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COMPACTNESS MODULO IN FIBREWISE IDEAL TOPOLOGICAL SPACE

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Abstract: The purpose of this paper is to study fibrewise compactness modulo in fibrewise ideal topological spaces which is called fibrewise *I*-compact space. Also we study some of their properties with some of the results in fibrewise compact space have been generalized of fibrewise I- compact space.

Keywords: *fibrewise I- compact, open cover, projection function.*

1. Introduction

A fibrewise topological space over B is a topological space X with a continuous function P: X $\rightarrow$B called the projection, and B called the base space. Most of the results can be found in James [6] 1984 ,[7] 1991 .The topic of ideal in topological space is introduced by Kuratowski[9] in 1933 and [10] in 1966 .In 1945 [14] introduced the concept of a local function and a kuratowskiclosure operator . If I is an ideal on a topological space (X,τ), then we can construct a topology on X which denoted by $\tau^*(I)$, that is in[8] . The concept of compactness modulo an ideal is introduced by Newcomb[12], and had been studied by Rancin[13] . In this paper we define a concept called fibrewise I-compact topological space over B. Also we study and discuss some of their properties, in particular we use the notion of fibrewise g-closed sets to prove theorem [3.6]with some corollaries.

2 . Some preliminary and Definitions

Definition2.1.[7]: Let B be any set. Then fibrewise set over B consists of a set X together with a function p: X$\rightarrow$B which is called the projection function.

For $b \in B$, the fibre over b is the subset $X_b = P^{-1}(b)$ of X . Also for each $B' \subset B$, then $X_{B'} = p^{-1}(B')$ is called a fibrewise set over B with the projection P . If $X' \subset X$ then$X'$is a fibrewise set over B with projection $P|_{X'}$.

Definition2.2. [7]: If X and Y are fibrewise sets over B, with projections p and q respectively, a function f: X$\rightarrow$ Y is said to be fibrewise function if qf =p, that is $f(X_b) \subseteq Y_b$ for each $b \in B$.

Definition2.3. [7]:Let B be a topological space. Then a fibrewise topology on a fibrewise set X over B is any topology on X for which the projection p is continuous.

A fibrewise topological space over the space B is defined to be a fibrewise set over B with fibrewise topology.

Definition2.4. [6]: Let X be a fibrewise topological space over B. Then X is fibrewise compact if for every fibre$X_b$of X,b∈B and every covering Γ of X_b by open sets of X there exists a neighborhood W of b in B and a finite subset Γ_0 of Γ covers X_W .

Definition2.5. [6]: Let X be fibrewise topological space over B and let A⊆ X . Then A is fibrewise compact subset of X if for every A_b ,b∈B where $A_b = A \cap X_b$ and every covering Γ of $A_b$by open sets of X , there exists a neighborhood W of b and a finite subset Γ_0 of Γ covers A_W where $A_W = A \cap X_W$.

Definition2.6. [1]: A fibrewise ideal on a fibrewise topological space (X, τ) over B is a nonempty collection I of subsets of X which satisfies

(i) A∈ I and B⊆A then B∈ I

(ii) A∈ I and B∈ I then A∪B∈ I

Lemma2.7. [1]: Let f : X→ Y be a fibrewise injection, where Xand Y are fibrewise sets over B . If I is any fibrewise ideal on X then f(I) $=\{f(I_1): I_1 \in I\}$is a fibrewise ideal on Y .

If Iis a fibrewise ideal on X and Y⊆X , then J= $\{Y \cap I_1 , I_1 \in I\}$ is a fibrewise ideal on Y .

If (X, τ) is a fibrewise topological space over B and I is a fibrewise ideal on X , then the triplet (X, τ, I) is called a fibrewise ideal topological space over B .

Definition2.8.[1]: Let (X, τ, I)be a fibrewise ideal topological space over B, then for any A∈P(X),
$A^*(I,\tau) = \{x \in X : A \cap U \notin I .\text{for each neighborhood } U \text{ of } x\}$is called a fibrewise local function of A with respect to I and τ,we will write A^* for $A^*(I , \tau)$.

Definition2.9. [1]: Let (X, τ, I) be a fibrewise ideal topological space over B . Then the map $cl^*( )$: P (X) → P (X) which is defined by $cl^*(A) = A \cup A^*$ for all A∈ P(X) is a kuratowski closure operator , we will denote by $\tau^*(I)$ the topology generated by $cl^*()$, that is $\tau^*(I)=\{U \subseteq X: cl^*(X - U) = X - U\}$ which is finer than τ where the collection β(I , τ) $=\{U - I_1: U \text{ is a neighborhood of } x , x \in X , I_1 \in I\}$ is a basis for a topology $\tau^*(I)$. We will write τ^* for $\tau^*(I)$.

Remark2.10.[1]:Let τ^*be the topology induced by the fibrewise ideal I on (X, τ). We note that this topology is finer than the topology τ and since τ is fibrewise topology, then the projection p : X →B is continuous relative to τ since $\tau^*$is finer that τ then the projection P is also continuous relative to τ^*this means that $\tau^*$is a fibrewise topology on X .

Example2.11.[1]: Let (X ,τ , I) be a fibrewise ideal topological space over B and A a subset of X , then :

(i) If I= {∅} , then $A^* = cl(A)$.
(ii) If I= P(X) , then $A^* = \emptyset$.

Note .$x \notin A^*$ if and only if (U- J)∩ A=∅, where U is aneighborhood of x and J∈I .

Lemma2.12. [1]:Let (X ,τ , I) be a fibrewise ideal topological space over B and let A be a subset of X , then :

(i) $A^* = cl(A^*) \subseteq cl(A)$;
(ii) A is fibrewise τ^* - closed if and only if $A^* \subseteq A$.

Definition2.13. [4]: Let (X, τ, I) be an ideal topological space. Then X is I – compact space if for every covering $\{U_\lambda, \lambda \in \Gamma\}$ of X by open sets of X there exists a finite subset Γ_0 of Γ such that $X-(\cup_{\lambda\epsilon\Gamma_0} U_\lambda)\epsilon I$.

3 .Fibrewise I –compact space

Definition3.1: A subset A of a topological space is said to be g-closed set if clA⊆U whenever, A⊆U and U∈τ .

Definition3.2: A subset A of a fibrewise topological space over B is said to be fibrewise g-closed set if $clA_b \subseteq U$ whenever, $b \in B, A_b \subseteq U$ and U∈τ .

Note: Every closed set A is fibrewise g-closed set.

Definition3.3: Let (X, τ, I) be a fibrewise ideal topological space over B. Then X is fibrewise I-compact space if for every fiber $X_b$of X ,b∈B,and every covering $\{U_\lambda, \lambda \in \Gamma\}$ of X_b by open sets of X,there exists a neighborhood W of b in B and a finite subset Γ_0 of Γ such that $X_W - (\cup_{\lambda\epsilon\Gamma_0} U_\lambda)\epsilon I$.

The space (X, τ, I) is called fibrewise I-compact space if X fibrewise I-compact .

Example3.4: If (X,τ) fibrewise topological space with the fibrewise ideal I= {∅},then (X ,τ) is fibrewise compact space if and only if is(X ,τ) fibrewise I-compact space .

Definition3.5: Let (X, τ, I) be a fibrewise ideal topological space over B and let A ⊆X .Then A is fibrewise I-compact subset if for every A_b of A , b ∈ B where $A_b = A \cap X_b$ and every covering $\{U_\lambda, \lambda \in \Gamma\}$of $A_b$by open sets of X , there exists a neighborhood W of b in B and a finite subset Γ_0 of Γ such that $A_W - (\cup_{\lambda\epsilon\Gamma_0} U_\lambda) \in I$ where $A_W = A \cap X_W$.

Theorem3.6: Every fibrewise g- closed subset of fibrewise I-compact space is fibrewiseI- compact.

Proof : Let A be fibrewise g- closed subset of (X , τ ,I) . Let $\{U_\lambda, \lambda\epsilon\Gamma\}$ be an open cover of A_b, b∈ B such that $A_b \subseteq \cup_{\lambda\epsilon\Gamma} U_\lambda$ .Since X is fibrewise I-compact then $\{U_\lambda, \lambda\epsilon\Gamma\} \cup (X_b - clA_b)$ is open cover of X_b, b∈B ,therefore there exists a neighborhood W of b in B and a finite subset Γ_0 of Γ such that either $X_W - (\{\cup_{\lambda\epsilon\Gamma_0} U_\lambda\} \cup (X_b - clA_b))\epsilon I$ or $X_W - (\cup_{\lambda\epsilon\Gamma_0} U_\lambda) \in I$ .either $[X_W - (\{\cup_{\lambda\epsilon\Gamma_0} U_\lambda\} \cup (X_b - clA_b))] \cap A$

$$\Rightarrow (X_W \cap A) - (\cup_{\lambda\epsilon\Gamma_0} U_\lambda)\epsilon I \Rightarrow A_W - (\cup_{\lambda\epsilon\Gamma_0} U_\lambda) \in I \text{ or}$$

$$[X_W - (\cup_{\lambda\epsilon\Gamma_0} U_\lambda)] \cap A \subset X_W - (\cup_{\lambda\epsilon\Gamma_0} U_\lambda) \in I.$$

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So $A \cap X_W - (\cup_{\lambda\epsilon\Gamma_0} U_\lambda) \in I$ and $A_W - (\cup_{\lambda\epsilon\Gamma_0} U_\lambda) \in I$. Hence A is fibrewise I-compact.

From Theorem3.6.we have get the following corollaries.

Corollary3.7: Every closed subset of fibrewise I - compact space is fibrewise I - compact.

Corollary3.8: If F is closed subset and K is fibrewise I - compact subset of X . Then $F \cap K$ is fibrewise I - compact.

Corollary 3.9: If A is a fibrewise I - compact in X and C is an open set such that $C \subseteq A$. Then $A - C$ is fibrewise I - compact.

Theorem3.10: Let f: $X \to Y$ be any continuous fibrewise bijection function where X, Y are fibrewise ideal topological spaces over B with fibrewise ideals I, f (I) on X, Y respectively. If X is fibrewise I –compact then Y is fibrewise I -compact.

Proof. Let $\{U_\lambda, \lambda\epsilon\Gamma\}$ be an open cover of $f(X_b) = Y_b, b\epsilon B$.Since f is continuous fibrewise function, then $\{f^{-1}(U_\lambda), \lambda\epsilon\Gamma\}$ is an open cover of $X_b = f^{-1}(Y_b)$. Since X is fibrewise I-compact there exists a neighborhood W of b in B and a finite subset $\Gamma_0$of Γ such that $X_W - \cup_{\lambda\epsilon\Gamma_0} f^{-1}(U_\lambda)\epsilon I$. Now $(f(X_W) - \cup_{\lambda\epsilon\Gamma_0} f^{-1}(U_\lambda))\epsilon f(I)$, but $f(X_W) - f(\cup_{\lambda\epsilon\Gamma_0} f^{-1}(U_\lambda)) \subset f(X_W - \cup_{\lambda\epsilon\Gamma_0} f^{-1}(U_\lambda))$ .This implies $f(X_W) - f\left(\cup_{\lambda\epsilon\Gamma_0} f^{-1}(U_\lambda)\right)\epsilon f(I)$ where f(I) is a fibrewise ideal on Y by Lemma 2.7. As$(f(X_W) - \cup_{\lambda\epsilon\Gamma_0} U_\lambda) \subseteq (f(X_W) - f\left(\cup_{\lambda\epsilon\Gamma_0} f^{-1}(U_\lambda)\right))$, so $(Y_W - (\cup_{\lambda\epsilon\Gamma_0} U_\lambda))\epsilon f(I)$. This means that Y is fibrewise I- compact.

Theorem 3.11: Let (X,τ,I) be any fibrewise ideal topological space over B and let A be a subset of X such that for every open set U with $A \subseteq U$ there is fibrewise I-compact set C with $A \subseteq C \subseteq U$. Then A is fibrewise I- compact.

Proof. Let $\{U_\lambda, \lambda\epsilon\Gamma\}$be a τ_A– open cover of A_b, Where $b \in B$,then there is open sets $\{V_\lambda, \lambda\epsilon\Gamma: V_\lambda\epsilon\tau\}$ in X such that $U_\lambda = V_\lambda \cap A$. By the given condition, there exists a fibrewise I- compact subset C of X such that $A \subseteq C \subseteq \cup_{\lambda\epsilon\Gamma} V_\lambda$.Then $\{V_\lambda \cap C, \lambda\epsilon\Gamma\}$ is a τ_C – open cover of $C_b, b\epsilon B$. As C is fibrewise I-compact, there exists a neighborhood W of b in B and a finite subset $\Gamma_0$of Γ such that $C_W - \cup_{\lambda\epsilon\Gamma_0} (V_\lambda \cap C)\epsilon I$ where $C_W = C \cap X_W, I_C = \{I_1 \cap C, I_1\epsilon I\}$ .Let $C_W - \cup_{\lambda\epsilon\Gamma_0} (V_\lambda \cap C) = I_1 \cap C$.

Since $C_W = \cup_{\lambda\epsilon\Gamma_0} (V_\lambda \cap C) \cup (I_1 \cap C)$, then $C_W \cap A = \left[\cup_{\lambda\epsilon\Gamma_0} (V_\lambda \cap C) \cup (I_1 \cap C)\right] \cap A \Rightarrow C_W \cap A = \left[\cup_{\lambda\epsilon\Gamma_0} (V_\lambda \cap C \cap A)\right] \cup [I_1 \cap C \cap A]$.

$\Rightarrow A_W = \left[\cup_{\lambda\epsilon\Gamma_0} (V_\lambda \cap A)\right] \cup [I_1 \cap A] \Rightarrow A_W - \cup_{\lambda\epsilon\Gamma_0} (V_\lambda \cap A) = I_1 \cap A\epsilon I_A$.

Implying that A is fibrewise I- compact.

Corollary3.12: If every open subset of X is fibrewise I- compact, then every subset of X contained in open subset is fibrewise I- compact.

Theorem3.13: If A and C are fibrewise I- compact subsets of ideal topological space (X,τ,I) over B ,then $A \cup C$ is fibrewise I-compact in X.

Proof. Let $\{U_\lambda, \lambda \epsilon \Gamma\}$ be an open cover of $(A \cup C)_b = A_b \cup C_b$ in X where $b \epsilon B$, then $\{U_\lambda, \lambda \epsilon \Gamma\}$ is open cover of A_b and C_b since A and C are fibrewise I-compact, there exist two neighborhoods W_1 and W_2 of b in B, $I_1, I_2 \epsilon I$ and finite subset Γ_0 and Γ_1 such that $A_{W_1} - (\cup_{\lambda_i \epsilon \Gamma_0} U_{\lambda_i}) = I_1$, where $A_{W_1} = A \cap X_{W_1}$ and $C_{W_2} - (\cup_{\lambda_k \epsilon \Gamma_1} U_{\lambda_k}) = I_2$ where

$C_{W_2} = C \cap X_{W_2}$, $A_{W_1} = [\cup_{\lambda_i \epsilon \Gamma_0} U_{\lambda_i}] \cup I_1$ and $C_{W_2} = [\cup_{\lambda_k \epsilon \Gamma_1} U_{\lambda_k}] \cup I_2$. Now $A_{W_1} \cup C_{W_2} = [\cup_{\lambda_i \epsilon \Gamma_0} U_{\lambda_i}] \cup [\cup_{\lambda_k \epsilon \Gamma_1} U_{\lambda_k}] \cup [I_1 \cup I_2] = \cup_{\lambda_i \epsilon \Gamma_0, \lambda_k \epsilon \Gamma_1} [U_{\lambda_i} \cup U_{\lambda_k}] \cup (I_1 \cup I_2)$, where $I_1 \cup I_2 \epsilon I$, and $W_1 \cap W_2$ is a neighborhood of b in B, so we have $(A_{W_1} \cup C_{W_2}) - \cup_{\lambda_i \epsilon \Gamma_0, \lambda_k \epsilon \Gamma_1} [U_{\lambda_i} \cup U_{\lambda_k}] \epsilon I$. Since $(A \cup C)_{W_1 \cap W_2} \subseteq A_{W_1} \cup C_{W_2}$, so $(A \cup C)_{W_1 \cap W_2} - \cup_{\lambda_i \epsilon \Gamma_0, \lambda_k \epsilon \Gamma_1} [U_{\lambda_i} \cup U_{\lambda_k}] \epsilon I$, that is $A \cup C$ is fibrewise I- compact.

Theorem3.14: Every fibrewise compact space X over B is fibrewise I- compact for any fibrewise ideal on X.

Proof. Let (X, τ) be a fibrewise compact space X over B, let I be any fibrewise ideal on X and $\{U_\lambda, \lambda \epsilon \Gamma\}$ open cover of X_b, $b \in B$, since (X, τ) is fibrewise compact space, so there exists a neighborhood W of b in B and a finite subset Γ_0 of Γ such that $X_W \subseteq (\cup_{\lambda \epsilon \Gamma_0} U_\lambda)$ and $X_W - (\cup_{\lambda \epsilon \Gamma_0} U_\lambda) = \emptyset \epsilon I$. Therefor X is fibrewise I- compact.

Theorem3.15: Let (X, τ, I) be an I- compact space, then (X, τ, I) is fibrewise I- compact space.

Proof. Let (X, τ, I) be I- compact space, let $\{U_\lambda, \lambda \epsilon \Gamma_j\}$ be an open cover of X_{b_j}, $b_j \in B$ $j \in J$, so $\cup_{\lambda \epsilon \Gamma_j} U_\lambda$, $j \epsilon J$ is open cover of X where $\cup_{\lambda \epsilon \Gamma_j} U_\lambda = \cup_{\lambda \epsilon \Lambda} U_\lambda$ (by taking $\cup_{j \epsilon J} \Gamma_j = \Lambda$, since (X, τ, I) is $I-$ compact space, there exists a finite subset Λ_0 of Λ such that $X - (\cup_{\lambda \epsilon \Lambda_0} U_\lambda) \epsilon I$, since $X_{W_j} \subseteq X$ for some neighborhood W_j of b_j in B $j \epsilon J$, so we have $X_{W_j} - (\cup_{\lambda \epsilon \Lambda_0} U_\lambda) \epsilon I$. Hence (X, τ, I) is fibrewise I- compact.

Theorem3.16: The following are equivalent for a fibrewise topological ideal space (X, τ, I):

(i) (X, τ, I) is fibrewise I- compact.

(ii) (X, τ^*, I) is fibrewise I-compact.

(iii) For any family $\{F_\lambda, \lambda \epsilon \Gamma\}$ of closed sets of X_b where $b \in B$ such that $\cap_{\lambda \epsilon \Gamma} F_\lambda = \emptyset$, there exists a neighborhood W of b in B and a finite subset Γ_0 of Γ such that $\cap_{\lambda \epsilon \Gamma_0} (F_W)_\lambda \epsilon I$ where $(F_W)_\lambda = X_W \cap F_\lambda$

Proof. (i) $\Rightarrow$ (ii) Let $\{U_\lambda, \lambda \epsilon \Gamma\}$ be a τ^*_open cover of X_b, $b \epsilon B$ such that $U_\lambda = V_\lambda - E_\lambda$, where V_λ are open sets in X and $E_\lambda \epsilon I$ for all $\lambda \epsilon \Gamma$. Now $\{V_\lambda, \lambda \epsilon \Gamma\}$ is open cover of X_b, so there exists a neighborhood W of b in B and a finite subset Γ_0 of Γ such that $X_W - (\cup_{\lambda \epsilon \Gamma_0} V_\lambda) \epsilon I$. This implies that $X_W - (\cup_{\lambda \epsilon \Gamma_0} V_\lambda) \subset [X_W - (\cup_{\lambda \epsilon \Gamma_0} V_\lambda)] \cup [\cup_{\lambda \epsilon \Gamma_0} E_\lambda] \epsilon I$. Therefore (X, τ^*, I) is fibrewise I- compact.

(ii) $\Longrightarrow$ (i) it follow from $\tau \subset \tau^*$.

(i) $\Rightarrow$ (iii) Let $\{F_\lambda, \lambda\epsilon\Gamma\}$ be a family of closed sets of X such that $\cap_{\lambda\epsilon\Gamma} F_\lambda = \emptyset$. Then $\{X - F_\lambda, \lambda\epsilon\Gamma\}$ is an open cover of X_b, bϵB . By (i) since (X,τ,I) is fibrewise I- compact ,there exists a neighborhood W of b in B and a finite subset $\Gamma_0$of Γ such that $X_W - (\cup_{\lambda\epsilon\Gamma_0} (X - F_\lambda))\epsilon I \Rightarrow X_W \cap (\cap_{\lambda\epsilon\Gamma_0} F_\lambda) = \cap_{\lambda\epsilon\Gamma_0} [X_W \cap F_\lambda] = \cap_{\lambda\epsilon\Gamma_0} F_{W_\lambda} \in I$.

(iii) $\Rightarrow$ (i)Let $\{U_\lambda, \lambda\epsilon\Gamma\}$ be an open cover of $X_b$where b∈B, then $\{X - U_\lambda, \lambda\epsilon\Gamma\}$ is a collection of closed sets and $\cap_{\lambda\epsilon\Gamma} (X - U_\lambda) = \emptyset$. Hence there exists a neighborhood W of b in B and a finite subset Γ_0 of Γ scuh that $\cap_{\lambda\epsilon\Gamma_0} [X_W \cap (X - U_\lambda)]\epsilon I$,that is $X_W - (\cup_{\lambda\epsilon\Gamma_0} U_\lambda)\epsilon$ I. This show that (X,τ,I) is fibrewise I- compact .

Conclusion

We defined and discussed compactness modulo in fibrewise ideal topological space which is called fibrewise I- compact with some results in fibrewise compact space which are generalized in fibrewise I- compact space.

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