful* if there exists a valid path from the source to the sink. The system is said to be *failed* if no such path exists. The *reliability of the system* is the probability that there exist one or more successful paths from the source to the sink [15].

A set of components, with the property that if all the components in the set are successful, then the system will be successful, is called *path* for the network. For example, in Fig. 1, the set $\{R_1, R_2, R_7\}$ is a path. A path with the property that the removal of any one component will cause the resulting set to not be a path, is called a *minimal path*.

Generally, if all the components in a minimal path are successful, while all other components have failed, the system will be successful. If any one of the components in the minimal path subsequently fails, then the system will fail. In terms of the network model, the minimal path corresponds to a simple path from the source to the sink in the network.

For example, in Fig. 1, the sets $\{R_1, R_5, R_7\}$, $\{R_2, R_6, R_8\}$, $\{R_1, R_3, R_6, R_8\}$, $\{R_1, R_3, R_4, R_7, R_6\}$, and $\{R_1, R_5, R_4, R_8\}$, $\{R_2, R_6, R_4, R_7\}$, $\{R_2, R_3, R_5, R_7\}$ and $\{R_2, R_3, R_5, R_8, R_4\}$ are minimal paths.

4 Sum of Disjoint Products Method (SDP)

Let A_i be the event that all the edges in min-path X_i are working. We can compute the reliability expression

$$R(G) = P(A_1) + P(\bar{A}_1 A_2) + ... + P(\bar{A}_1 \bar{A}_2 ... \bar{A}_{m-1} A_m), \tag{4.1}$$

where m is the total number of min-paths, $\bar{A}_i$ denotes the complement of the event A_i, and P is the probability function. SDP methods involve adding probabilities; however, the calculation of each constituent probability is generally involved [10]. It is also important to emphasize that the effectiveness of these methods can be highly dependent on the specific ordering given to the events A_i. In order to apply the SDP technique, the problem is to compute $P(\alpha)$, where $\alpha = \bar{A}_1 \bar{A}_2 ... \bar{A}_{i-1} \bar{A}_i$. We define by α_i the event that all components in the min-path X_i are working and each min-path $X_1, X_2, \ldots, X_{i-1}$ contains at least one failed component. If A_j and A_i are independent for $1 \le j \le i$, then we express events in terms of min-paths

$$P(\alpha_i) = \prod_{k \in X_i} p_k \prod_{j=1}^{i-1} [1 - \prod_{s \in X_i} p_s], \tag{4.2}$$

where p_k is the reliability of edge k. This equation defines in which way each min-path X contributes to the final reliability, considering a parallel structure or disjoint path (without common elements). If the min-paths share some common edges, then the solution of $P(\alpha_i)$ appears as involved. In this case we compute $P(\alpha_i)$ as

$$P(\alpha_i) = P(A_i) P(\bar{A}_1 | A_i) P(\bar{A}_2 | A_i A_1) ... P(\bar{A}_{i-1} | A_i A_1 A_2 \ldots A_{i-2}). \tag{4.3}$$

Let D_j be a conditional event such that min-path X_j is down given, assuming that the min-paths $X_i, X_1, ... X_{j-1}$ are working. We have

$$P(\alpha_i) = P(A_j) \prod_{j=1}^{i-1} P(D_j), \tag{4.4}$$

where $P(A_i)$ represents the probability of the event A_i, and can be simply computed since failures are assumed to be statistically independent. On the contrary, the evaluation of $P(D_j)$ is a complex problem in the system reliability field.

Theorem 4.1. *If $R_1(t), R_2(t), R_3(t), R_4(t),\ R_5(t), R_6(t), R_7(t), R_7(t), R_8(t)$ are reliabilities of arcs (paths) in a bridge system (fig 1), then the pullback reliability $R_S(t)$ of the system (by using sum of disjoint products method) is*

$$\begin{aligned} R_S(t) = (&R_1R_5R_7 + R_2R_6R_8 + R_2R_3R_5R_7 + R_1R_3R_6R_8 + R_1R_4R_5R_8 \\ &+ R_2R_4R_6R_7 - R_1R_2R_3R_5R_7 - R_1R_2R_3R_6R_8 + R_1R_3R_4R_6R_7 \\ &+ R_2R_3R_4R_5R_8 - R_1R_4R_5R_7R_8 - R_2R_4R_6R_7R_8 \\ &- R_1R_2R_3R_4R_5R_8 - R_1R_2R_3R_4R_6R_7 - R_1R_2R_4R_5R_6R_7 \\ &- R_1R_2R_4R_5R_6R_8 - R_1R_3R_4R_5R_6R_7 - R_1R_3R_4R_5R_6R_8 \\ &- R_2R_3R_4R_5R_6R_7 - R_2R_3R_4R_5R_6R_8 - R_1R_2R_5R_6R_7R_8 \\ &- R_1R_3R_4R_6R_7R_8 - R_2R_3R_4R_5R_7R_8 - R_1R_3R_5R_6R_7R_8 \\ &- R_2R_3R_5R_6R_7R_8 + 2R_1R_2R_3R_4R_5R_6R_7 + 2R_1R_2R_3R_4R_5R_6R_8 \\ &+ R_1R_2R_3R_4R_5R_7R_8 + R_1R_2R_3R_4R_6R_7R_8 + 2R_1R_2R_3R_5R_6R_7R_8 \\ &+ 2R_1R_2R_4R_5R_6R_7R_8 + 2R_1R_3R_4R_5R_6R_7R_8 \\ &+ 2R_2R_3R_4R_5R_6R_7R_8 - 4R_1R_2R_3R_4R_5R_6R_7R_8)(t). \end{aligned} \tag{4.5}$$

Proof. We start from the graph in Fig. 1, we denote $R_i = Pr\{r_i = 1\}$, $F_i = Pr\{r_i = 0\}$ and we use min-path sets; then the structure function of the given reliability system is

$$\begin{aligned} \Phi(r) =&P(r_1r_5r_7) + P(\overline{r_1r_5r_7}\ r_2r_6r_8) + P(\overline{r_1r_5r_7}\ \overline{r_2r_6r_8}\ r_1r_3r_6r_8) + P(\overline{r_1r_5r_7} \\ &\overline{r_2r_6r_8}\ \overline{r_1r_3r_6r_8}\ r_1r_3r_4r_7r_6) + P(\overline{r_1r_5r_7}\ \overline{r_2r_6r_8}\ \overline{r_1r_3r_6r_8}\ \overline{r_1r_3r_4r_7r_6} \\ &r_1r_5r_4r_8) + P(\overline{r_1r_5r_7}\ \overline{r_2r_6r_8}\ \overline{r_1r_3r_6r_8}\ \overline{r_1r_3r_4r_7r_6}\ \overline{r_1r_5r_4r_8}\ r_2r_6r_4r_7) \\ &+ P(\overline{r_1r_5r_7}\ \overline{r_2r_6r_8}\ \overline{r_1r_3r_6r_8}\ \overline{r_1r_3r_4r_7r_6}\ \overline{r_1r_5r_4r_8}\ \overline{r_2r_6r_4r_7}\ r_2r_3r_5r_7) \\ &+ P(\overline{r_1r_5r_7}\ \overline{r_2r_6r_8}\ \overline{r_1r_3r_6r_8}\ \overline{r_1r_3r_4r_7r_6}\ \overline{r_1r_5r_4r_8}\ \overline{r_2r_6r_4r_7}\ \overline{r_2r_3r_5r_7} \\ &\quad r_2r_3r_5r_8r_4). \end{aligned} \tag{4.6}$$

If we replace $r_i = R_i(t)$, with $1 - r_i(t) = F_i$, we have

$$
\begin{aligned}
R_{S_I}(t) = {} & R_1(t)R_5(t)R_7(t) + (1 - R_1(t)R_5(t)R_7(t))R_2(t)R_6(t)R_8(t) \\
& + (1 - R_1(t)R_5(t)R_7(t))(1 - R_2(t)R_6(t)R_8(t))R_1(t)R_3(t)R_6(t)R_8(t) \\
& + (1 - R_1(t)R_5(t)R_7(t))(1 - R_2(t)R_6(t)R_8(t))(1 - R_1(t)R_3(t)R_6(t)R_8(t)) \\
& R_1(t)R_3(t)R_4(t)R_7(t)R_6(t) + (1 - R_1(t)R_5(t)R_7(t))(1 - R_2(t)R_6(t)R_8(t)) \\
& (1 - R_1(t)R_3(t)R_6(t)R_8(t))(1 - R_1(t)R_3(t)R_4(t)R_7(t)R_6(t))R_1(t)R_5(t) \\
& R_4(t)R_8(t) + (1 - R_1(t)R_5(t)R_7(t))(1 - R_2(t)R_6(t)R_8(t))(1 - R_1(t)R_3(t) \\
& R_6(t)R_8(t))(1 - R_1(t)R_3(t)R_4(t)R_7(t)R_6(t))(1 - R_1(t)R_5(t)R_4(t)R_8(t)) \\
& R_2(t)R_6(t)R_4(t)R_7(t) + (1 - R_1(t)R_5(t)R_7(t))(1 - R_2(t)R_6(t)R_8(t)) \\
& (1 - R_1(t)R_3(t)R_6(t)R_8(t))(1 - R_1(t)R_3(t)R_4(t)R_7(t)R_6(t))(1 - R_1(t) \\
& R_5(t)R_4(t)R_8(t))(1 - R_2(t)R_6(t)R_4(t)R_7(t))R_2(t)R_3(t)R_5(t)R_7(t) \\
& + (1 - R_1(t)R_5(t)R_7(t))(1 - R_2(t)R_6(t)R_8(t))(1 - R_1(t)R_3(t)R_6(t)R_8(t)) \\
& (1 - R_1(t)R_3(t)R_4(t)R_7(t)R_6(t))(1 - R_1(t)R_5(t)R_4(t)R_8(t))(1 - R_2(t)R_6(t) \\
& R_4(t)R_7(t))(1 - R_2(t)R_3(t)R_5(t)R_7(t))R_2(t)R_3(t)R_5(t)R_8(t)R_4(t).
\end{aligned}
\tag{4.7}
$$

□

By computations and using the *expectation* [5, 6], we obtain a very long reliability polynomial, and this lead to difficulty in computations and geometrically interpretation. Due these reasons, we shall introduce an equivalent reliability polynomial which is more simple. Perhaps, the best way to do this is to use Delta-Star Technique.

5 Delta-Star technique for simplified equivalent reliability polynomial

We start from the simplest and very practical method to compute reliability of (bridge structures) bridge networks. This technique transforms a bridge network to its equivalent series and parallel form.

In other words, we use a diffeomorphism to change the initial reliability polynomial into a simplified form. This diffeomorphism must change the unit hypercube into a subset of unit hypercube.

Computationally, this method have some advantages (see also [2]). The advantages of using this method is for once a bridge network is transformed to its equivalent parallel and series form, the network reduction approach can be applied to obtain network reliability. Nonetheless, the delta-star method can easily handle networks containing more than one bridge configurations. Furthermore, it can be applied to bridge networks composed of devices having two mutually exclusive failure modes.

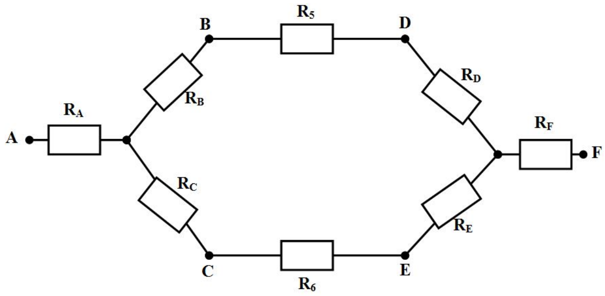

FIGURE 2. A simple network

To analyze a complex system (bridge structure) as presented in the Fig. 1, the equivalent reliability equations for the network reliability between delta configurations in which $A, B; B, C;$ and A, C, in addition to $D, F; D, E$ and E, F can be phrased into the following expressions

$$R_A R_B = 1 - (1 - R_{AB})(1 - R_{AC} R_{BC}) \tag{5.1}$$

$$R_B R_C = 1 - (1 - R_{BC})(1 - R_{AC} R_{AB}) \tag{5.2}$$

$$R_A R_C = 1 - (1 - R_{AC})(1 - R_{AB} R_{BC}). \tag{5.3}$$

Solving Equations (5.1)-(5.3), we obtain the following star equivalent reliabilities

$$R_A = \sqrt{\frac{[1 - (1 - R_{AB})(1 - R_{AC} R_{BC})][1 - (1 - R_{AC})(1 - R_{AB} R_{BC})]}{1 - (1 - R_{BC})(1 - R_{AC} R_{AB})}}$$

$$R_B = \sqrt{\frac{[1 - (1 - R_{AB})(1 - R_{AC} R_{BC})][1 - (1 - R_{BC})(1 - R_{AC} R_{AB})]}{1 - (1 - R_{AC})(1 - R_{AB} R_{BC})}}$$

$$R_C = \sqrt{\frac{[1 - (1 - R_{BC})(1 - R_{AC} R_{AB})][1 - (1 - R_{AC})(1 - R_{AB} R_{BC})]}{1 - (1 - R_{AB})(1 - R_{AC} R_{BC})}}.$$

The transformation delta-star equations applied to R_A, R_B, R_C and R_D, R_E, R_F, gives a simple configuration, so by using the above results, the equivalent network to Fig. 1 complex system is shown in Fig. 2. The reliability polynomial of the system in Fig. 2 is

$$R_{S_{II}} = R_A R_F [1 - (1 - R_B R_5 R_D)(1 - R_C R_6 R_E)]$$

or

$$R_{S_{II}} = R_5 R_A R_B R_D R_F + R_6 R_A R_C R_E R_F - R_5 R_6 R_A R_B R_C R_D R_E R_F. \tag{5.4}$$

Lemma 5.1. *The algebraic diffeomorphism described by the formulas 5.1-5.3 changes the unit hypercube into a subset of unit hypercube.*

Proof. To simplify, let us denote $R_{AB} = c, R_{BC} = a, R_{AC} = b$, $a, b, c \in [0, 1]$, $a \neq 0$ or $bc \neq 0$. We introduce the function

$$f(a, b, c) = \frac{(1 - (1 - c)(1 - ba))(1 - (1 - b)(1 - ca))}{1 - (1 - a)(1 - bc)}.$$

Obviously, $f \geq 0$. We shall show that $f \leq 1$. First we remark that

$$f(a, b, c) = \frac{(c + ab - abc)(b + ac - abc)}{a + bc - abc}.$$

Case $b \neq 0$**,** $c \neq 0$**,** $b \neq 1$**,** $c \neq 1$**, i.e.,** $b, c \in (0, 1)$. We fix b and c, and accept a as variable. The inequality $f(a, b, c) \leq 1$ is equivalent to

$$g(a) = (c + ab - abc)(b + ac - abc) - (a + bc - abc) \leq 0.$$

The function $g : R \to R$ is of degree two, where the coefficient of a^2 is

$$b(1 - c)c(1 - b) > 0.$$

We can justify that $g(1) < 0$. Due this inequality, the discriminant Δ is strictly positive. Consequently the equation $g(a) = 0$ has two real distinct roots, let say $a_1 < a_2$. The sign of the function $g(a)$ is given in the table

a	$-\infty$		a_1		a_2		∞
g(a)		+	0	-	0	+	

The inequality $g(1) < 0$ implies $1 \in (a_1, a_2)$. On the other hand, $g(0) = 0$ implies $0 = a_1$. From $a_1 = 0 < 1 < a_2$, we find $[0, 1] \subset [a_1, a_2]$.

The cases $b = 0$, $b = 1$, $c = 0$, $c = 1$ are treated similarly. □

Theorem 5.2. *The reliability polynomials 4.5 and 5.4 are equivalent via the algebraic diffeomorphism described by the formulas 5.1-5.3 and their similar formulas.*

An algebraic diffeomorphism allows us to pull back and push forward arbitrary reliability polynomials, it provides another way of comparing polynomials at different points on a manifold. Diffeomorphism invariance is the technical implementation of a physical idea into mathematics, due to Einstein.

Generally, a polynomial whose coefficients are in Z, and whose indeterminates $X_1, ..., X_n$ are boolean variables is called *reliability polynomial.* The set of reliability polynomials is the quotient of the ring $Z[X_1, ..., X_n]$ by the ideal $(X_1^2 - X_1, \cdots, X_n^2 - X_n)$.

Open Problem. Given two reliability polynomials P and Q, find an affine diffeomorphism φ, defined by a matrix A, with $||A||_\infty \leq 1$, such that $P \circ \varphi = Q$ (computations must have boolean sense). Equivalently, if $\varphi(x) = Ax + b$, then $\nabla Q(x) = \nabla P(\varphi(x)) \cdot A$.

Generally, the convenient algebraic diffeomorphism is those which realise three properties: (i) it moves the unit hypercube into a subset of unit hypercube; (ii) the number of terms in Q is smaller than the number of terms in P; (iii) the degree of Q is smaller than or equal to the degree of P.

Through the above, it is enough to study the geometric properties of reliability polynomial (5.4) to know the characteristics that help in the understanding the behavior of the system.

5.1 Approximate computations

If we assume that the reliability of all components in Fig. 1 are independent and identical as for example $(R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8)$=(0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5), and we apply these values in the formula (4.5), we obtain $R_S = 0.3281$, also if we substituting the given data (R_5, R_6, R_A, R_B, R_C, R_D, R_E, R_F) =(0.5, 0.5, 0.7906, 0.7906, 0.7906, 0.7906, 0.7906, 0.7906) into formula (5.4), taking into account computing equations (5.1) through (5.3), we get the approximation $R_{S_{II}} = 0.3296$.

If we take the components in electrical circuit are all independent non-identical units, as for example, $(R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8)$=(0.9, 0.85, 0.7, 0.65, 0.55, 0.75, 0.95, 0.6), and apply substitution these values in formula (4.5), we obtain $R_S = 0.7945$. While when we take the values (R_5, R_6, R_A, R_B, R_C, R_D, R_E, R_F)=(0.55, 0.75, 0.9874 , 0.9717, 0.9565, 0.9861, 0.8615, 0.9832), into formula (5.4), we find the approximation
$R_{S_{II}} = 0.7954$.

It should be noted that in the first case, when all units of electric circuit in Fig. 1 are independent and identical, the system will drift to failure, so must take into account that the units must be operate more than usual activity in order to ensure the success of the work of an electrical circuit.

6 Equi-reliable hypersurfaces

First, we shall rewrite the reliability polynomial (5.4) by replacing the indices A, B, C, D, E, F with numbers,

$$R_1 = R_A; R_2 = R_B; R_3 = R_C; R_4 = R_D; R_7 = R_E; R_8 = R_F.$$

In this way we obtain the polynomial [6]

$$P = R_1R_2R_4R_5R_8 + R_1R_3R_6R_7R_8 - R_1R_2R_3R_4R_5R_6R_7R_8. \tag{6.1}$$

Introducing the submersion

$$\varphi : X_1 = R_1R_2R_4R_5R_8, X_2 = R_1R_3R_6R_7R_8, X_3 = R_1R_2R_3R_4R_5R_6R_7R_8,$$

the polynomial P is changed into a linear form

$$F = X_1 + X_2 - X_3$$

and $P = F \circ \varphi$. The constant level sets of F are parallel to one diagonal hyperplane of R^3. The submersion φ change the unit hypercube into a unit hypercube.

In $\mathbb{R}^8$, let us consider the constant level algebraic hypersurfaces

$$c = R_1R_2R_4R_5R_8 + R_1R_3R_6R_7R_8 - R_1R_2R_3R_4R_5R_6R_7R_8,$$

which will be called *equi-reliable hypersurfaces* [6, 16].

6.1 Critical Points

The critical points of the polynomial $R_{S_{II}} = P$ determine a *variety*. Indeed the vector equation $\nabla P = 0$ is equivalent to the system

$$\frac{\partial P}{\partial R_1} = 0, \frac{\partial P}{\partial R_2} = 0, \frac{\partial P}{\partial R_3} = 0, \frac{\partial P}{\partial R_4} = 0,$$

$$\frac{\partial P}{\partial R_5} = 0, \frac{\partial P}{\partial R_6} = 0, \frac{\partial P}{\partial R_7} = 0, \frac{\partial P}{\partial R_8} = 0$$

or equivalently

$$R_2R_4R_5R_8 + R_3R_6R_7R_8 - R_2R_3R_4R_5R_6R_7R_8 = 0,$$

$$R_1R_4R_5R_8 - R_1R_3R_4R_5R_6R_7R_8 = 0,$$

$$R_1R_6R_7R_8 - R_1R_2R_4R_5R_6R_7R_8 = 0,$$

$$R_1R_2R_3R_5R_6R_7R_8 = 0,$$
$$R_1R_2R_4R_8 - R_1R_2R_3R_4R_6R_7R_8 = 0$$
$$R_1R_3R_7R_8 - R_1R_2R_3R_4R_5R_7R_8 = 0,$$
$$R_1R_3R_6R_8 - R_1R_2R_3R_4R_5R_6R_8 = 0,$$
$$R_1R_2R_4R_5 + R_1R_3R_6R_7 - R_1R_2R_3R_4R_5R_6R_7 = 0.$$

The solutions are

$$(R_1 = 0, R_2, R_3, R_4, R_5, R_6, R_7, R_8 = 0),$$
$$(R_1 = 0, R_2 = 0, R_3 = 0, R_4, R_5, R_6, R_7, R_8),$$
$$(R_1 = 0, R_2 = 0, R_3, R_4, R_5, R_6 = 0, R_7, R_8),$$
$$(R_1 = 0, R_2 = 0, R_3, R_4, R_5, R_6, R_7 = 0, R_8).$$

In this context, we can state the following

Theorem 6.1. *(i) The Hessian of the reliability polynomial P is degenerate.*
(ii) All the critical points are saddle points.

Proof. (i) The function P is harmonic. Consequently $\operatorname{tr} Hess\ P = 0$. □

Consequently the true optimization problems involving the previous polynomial are of the type min max, max min or optimizations with constraints.

The restrictions of the polynomial P to each variable is an affine function. Consequently, all the geometrical approximations of order greater than 2 reduces to the linear approximation (tangent plane).

6.2 Equi-reliable geodesics

For a constant level set, the geodesics are curves for which the acceleration is collinear to the normal.

Let $(R_1(t), R_2(t), R_3(t), R_4(t), R_5(t), R_6(t), R_7(t), R_8(t))$ be a curve with unit speed, i.e., $\dot{R}_1^2 + \cdots + \dot{R}_8^2 = 1$, situated in a constant level set attached to the reliability polynomial. The equations of geodesics are

$$\begin{array}{rcl}
\ddot{R}_1(t) & = & \mu(t)\ (R_2R_4R_5R_8 + R_3R_6R_7R_8 - R_2R_3R_4R_5R_6R_7R_8)(t) \\
\ddot{R}_2(t) & = & \mu(t)\ (R_1R_4R_5R_8 - R_1R_3R_4R_5R_6R_7R_8)(t) \\
\ddot{R}_3(t) & = & \mu(t)\ (R_1R_6R_7R_8 - R_1R_2R_4R_5R_6R_7R_8)(t) \\
\ddot{R}_4(t) & = & \mu(t)\ (R_1R_2R_3R_5R_6R_7R_8)(t) \\
\ddot{R}_5(t) & = & \mu(t)\ (R_1R_2R_4R_8 - R_1R_2R_3R_4R_6R_7R_8)(t) \\
\ddot{R}_6(t) & = & \mu(t)\ (R_1R_3R_7R_8 - R_1R_2R_3R_4R_5R_7R_8)(t) \\
\ddot{R}_7(t) & = & \mu(t)\ (R_1R_3R_6R_8 - R_1R_2R_3R_4R_5R_6R_8)(t) \\
\ddot{R}_8(t) & = & \mu(t)\ (R_1R_2R_4R_5 + R_1R_3R_6R_7 - R_1R_2R_3R_4R_5R_6R_7)(t).
\end{array}$$

Locally a constant level set is represented by the graph

$$R_8 = \chi(R_1, R_2, R_3, R_4, R_5, R_6, R_7)$$

and

$$\dot{R}_8 = \frac{\partial\chi}{\partial R_1}\dot{R}_1 + ... + \frac{\partial\chi}{\partial R_7}\dot{R}_7$$

The previous theory is equivalent to the problem:

Find

$$\min \int_{t_0}^{t_1} (\dot{R}_1^2 + \cdots + \dot{R}_8^2)(t)dt$$

subject to

$$P(R_1, \cdots, R_8) = c.$$

Along each geodesic the reliability is constant. Consequently, the previous geodesics are locally the shortest paths (evolutions) that are equi-reliable.

Let $\gamma(t)$ be a geodesic. The reliability curve $\gamma(t)exp(-\lambda\, t); \lambda > 0$ is a decay curve which realize a distance between two states situated in different constant level sets. This curve is necessary when we built the pullback reliability (when we want to compute mean time to failure (MTTF)).

6.3 Optimal control

Consider a curve of reliability $R_i = R_i(t)$, $i = 1, ..., 8$. Introduce the control $u(t) = (u_i(t))$, with $|u_i(t)| \leq 1$. A very important problem for practice can be formulated as optimal control problem ([9]): Find

$$\max_u \int_{t_0}^{t_1} P(R(t))dt$$

subject to

$$\ddot{R}_i(t) + \lambda_i \dot{R}_i(t) + \omega_i^2 R_i(t) = u_i(t).$$

7 Diagonal Reliability Polynomials

Let us analyze how many diagonal polynomials are induced by the reliability polynomial (6.1). A diagonal polynomial is a restriction of a reliability polynomial to a diagonal of the space R^8, i.e., identifying some variables.

Theorem 7.1. *The reliability polynomial* (6.1)*, where* $R_5 \neq R_6$ *induces*

$$2^6C_6^0 = 64,\ 3^5C_6^1 = 1458,\ 4^4C_6^2 = 3840,$$

$$5^3C_6^3 = 2500,\ 6^2C_6^4 = 540,\ 7^1C_6^5 = 42,\ 8^0C_6^6 = 1$$

diagonal polynomials in $2, \ldots, 8$ *variables.*

Proof. The diagonal polynomials associated to the reliability polynomial (6.1) are counted as follows:

- 2 variables: if $R_5 = x$, and $R_6 = y$, we have 2^6 polynomials.
- 3 variables: if $R_5 = x$, $R_6 = y$, $R_1 = z$, then the number of polynomials is $\sum_{k=1}^{5} C_5^k 2^{5-k} = (2+1)^5 = 243$. But the choosing of z can be made in 6 ways. Hence, we find 1458 polynomials.
- $4 - 8$ variables: similar ideas.

□

8 Mean time to failure

Let us consider again the reliability polynomial (6.1). Suppose (the "probabilities") $R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8$ are related by exponential decay curve (see [3])

$$R_i = a_i e^{-b_i t},\ 0 < a_i \leq 1,\ b_i \geq 0,\ i = 1, ..., 8,$$

or explicitly

$$R_1 = a_1 e^{-b_1 t}, R_2 = a_2 e^{-b_2 t}, R_3 = a_3 e^{-b_3 t}, R_4 = a_4 e^{-b_4 t},$$

$$R_5 = a_5 e^{-b_5 t}, R_6 = a_6 e^{-b_6 t}, R_7 = a_7 e^{-b_7 t}, R_8 = a_8 e^{-b_8 t}.$$

There are several types of problems in science in which experimental observations may best be represented by a linear combination of such exponentials.

First we compute the pullback of the reliability polynomial [4],

$$R_1R_2R_4R_5R_8 = a_1a_2a_4a_5a_8 e^{-(b_1+b_2+b_4+b_5+b_8)t},$$
$$R_1R_3R_6R_7R_8 = a_1a_3a_6a_7a_8 e^{-(b_1+b_3+b_6+b_7+b_8)t},$$
$$R_1R_2R_3R_4R_5R_6R_7R_8 = a_1a_2a_3a_4a_5a_6a_7a_8 e^{-(b_1+b_2+b_3+b_4+b_5+b_6+b_7+b_8)t},$$

$$
\begin{aligned}
P =& a_1a_2a_4a_5a_8e^{-(b_1+b_2+b_4+b_5+b_8)t} + a_1a_3a_6a_7a_8e^{-(b_1+b_3+b_6+b_7+b_8)t} \\
&- a_1a_2a_3a_4a_5a_6a_7a_8e^{-(b_1+b_2+b_3+b_4+b_5+b_6+b_7+b_8)t}.
\end{aligned}
\tag{8.1}
$$

Using the pullback (8.1) and the definition of MTTF, we obtain

$$
\begin{aligned}
MTTF = \int_0^\infty &[a_1a_2a_4a_5a_8e^{-(b_1+b_2+b_4+b_5+b_8)t} + a_1a_3a_6a_7a_8e^{-(b_1+b_3+b_6+b_7+b_8)t} \\
&- a_1a_2a_3a_4a_5a_6a_7a_8e^{-(b_1+b_2+b_3+b_4+b_5+b_6+b_7+b_8)t}]dt.
\end{aligned}
\tag{8.2}
$$

Evaluating the integral, we find

Theorem 8.1. *The mean time to failure is*

$$
\begin{aligned}
MTTF =& \frac{a_1a_2a_4a_5a_8}{b_1+b_2+b_4+b_5+b_8} + \frac{a_1a_3a_6a_7a_8}{b_1+b_3+b_6+b_7+b_8} \\
&- \frac{a_1a_2a_3a_4a_5a_6a_7a_8}{b_1+b_2+b_3+b_4+b_5+b_6+b_7+b_8}.
\end{aligned}
\tag{8.3}
$$

The function $(a_1, ..., a_8) \to MTTF(a_1, ..., a_8)$ is a signomial. The term "signomial" was introduced by Richard J. Duffin and Elmor L. Peterson in their seminal joint work on general algebraic optimization published in the late 1960s and early 1970s. Although nonlinear optimization problems with constraints and/or objectives defined by signomials are normally harder to solve than those defined by only posynomials (because, unlike posynomials, signomials are not guaranteed to be globally convex), signomial optimization problems often provide a much more accurate mathematical representation of real-world nonlinear optimization problems.

The function $(b_1, ..., b_8) \to MTTF(b_1, ..., b_8)$ is a passing to frequency in the sense of Laplace transform.

8.1 Interpretation by Laplace transform

In the formula (8.3), we recognize Laplace transforms of original functions

$$a_1(t)a_2(t)a_4(t)a_5(t)a_8(t),$$

$$a_1(t)a_3(t)a_6(t)a_7(t)a_8(t),$$

$$a_1(t)a_2(t)a_3(t)a_4(t)a_5(t)a_6(t)a_7(t)a_8(t),$$

to frequency

$$b_1 + b_2 + b_4 + b_5 + b_8,$$

$$b_1 + b_3 + b_6 + b_7 + b_8,$$
$$b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7 + b_8.$$

The meaning of the integrals depend on types of functions of interest. A necessary condition for existence of the integrals is that the functions must be locally integrable on $[0, \infty)$.

On the other hand the, if $P(R_1, ..., R_8)$ is the previous reliability polynomial, then its 8-dimensional Laplace transform is defined by

$$\begin{aligned} L(p) &= \int_0^\infty \cdots \int_0^\infty P(R_1, ..., R_8)\, e^{-p_1R_1-\cdots-p_8R_8}\, dR_1 \cdots dR_8 \\ &= \frac{1}{p_1 \cdots p_8}\, P\left(\frac{1}{p_1}, \cdots, \frac{1}{p_8}\right). \end{aligned}$$

where $p = (p_1, ..., p_8)$. In the previous case, we obtain the Laplace transform

$$L(p) = \frac{p_3p_6p_7 + p_2p_4p_5 - 1}{p_1^2 \cdots p_8^2}.$$

9 Exponential decay curves contained in reliability hypersurface

The graph of the reliability polynomial is a hypersurface in R^9 of Cartesian explicit equation (6.1), called *reliability hypersurface*.

Problem. How should the components R_i depend exponentialy on time, so that the reliability of the system be also exponential in time? (Geometrically, this means to find all the exponential decreasing curves which are contained in the reliability hypersurface).

Suppose $R_{S_{II}} = P = R_9 = a_9e^{-b_9t}$ and we look if the exponential decay curves (see [3])

$$R_i = a_ie^{-b_it},\ 0 < a_i \leq 1,\ b_i \geq 0, i = 1, ..., 9,$$

are included in reliability hypersurface

$$R_9 = R_1R_2R_4R_5R_8 + R_1R_3R_6R_7R_8 - R_1R_2R_3R_4R_5R_6R_7R_8.$$

In other words, we impose the identification

$$\begin{aligned} a_9e^{-b_9t} &= a_1a_2a_4a_5a_8e^{-(b_1+b_2+b_4+b_5+b_8)t} + a_1a_3a_6a_7a_8e^{-(b_1+b_3+b_6+b_7+b_8)t} \\ &- a_1a_2a_3a_4a_5a_6a_7a_8e^{-(b_1+b_2+b_3+b_4+b_5+b_6+b_7+b_8)t}, \\ &= a_1a_8e^{-(b_1+b_8)t}\big[a_2a_4a_5e^{-(b_2+b_4+b_5)t} + a_3a_6a_7e^{-(b_3+b_6+b_7)t} \\ &- a_2a_3a_4a_5a_6a_7e^{-(b_2+b_3+b_4+b_5+b_6+b_7)t}\big]. \end{aligned} \tag{9.1}$$

For simplify the computations, we denote

$$A_0 = a_9, A_1 = a_1a_2a_4a_5a_8, A_2 = a_1a_3a_6a_7a_8, A_3 = a_1a_2a_3a_4a_5a_6a_7a_8$$

and

$$B_0 = b_9, B_1 = b_1 + b_2 + b_4 + b_5 + b_8, B_2 = b_1 + b_3 + b_6 + b_7 + b_8,$$

$$B_3 = b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7 + b_8.$$

Hence, the identity can be written in the form

$$A_0e^{-B_0t} = A_1e^{-B_1t} + A_2e^{-B_2t} - A_3e^{-B_3t}. \tag{9.2}$$

On the other hand a set $\{e^{\beta_i t}, i = 1, ..., n, \beta_i \neq \beta_j\}$ is linearly independent. Consequently, in order to exists A_0, B_0 which realizes the identity, we need to consider two cases:

Case 1 $(B_0 = B_1 = B_2 = B_3, A_0 = A_1 + A_2 - A_3)$. Explicitly,

$$b_9 = b_1 + b_2 + b_4 + b_5 + b_8 = b_1 + b_3 + b_6 + b_7 + b_8$$

$$= b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7 + b_8,$$

$$a_9 = a_1a_2a_4a_5a_8 + a_1a_3a_6a_7a_8 - a_1a_2a_3a_4a_5a_6a_7a_8.$$

It follows

$$b_9 = b_1 + b_8,\ b_2 = b_3 = b_4 = b_5 = b_6 = b_7 = 0;$$

$$a_9 = a_1a_8(1 - (a_2a_4a_5 - 1)(a_3a_6a_7 - 1)) = P(a).$$

Corollary 9.1. *In the foregoing conditions, the $MTTF$ is reduced to*

$$MTTF = \frac{a_9}{b_9} = \frac{P(a)}{b_1 + b_8}.$$

Case 2.a $(B_1 = B_3, A_1 = A_3$ and $B_0 = B_2, A_0 = A_2)$.
If $(B_1 = B_3, A_1 = A_3) \Rightarrow$

$$\begin{aligned} b_1 + b_2 + b_4 + b_5 + b_8 =& b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7 + b_8 \\ 0 =& b_3 + b_6 + b_7, \qquad b_3 = b_6 = b_7 = 0; \\ a_1a_2a_4a_5a_8 =& a_1a_2a_3a_4a_5a_6a_7a_8 \\ 1 =& a_3a_6a_7. \end{aligned}$$

We use these values to solve the system $(B_0 = B_2, A_0 = A_2)$. We find

$$b_9 = b_1 + b_8,\ a_9 = a_1a_8. \tag{9.3}$$

Corollary 9.2. *In the foregoing conditions, the $MTTF$ is reduced to*

$$MTTF = \frac{a_9}{b_9} = \frac{a_1 a_8}{b_1 + b_8}.$$

Case 2.b ($B_2 = B_3, A_2 = A_3$ and $B_0 = B_1, A_0 = A_1$).
If $(B_2 = B_3, A_2 = A_3) \Rightarrow$

$$\begin{aligned} b_1 + b_3 + b_6 + b_7 + b_8 =& b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7 + b_8 \\ 0 =& b_2 + b_4 + b_5, \qquad b_2 = b_4 = b_5 = 0; \\ a_1 a_3 a_6 a_7 a_8 =& a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 \\ 1 =& a_2 a_4 a_5, \end{aligned}$$

similarly, we solve the following system of parameters equations by using the above data, i.e., solve $(B_0 = B_1, A_0 = A_1)$

$$b_9 = b_1 + b_8,\ a_9 = a_1 a_8. \tag{9.4}$$

Corollary 9.3. *In the foregoing conditions, the $MTTF$ is reduced to*

$$MTTF = \frac{a_9}{b_9} = \frac{a_1 a_8}{b_1 + b_8}.$$

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