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Regional Boundary Asymptotic Gradient reduced order observer

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Abstract: The purpose of this paper is to introduce the notion of regional boundary asymptotic gradient reduced order observer ($\Gamma^* AGRO$-observer) in distributed parameter systems. More precisely, we explore and discuss the existing of this approach in which estimates a considered sub-region Γ^* for the considered domain boundary. Thus, we show that the approach is enables to build the unknown part of the state gradient when the output function gives part of information about the region state. Furthermore, the characterization of this notion depend on regional boundary gradient strategic sensors (RBG-strategic sensors) concept in order that regional boundary asymptotic gradient reduced order observability ($\Gamma^* AGRO$-observability) to be achieved and analyzed. Finally, an application is presented to various situations of strategic sensors for internal case.
Keywords: $\Gamma^* G$-strategic sensors, $\Gamma^* AGRO$-observability, $\Gamma^* AGRO$-observers, Exchange system.

1. Introduction
The main idea of observer theory concepts are focused on reconstruction a dynamic system which is estimated the state of the original system using only the measured input and output function [1-2]. When the measurement function gives information about some state of the original system in this case there is a necessity to introduce an estimator which enables to reconstruct the unknown part of state vector and then, this problem is said to be reduced order observer [3-5]. The, asymptotic observer theory explored by Luenberger in [1,6] for finite dimensional linear systems and extended to distributed parameter systems govern by strongly continuous semi-group in Hilbert space by Gressang and Lamont as in [7]. The study of this approach via another variable like sensors and actuators developed by El-Jai *et al.* as in ref. s [3-8] in order to achieve asymptotic observability. One of the most important approach in system theory is focused on reconstruction the state of the system from knowledge of dynamic system and the output function on a sub region ω of a spatial domain Ω. Thus, this problem is called regional observability problem has been received much attention as in [9-11]. An extension of this notion has been given in [12-13] to the regional gradient case. The regional asymptotic notion has been introduced and developed by Al-Saphory and El Jai in [14-15]. Thus, this notion consists in studying the asymptotic behavior of the system in an internal sub-region ω of a spatial domain Ω. Thus, the asymptotic regional state reconstruction studied and developed in [16-18] and extended to the regional asymptotic gradient reduced order observer ($RAGRO$-observer) which allows to estimate the state gradient of the original system.
The purpose of this paper is to study and examine the concept of RBAGRO-observer by using the choice of sensors. The principle reason for considering this case is that, in first time the existent of a dynamical system which is observed asymptotically the gradient of the system state on some boundary region $\Gamma^* \subset \partial\Omega$ [19-21]. In second

time, of energy exchange problem, where the objective is to calculate the energy exchanged between a casting plasma on a plane target which is perpendicular to the direction of the flow from measurements carried out by internal thermocouples (Figure 1).

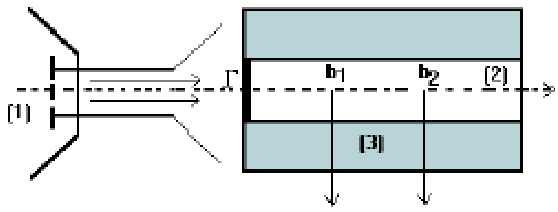

Fig. 1: Model of energy exchanged problem on Γ^*

where (1) is the torch of plasma, (2) is the probe of (steal), (3) is the insulator, Γ^* is the face of exchange and b_1, b_2 sensor locations. The outline of this paper is organized as follows: Section 2 is devoted to the problem statement and some basic concept related to the regional boundary asymptotic gradient observability ($RBAG$-observability) and regional boundary asymptotic gradient detectability ($RBAG$-detectability). In section 3, we introduce $RBAGRO$-observer notion for a distributed parameter system in terms of regional asymptotic gradient reduced order detectability and reduced order strategic sensors. In the last section, we illustrate applications with different domains and circular strategic sensors of two-phase exchange systems.

2. Problem formulation and preliminaries

This section present considered system and formulation of problem with some definitions and characterizations which is related to the present work.

2.1 Problem statement

Let Ω be a regular, bounded and open subset of $\mathcal{R}^n$, with boundary $\partial\Omega$ and Γ^* be a region subset of $\partial\Omega$. We denoted $\Pi = \Omega \times]0,\infty[$ and $\Xi = \partial\Omega \times]0,\infty[$. Consider the parabolic system which is described by the following state space equation

$$\begin{cases} \frac{\partial w}{\partial t}(\zeta,t) = \mathcal{A}\, w(\zeta,t) + B\, u(t) & \Pi \\ w\,(\zeta,0) = w_0(\zeta) & \bar{\Omega} \\ w\,(\mu,t) = 0 & \Xi \end{cases} \tag{1}$$

together with the output function

$$y\,(.,t) = C\,w\,(.,t) \tag{2}$$

• The separable Hilbert spaces are $\mathbb{W}$, $\mathbb{U}$ and $\mathbb{Y}$ where $\mathbb{W} = H^1(\bar{\Omega})$ is the state space, $\mathbb{U} = L^2(0,\infty,\mathcal{R}^p)$ is the control space and $\mathbb{Y} = L^2(0,\infty,\mathcal{R}^q)$ is the observation space, where p and q are the numbers of actuators and sensors.

• $\mathcal{A} = \sum_{i,j=1}^{n} \frac{\partial}{\partial w_j}(a_{ij}\frac{\partial}{\partial w_j})$ with $a_{ij} \in D(\mathcal{A})$ (domain of $\mathcal{A}$) is a second order linear differential operator, which generates a strongly continuous semi-group $(S_{\mathcal{A}}(t)\,)_{t\geq 0}$ on the state space $\mathbb{W}$ and is self-adjoint with compact resolvent [22].

• The operators $B \in L(\mathcal{R}^p,\mathbb{W})$ and $C \in L(\mathbb{W},R^q)$ depend on the structure of actuators and sensors as in [23] **(**figure 2**)**

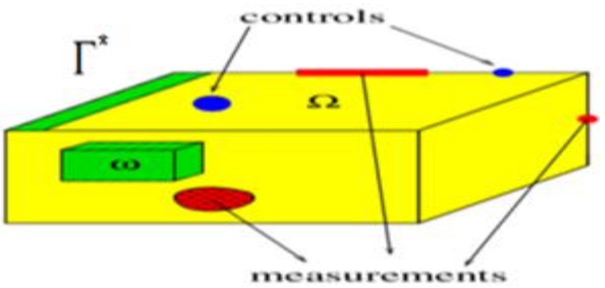

Fig. 2: General mathematical model

•The mathematical model in figure 2 is more general and complex than the spatial case of real model in figure 1.

• Under the given assumptions above, the system (1) has a unique solution given by the following form [24-25].

$$w(\zeta,t) = S_{\mathcal{A}}(t)w_0(\zeta) + \int_0^t S_{\mathcal{A}}(t-s)B\,u(s)\,ds \tag{3}$$

• The problem is how to reconstruct a dynamical system may be called estimator for the current state gradient in a given region on Γ^*, and to give a sufficient condition for the existence of a $RBAGRO$-observer.
• The initial state w_0 and its gradient ∇w_0 are supposed to be unknown, the problem concerns the building of the initial gradient ∇w_0 on the region Γ^* of the system domain $\partial\Omega$.

• Now, we reflect the operator K given by the form

$$K: \mathbb{W} \rightarrow \mathbb{Y}$$
$$w \rightarrow C\,S_{\mathcal{A}}(.)\,w$$

where K is a bounded linear operator as in [8, 26, 28]. And the adjoint operator $K^*$of K is defined by $K^*: \mathbb{Y} \rightarrow \mathbb{W}$, and represented by the form

$$K^*\,y^* = \int_0^t S^*_{\mathcal{A}}(s)\,C^*\,y^*(s)\,ds$$

• The operator ∇ denotes the gradient is given by

$$\begin{cases} \nabla: H^1(\Omega) \rightarrow (H^1(\Omega)\,)^{\,n} \\ w \rightarrow \nabla \mathrm{w} = \left(\frac{\partial w}{\partial \zeta_1}, \dots, \frac{\partial w}{\partial \zeta_n}\right) \end{cases}$$

and, the adjoint of ∇ denotes by ∇^* is given by

$$\begin{cases} \nabla^*: (H^1(\Omega)\,)^{\,n} \rightarrow H^1(\Omega) \\ w \rightarrow \nabla^* \mathrm{w} = v \end{cases}$$

where v is a solution of the Dirichlet problem

$$\begin{cases} \Delta v = -div(w) & \Omega \\ v = 0 & \partial\Omega \end{cases}$$

Thus, the extension of the trace operator [27] which is denoted by γ defined as

$$\gamma: (H^1(\Omega))^n \rightarrow (H^{1/2}(\partial\Omega))^n$$

and the adjoint is given by γ^*.
• For a region Γ^* of $\partial\Omega$, we define the gradient restriction operator by the form

$$\chi_{\Gamma^*}: (H^{1/2}(\partial\Omega))^n \longrightarrow (H^{1/2}(\Gamma^*))^n$$

where the adjoint of w_{Γ^*} denotes by $w^*_{\Gamma^*}$ is defined by

$\chi^*_{\Gamma^*}: (H^{1/2}(\Gamma^*))^n \longrightarrow (H^{1/2}(\partial\Omega))^n$

• Finally, we denote the operator $H_{\Gamma^* RBG} = \chi_{\Gamma^*}\gamma\nabla K^*$ from $\mathbb{Y}$ into $(H^{1/2}(\Gamma^*))^n$ and the adjoint of this operator given by $H^*_{\Gamma^*} = K\nabla^*\gamma^*\chi^*_{\Gamma^*}$.

Now, the problem is how to build an approach which observe (estimates) regional state gradient on a region Γ^* of the boundary $\partial\Omega \subset \bar{\Omega}$ asymptotically by using a dynamic system (an observer) in reduced order case only may be called reduced-order observer in region Γ^*. The important of an observer is that to estimates all the gradient of state variables, regardless of whether some are available for direct measurements or not [1].

2.2 Γ^*G-observability and Γ^*AG-detectability

In this subsection devotes the relation between concept of Γ^*G-observability and Γ^*AG-detectability on Γ^*. As well known the observability [23-26] and asymptotic observability [3-5, 8 28] are important concepts to estimate the unknown state of the considered dynamic system from the input and output functions. Thus, These notions are studied and introduced to the DPS_S analysis with different characterizations by El-Jai, Zerrik and Al-Saphory *et al.* in many paper for example [9-15, 29-33] in connection with strategic sensors.

• The systems (1)-(2) are said to be exactly regionally boundary gradient observable ($\mathbb{E}\Gamma^*$G-observable) on Γ^* if

$$Im\, H = Im\chi_{\Gamma^*}\gamma\nabla K^* = (H^{1/2}(\Gamma^*))^n$$

• The systems (1)-(2) are said to be weakly regionally boundary gradient observable ($\mathbb{W}\Gamma^*$G-observable) on Γ^* if

$$\overline{Im\, H} = \overline{Im\chi_{\Gamma^*}\gamma\nabla K^*} = (H^{1/2}(\Gamma^*))^n$$

It is equivalent to say that the systems (1)-(2) are $\mathbb{W}\Gamma^*$G-observable if

$$Ker\, H^* = ker K\ \nabla^*\gamma^*\chi_{\Gamma^*} = \{0\}$$

• If the systems (1)-(2) are is $\mathbb{W}\Gamma^*$G-observable, then $w_0(\zeta, 0)$ is given by

$$w_0 = (K^*K)^{-1}K^*y = K^\dagger y,$$

where $K^\dagger$ is the pseudo-inverse of the operator K [10-11].

• A sensor (D, f) is regional boundary gradient strategic (Γ^*G-strategic) if the observed system $\mathbb{W}\Gamma^*$G-observable.

• The measurements can be obtained by the use of zone or pointwise sensors, which may be located in Ω or $\partial\Omega$ [28].

• Then, according to the choice of the parameters D_i and f_i, we have different types of sensors:

• It may be zone, if $D_i \subset \bar{\Omega}$ and $f_i \in L^2(D_i)$. In this case, the operator C is bounded and the output function (2) may be given by the form

$$y(t) = \int_{D_i} w(\zeta, t) f_i(\zeta)\, d\zeta \quad (4)$$

• It may be pointwise, if $D_i = \{b_i\}$ with $b_i \in \bar{\Omega}$ and $f = \delta(. - b_i)$ where δ is a Dirac mass concentrated in b [14, 20, 24]. In this case, the operator C is a bounded and the output function (2) may be given by the form

$$y(t) = \int_{\Omega} w(\zeta, t)\delta_{b_i}(\zeta - b_i)\, d\zeta \quad (5)$$

• It may be boundary zone, if $\Gamma^*_i \subset \partial\Omega$ and $f_i \in L^2(\Gamma^*_i)$, the output function (2) may be given by the form

$$y(t) = \int_{\Gamma^*} w(\mu, t)\, f_i(\mu)\, d\mu \quad (6)$$

Definition 2.1: The semi-group $(S_{\mathcal{A}}(t))_{t\geq 0}$ is regionally boundary asymptotically gradient stable (Γ^*AG-stable) on Γ^*, if and only if for some positive constants M_{Γ^*}, α_{Γ^*}, we have

$$\|w_{\Gamma^*}\gamma\nabla S_{\mathcal{A}}(.)\|_{L((H^{1/2}(\Gamma^*))^n, H^1(\bar{\Omega}))} \leq M_{\Gamma^*}\, e^{\alpha_{\Gamma^*}}, \forall t \geq 0.$$

Remark 2.2: If the semi-group $(S_{\mathcal{A}}(t))_{t\geq 0}$ is Γ^*AG-stable on $(H^{1l2}(\Gamma^*))^n$, then for all $w_o \in H^1(\Omega)$, the solution of associated system satisfies

$$\lim_{t\to\infty}\|\chi_{\Gamma^*}\gamma\nabla w(.,t)\|_{(H^{1/2}(\Gamma^*))^n} = \lim_{t\to\infty}\|\chi_{\Gamma^*}\gamma\nabla S_{\mathcal{A}}(t)w_0\|_{(H^{1/2}(\Gamma^*))^n} = 0 \quad (7)$$

Definition 2.3: The system (1) is said to be Γ^*AG-stable on Γ^* if the operator $\mathcal{A}$ generates a semi-group which is Γ^*AG-stable on the space $(H^{1/2}(\Gamma^*))^n$.

Definition 2.4: The system (1)-(2) is said to be regionally boundary asymptotically gradient detectable (Γ^*AG-detectable), if there exists an operator $H_{\Gamma^*AG}: \mathcal{R}^q \to (H^{1/2}(\Gamma^*))^n$, such that the operator $(\mathcal{A} - H_{\Gamma^*AG}C)$ generates a strongly continuous semi-group $(S_{H_{\Gamma^*AG}}(t))_{t\geq 0}$, which is Γ^*AG-stable on $(H^{1/2}(\Gamma^*))^n$.

Proposition 2.5: If the system (1)-(2) is Γ^*G-observable on Γ^*, then it is Γ^*AG-detectable on Γ^*. This results gives the following inequality: $\exists\, k_{\Gamma^*AG} > 0$, such that

$$\|\chi_{\Gamma^*}\gamma\nabla S_{\mathcal{A}}(.)w\|_{(H^{1/2}(\Gamma^*))^n} \leq k_{\Gamma^*AG}\|CS_{\mathcal{A}}(.)w\|_{L^2(0,\infty,\mathbb{Y})},$$

for all $w \in (H^{1/2}(\Gamma^*))^n$.

Proof: We conclude the proof of this proposition is conclude from the results on observability considering $\chi_{\Gamma^*}\nabla K^*$. We have the following forms [25, 28]

1. $Imf \subset Img$.
2. There exists $k > 0$, such that

$$\|f^*w^*\|_{E^*} \leq k\,\|g^*w^*\|_{F^*}, \text{for all } w^* \in G^*$$

From the right hand said of above inequality $k_{\Gamma^*AG}\,\|g^*w^*\|_{F^*}$, there exists $M_{\Gamma^*AG}, \omega_{\Gamma^*AG} > 0$ with $k_{\Gamma^*AG} < M_{\Gamma^*AG}$, such that

$$k_{\Gamma^*AG}\,\|g^*\mathrm{w}\|_{F^*} \leq M_{\Gamma^*AG}e^{-\omega_{\Gamma^*AG}t}\|w^*\|_{F^*}$$

where E, F and G be a reflexive Banach spaces and $f \in L(E,G)$, $g \in L(F,G)$. If we apply this result, considered

$$E = G = (H^{1/2}(\Gamma^*))^n,\ F = \mathbb{Y},\ f = Id_{(H^{1/2}(\Gamma^*))^n}$$

and

$$g = S^*_{\mathcal{A}}(.)\chi^*_{\Gamma^*}\gamma^*\nabla^*C^*$$

where $S_{\mathcal{A}}(.)$ is a strongly continuous semi-group generates by $\mathcal{A}$, which is Γ^*AG-stable on Γ^*, then it is Γ^*AG-detectable on Γ^*. Thus, the notion of Γ^*AG-detectability is a weaker property than the Γ^*G-observability [21-29].

Remark 2.6: The important purpose of Γ^*AG – detectability that is related to the possibility for defining a Γ^*AG-estimator of the system state from the knowledge of the output function and input.

3. Γ^*AGRO-observer

In this section , we need some of additional assumptions, which we explain in chapter one section 1.4 for the systems state (1)-(2).

3.1 Regionally boundary asymptotic gradient reduced-order

Let us consider $\mathbb{W}=\mathbb{W}_1 \oplus \mathbb{W}_2$ where $\mathbb{W}_1$ and $\mathbb{W}_2$ are subspace of $\mathbb{W}$. Under the hypothesis in ref. [7,17-22, 29, 31-33] , we have the dynamical system given by

$$\begin{cases} \frac{\partial w_2}{\partial t}(\xi,t) = A_{21}w_1(\zeta,t) + A_{22}w_2(\zeta,t) + B_2\,u(t) & \Pi \\ w_2(\zeta,0) = w_{2_0}(\zeta) & \Omega \\ w_2(\mu,t) = 0 & \Xi \end{cases} \tag{8}$$

Augmented with the output function

$$y(.,t) = C\,w_1(\zeta,t). \tag{9}$$

The problem consists in constructing a regional asymptotic gradient estimator that enables one to estimate the unknown part $w_2(\zeta,t)$ equivalent; now to define the dynamical system for (9). Thus, equations (8)-(9) allow the following system:

$$\begin{cases} \frac{\partial a}{\partial t}(\zeta,t) = A_{22}a(\zeta,t) + [B_2u(t) + A_{21}y(\zeta,t)] & \Pi \\ a(\zeta,0) = a_0(\zeta) & \Omega \\ a(\mu,t) = 0 & \Xi \end{cases} \tag{10}$$

with the output function

$$\tilde{y}(.,t) = A_{12}a(\zeta,t). \tag{11}$$

where the state a in system (10) plays the role of the state w_2 in system (8).

3.2 Γ^*AGRO-observability and Γ^*AGRO-detectability

As in ref. [4] we can extend these result to the case of regional asymptotic gradient reduced ordered system from regional observability and Γ^*AG- detectability. In this case, the equation(1)-(2) it can be given by defining the following operator

$$\mathcal{K}: w_2 \to \mathcal{K}w_2 = \mathcal{A}_{12}S_{\mathcal{A}_{22}}(t)\, w_2 \in \mathbb{Y}\ ,$$

then

$$y(.,t) = \mathcal{K}w_{2_0}(.), \text{ with the adjoint } \mathcal{K}^*: \mathbb{Y} \to w_2$$

such that

$$\mathcal{K}^*y^*(.,t) = \int_0^t S_{\mathcal{A}_{22}}(s)\mathcal{A}^*_{12}\, y^*(.,s)\, ds.$$

Let $\Gamma^* \in \partial\Omega$ and $\chi_{\Gamma^*}:(H^{1/2}(\Gamma^*))^n \to (H^{1/2}(\Gamma^*))^n = w_2\ ,\ w_2 \to w_{\Gamma^*}w_2 = w_2|_{\Gamma^*}$

where $w_2|_{\Gamma^*}$ is the restriction of the state w_2 to Γ^*.

Definition 3.1: The systems (10)-(11) are called exactly regionally boundary asymptotic gradient reduced order observable (or $\mathbb{E}\Gamma^*AGRO$-observable) if

$$Im\,\chi_{\Gamma^*}\gamma_0\mathcal{K}^* = (H^{1/2}(\Gamma^*))^n = w_2$$

Definition 3.2: The systems (10)-(11) are called weakly regionally boundary asymptotic gradient reduced order observable (or $\mathbb{W}$ Γ^*AGRO-observable) if

$$\overline{Im\, w_{\Gamma^*}\gamma\mathcal{K}^*} = (H^{1/2}(\Gamma^*))^n = w_2$$

This equation $\overline{Im\chi_{\Gamma^*}\gamma\mathcal{K}^*}$ is equivalent to $Ker\ \mathcal{K}\gamma^*{\chi^*}_{\Gamma^*} = \{0\}$

Definition 3.3: The suite of sensors (zone or pointwise) $(D_i, f_i)_{1\leq i\leq q}$ are called regional boundary asymptotic gradient reduced order strategic sensors (or Γ^*AGRO-strategic sensors) if the systems (10)-(11) are $\mathbb{W}\Gamma^*AGRO$-observable.

Remark 3.4: We know the semi-group $(S_{\mathcal{A}_{22}}(t))_{t\geq 0}$ on Hilbert space $H^{1/2}(\Gamma^*)$ is said to be Γ^*AGRO-stable [20-21], if there exists $M_{\mathcal{A}_{22}}, \alpha_{\mathcal{A}_{22}} > 0$ such that

$$\|S_{\mathcal{A}_{22}}(t)\|_{(H^{1/2}(\Gamma^*))^n} \leq M_{\mathcal{A}_{22}} e^{-\alpha_{\mathcal{A}_{22}} t}, t \geq 0 \tag{12}$$

Remark 3.5: The relation (12) is true on a given sub-domain $\Gamma^* \subset \partial\Omega$, i.e.

$$\left\|\chi_{\Gamma^*} S_{\mathcal{A}_{22}}(t)\right\|_{L\left(H^{1/2}(\Gamma^*))^n, H^{1/2}(\Gamma^*)\right)} \leq M_{\mathcal{A}_{22}}\, e^{-\alpha_{\mathcal{A}_{22}} t}\ , t \geq 0. \tag{13}$$

and then

$$\lim_{t\to\infty} \|w_2(.,t)\|_{(H^{1/2}(\Gamma^*))^n} = 0$$

Now, we refer to this as regional boundary asymptotic gradient reduced order stability (or Γ^*AGRO- stability).

Definition 3.6: The system (10) is said to be regional boundary asymptotic gradient reduced order stability (or Γ^*AGRO- stable) if the operator $\mathcal{A}_{22}$ generates a semi-group which is Γ^*AGRO - stable.

Definition 3.7: The systems (10)-(11) are said to be regional boundary asymptotic gradient reduced order detectability (or Γ^*AGRO - detectable) if there exists an operator $\mathcal{H}_{\Gamma^*}: \mathbb{R}^q \to (H^{1/2}(\Gamma^*))^n$ such that $(\mathcal{A}_{22} - \mathcal{H}_{\Gamma^*}\mathcal{A}_{12})$ generates a strongly continuous semi-group $(S_{\mathcal{A}_{22}}(t))t \geq 0$, which is $\Gamma^*AGRO -$ stable.

Therefore, we have the dynamical system for (10)-(11) may be given by

$$\begin{cases} \frac{\partial \hat{z}}{\partial t}(\zeta,t) = \mathcal{A}_{22}\hat{z}(\zeta,t) + [B_2u(t) + \mathcal{A}_{21}y(\zeta,t)] + \mathcal{H}_{\Gamma^*AG}(\tilde{y}(.,t) - \mathcal{A}_{12}\hat{z}(\zeta,t) & \Pi \\ \hat{z}(\zeta,0) = \hat{z}_0(\zeta) & \Omega \\ \hat{z}(\mu,t) = 0 & \Xi \end{cases} \tag{14}$$

Where $(\mathcal{A}_{22} - \mathcal{H}_{\Gamma^*AG}\mathcal{A}_{12})$ generates a strongly continuous semi-group $(S_{\mathcal{A}_{22}}(t))t\geq 0$ which is Γ^*AGRO- stable on the Hilbert space $\mathbb{W}_2 \subset \mathbb{W}=(H^{1/2}(\Gamma^*))^n$, $(B2 - \mathcal{H}_{\Gamma^*AG}B1) \in L(\mathbb{R}^p, \mathbb{W}_2)$

and

$(\mathcal{A}_{22}\mathcal{H}_{\Gamma^*AG} - \mathcal{H}_{\Gamma^*AG}\mathcal{A}_{12}\mathcal{H}_{\Gamma^*AG} - \mathcal{H}_{\Gamma^*AG}\mathcal{A}_{11} + \mathcal{A}_{21}) \in L(\mathbb{R}^p, \mathbb{W}_2)$ as in [20].

The importance of reduced Γ^*AGRO - detectability is possible to define a reduced Γ^*AGRO-estimator for system state may be given by the following important result.

Theorem 3.8: If there are q sensors $(D_i, f_i)_{1\le i\le q}$ and the spectrum of $\mathcal{A}_{22}$ contains J eigenvalues with non-negative real parts. The systems (10)-(11) are Γ^*AGRO-detectable if and only if

1. $q \ge m_2$

2. $rank\ G_{2_i} = m_{2_i}, \forall i, i = 1, \dots, J$ with

$$G_2 = G_{2_{ij}} = \begin{cases} < \varphi_j(.), f_i(.) >_{L^2(D_i)} & \text{for zone sensors} \\ \varphi_j(b_i) & \text{for pointwise sensors} \end{cases}$$

where $\sup\ m_{2_i} = m_2 < \infty\ and\ j = 1, \dots, \infty.$

Proof : The prove is developed to the case of zone sensors in the following stapes:

First: The system (10) can be decomposed by the projections $\mathcal{P}$ and $I - \mathcal{P}$, on two parts, unstable and stable under the assumptions of section 3.1, where $\mathcal{P}$ and $(I - \mathcal{P})$ play the role of projection as E_1, E_2 [7]. The state vector may be given by

$$w_2(\zeta, t) = [w_{2_1}(\zeta, t) w_{2_2}(\zeta, t)]^{tr},$$

where $w_{2_1}(\zeta, t)$ is the state component of the unstable part of system (10), that may be written in the form

$$\begin{cases} \frac{\partial w_{2_1}}{\partial t}(\zeta, t) = \mathcal{A}_{22_1} w_{2_1}(\zeta, t) + \mathcal{P}[\mathcal{A}_{21_1} w_{1_1}(\zeta, t) + B_2 u(t)] & \Pi \\ w_{2_1}(\zeta, 0) = w_{2_{1_0}}(\zeta) & \Omega \\ w_{2_1}(\mu, t) = 0 & \Xi \end{cases} \quad (15)$$

and $w_{2_2}(\zeta, t)$ is the component state of the stable part of system(10), given by

$$\begin{cases} \frac{\partial w_{2_2}}{\partial t}(\zeta, t) = \mathcal{A}_{22_2} w_{2_2}(\zeta, t) + (I - \mathcal{P})\ [\mathcal{A}_{21_2} w_{1_2}(\zeta, t) + B_2 u(t)] & \Pi \\ w_{2_2}(\zeta, 0) = w_{2_{2_0}}(\zeta) & \Omega \\ w_{2_2}(\mu, t) = 0 & \Xi \end{cases} \quad (16)$$

The operator $\mathcal{A}_{22_1}$ is represented by a matrix of order $\left(\sum_{i=1}^{J} m_{2_i}, \sum_{i=1}^{J} m_{2_i}\right)$ given by

$$\mathcal{A}_{22_1} = diag[\lambda_{2_1}, \dots, \lambda_{2_1}, \dots, \lambda_{2_J}, \dots, \lambda_{2_J}] \text{ and } \mathcal{P}B_2 = [G_{2_1}^{tr}, G_{2_2}^{tr}, \dots, G_{2_J}^{tr}]$$

From condition (2) of this theorem, then the suite of sensors $(D_i, f_i)_{1\le i\le q}$ is Γ^*AGRO-strategic for the unstable part of the system (10), the subsystem (15) is weakly regionally boundary asymptotic gradient reduced order-observable in Γ^* (or $\mathbb{W}$ Γ^*AGRO- observable) and since it is finite dimensional, then it is exactly regionally boundary asymptotic gradient reduced order-observable in Γ^* (or $\mathbb{E}$ Γ^*AGRO-observable).

Therefore it is Γ^*AGRO- detectable, and hence there exists an operator $\mathcal{H}^1_{\Gamma^*AG}$ such that $(\mathcal{A}_{22_1} - \mathcal{H}^1_{\Gamma^*AG}\mathcal{A}_{12_1})$ which satisfies the following

$\exists M^1_{\Gamma^*AG}, \alpha^1_{\Gamma^*AG} > 0$ such that $\|e^{(\mathcal{A}_{22_1} - \mathcal{H}^1_{\Gamma^*AG}\mathcal{A}_{12_1})t}\|_{(H^{1/2}(\Gamma^*))^n} \leq M^1_{\Gamma^*AG} e^{-\alpha^1_{\Gamma^*AG}(t)}$

and we have

$$\|w_{2_1}(\zeta,t)\|_{(H^{1/2}(\Gamma^*))^n} \leq M^1_{\Gamma^*AG} e^{-\alpha^1_{\Gamma^*AG}(t)} \|\mathcal{P} w_{2_0}(.)\|_{(H^{1/2}(\Gamma^*))^n}$$

Since the semi-group generated by the operator $\mathcal{A}_{22_2}$ is Γ^*AGRO-stable, $\exists M^2_{\Gamma^*AG}, \alpha^2_{\Gamma^*AG} > 0$ such that

$$\|w_{2_2}(\zeta,t)\|_{(H^{1/2}(\Gamma^*))^n} \leq M^2_{\Gamma^*AG} e^{-\alpha^2_{\Gamma^*AG}(t)} \|(I-\mathcal{P}) w_{2_0}(.)\|_{(H^{1/2}(\Gamma^*))^n}$$

$$+ \int_0^t M^2_{\Gamma^*AG} e^{-\alpha^2_{\Gamma^*AG}(t-\tau)} \|(I-\mathcal{P}) w_{2_0}(.)\|_{(H^{1/2}(\Gamma^*))^n} \|u(\tau)\| d\tau$$

therefore $w_2\,(\zeta,t) \to 0$ when $t \to \infty$. Thus, the systems (10)-(11) are Γ^*AGRO-detectable.

Second: If the systems (10)-(11) are Γ^*AGRO-detectable, then $\exists \mathcal{H}_{\Gamma^*AG} \in L(L^2(0,\infty,\mathbb{R}^q), (H^{1/2}(\Gamma^*))^n)$ such that $(\mathcal{A}_{22} - \mathcal{H}_{\Gamma^*AG}\mathcal{A}_{12})$ generates an Γ^*AGRO-stable, strongly continuous semi-group $(S_{\mathcal{A}_{22}}(t))_{t\geq 0}$ on the space $H^{1/2}(\Gamma^*)$ which satisfies the following

$$\exists M_{\Gamma^*AG}, \alpha_{\Gamma^*AG} > 0 \text{ such that } \|\chi_{\Gamma^*AG} S_{\mathcal{A}_{22}}(t)\|_{(H^{1/2}(\Gamma^*))^n} \leq M_{\Gamma^*AG} e^{-\alpha_{\Gamma^*AG}(t)}$$

Thus the unstable subsystem (15) is Γ^*AGRO-detectable. Since this subsystem is of finite dimensional, then it is exactly Γ^*AGRO -observable. Therefore (15) is weakly Γ^*AGRO -observable and hence it is reduced Γ^*AGRO -strategic , $i.e.$

$$[\mathcal{K}\chi^*_{\Gamma^*} w^*_2(.,t) = 0 \Longrightarrow w^*_2(.,t) = 0]. \text{ For } w^*_2(.,t) \in H^{1/2}(\Gamma^*)$$

We have

$$[\mathcal{K}\chi^*_{\Gamma^*AG} w^*_2(.,t) = (\textstyle\sum_{j=1}^J e^{\lambda_j t} \langle \varphi_j(.), w^*_2(.,t)\rangle_{(H^{1/2}(\Gamma^*))^n} \langle \varphi_j(.), f_i(.)\rangle_{(H^{1/2}(\Gamma^*))^n})_{1\leq i\leq q}$$

If the unstable system (15) is not Γ^*AGRO -strategic, $\exists w^*_2(.,t) \in (H^{1/2}(\Gamma^*))^n$ such that

$$\mathcal{K}\chi^*_{\Gamma^*AG} w^*_2(.,t) = 0,$$

this leads to

$$\textstyle\sum_{j=1}^J \langle \varphi_j(.), w^*_2(.,t)\rangle_{(H^{1/2}(\Gamma^*))^n} \langle \varphi_j(.), f_i(.)\rangle_{(H^{1/2}(\Gamma^*))^n} = 0$$

the state vectors w_{2_i} may be given

$$w_{2_i}(.,t) = [\langle \varphi_j(.), w^*_2(.,t)\rangle_{(H^{1/2}(\Gamma^*))^n} \langle \varphi_j(.), w^*_2(.,t)\rangle_{(H^{1/2}(\Gamma^*))^n}]^{tr} \neq 0$$

We then obtain $G_{2_i} w_{2_i} = 0, \forall i, i = 1, \ldots, J$ and therefore $rank\ G_{2_i} \neq m_{2_i}$.

Here, we construct the $\Gamma^{*}AGRO$ - estimator for parabolic distributed parameter system (1), we need to present the following remarks

Remark 3.9: Now, choose the following decomposition:

$$\hat{z} = \begin{bmatrix} \hat{z}_1 \\ \hat{z}_2 \end{bmatrix} = \begin{bmatrix} y \\ \varphi + \mathcal{H}_{\Gamma^* AGRO} y \end{bmatrix}$$

Which estimates asymptotic gradient the state vector

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

then, the dynamical system (14) is given by the following system:

$$\begin{cases} \frac{\partial \varphi}{\partial t}(\zeta,t) = (\mathcal{A}_{22} - \mathcal{H}_{\Gamma^* AGRO}\mathcal{A}_{12})\varphi\,(\zeta,t) + (\mathcal{A}_{22}\mathcal{A}_{12}\mathcal{H}_{\Gamma^* AGRO} & \\ -\mathcal{H}_{\Gamma^* AGRO}\mathcal{A}_{11} + \mathcal{A}_{21})y(\zeta,t) + (B_2 - \mathcal{H}_{\Gamma^* AGRO}B_1)u(t) & \Pi \\ \varphi\,(\zeta,0) = \varphi_0(\zeta) & \Omega \\ \varphi\,(\mu,t) = 0 & \Xi \end{cases} \tag{17}$$

which defines an $\Gamma^{*}AGRO$- estimator for $T_{\Gamma^* AGRO} w_2(\zeta,t)$ if

1. $\lim_{t\to\infty} \|\varphi\,(\zeta,t) - T_{\Gamma^* AGRO} w_2(\zeta,t)\|_{(H^{1/2}(\Gamma^*))^n} = 0$

2. $T_{\Gamma^* AGRO}: D(\mathcal{A}_{22}) \to D(\mathcal{A}_{22} - \mathcal{H}_{\Gamma^* AGRO}\mathcal{A}_{12})$ where $T_{\Gamma^* AGRO} = w_{\Gamma^* AGRO} T$ and $\varphi(\zeta,t)$ is the solution of

system (17).

Remark 3.10: The dynamical system (17) estimates the regional boundary asymptotic gradient reduced order state of the system (1) if the following conditions satisfies:

1. $\exists L_{\Gamma^* AGRO} \in L(0, H^{1/2}(\Gamma^*))^n)$ and $M_{\Gamma^* AGRO} \in L(H^{1/2}(\Gamma^*))^n$ such that:

$$L_{\Gamma^* AGRO}\mathcal{A}_{12} + M_{\Gamma^* AGRO} T_{\Gamma^* AGRO} = I_{\Gamma^* AGRO}$$

2. $T_{\Gamma^* AGRO}\mathcal{A}_{22} - (\mathcal{A}_{22} - \mathcal{H}_{\Gamma^* AGRO}\mathcal{A}_{12})T_{\Gamma^* AGRO} = \mathcal{H}_{\Gamma^* AGRO}\mathcal{A}_{12}$ and $(B_2 - \mathcal{H}_{\Gamma^* AGRO}B_1) = T_{\Gamma^* AGRO}B_2$

3. The system (17) defines an $\Gamma^{*}AGRO$- estimator for the system (1).

4. If $\mathbb{W} = \mathbb{W}_2$ and $T_{\Gamma^* AGRO} = I_{\Gamma^* AGRO}$ then, in the above case, we have

$$\mathcal{A}_{22} - (\mathcal{A}_{22} - \mathcal{H}_{\Gamma^* AGRO}\mathcal{A}_{12}) = \mathcal{H}_{\Gamma^* AGRO}\mathcal{A}_{12}$$

Remark 3.11: The system (1) is $\Gamma^{*}AGRO$-observable if there exists an $\Gamma^{*}AGRO$ - estimators (17) which estimates the regional boundary asymptotic gradient reduced order state the system. Now, we present the sufficient condition of the regional boundary asymptotic gradient reduced order observability notion as in the following main result.

Theorem 3.12: If the systems (10)-(11) are $\Gamma^{*}AGRO$- detectable, then it is $\Gamma^{*}AGRO$ - observable by the dynamical system (16), that means

$$limt \to \infty \|(\varphi(\zeta,t) + \mathcal{H}_{\Gamma^* AG} y(\zeta,t)) - w_2(\zeta,t)\|_{(H^{1/2}(\Gamma^*))^n} = 0,$$

Proof: The solution of the dynamical system (14) is given by

$$\hat{z}(\zeta,t) = S_{\mathcal{H}_{\Gamma^* AG}}(t)\hat{z}_0(\zeta) + \int_0^t S_{\mathcal{H}_{\Gamma^* AG}}(t-\tau)\,[B_2 u(\tau) + \mathcal{A}_{21} y(\zeta,\tau) + \mathcal{H}_{\Gamma^* AG}\tilde{y}(\zeta,\tau)]d\tau \quad (18)$$

From the equation (11), we have

$$\tilde{y}(\zeta,t) = \mathcal{A}_{12} a(.,t) = \frac{\partial w_1}{\partial t}(\zeta,t) - \mathcal{A}_{11} w_1(\zeta,t) - B_1 u(t) \quad (19)$$

By using (18) and (19), we obtain

$$\hat{z}(\zeta,t) = S_{\mathcal{H}_{\Gamma^* AG}}(t)\hat{z}_0(\zeta) + \int_0^t S_{\mathcal{H}_{\Gamma^* AG}}(t-\tau)\mathcal{H}_{\Gamma^* AG}\frac{\partial w_1}{\partial t}(\zeta,\tau)d\tau + \int_0^t S_{\mathcal{H}_{\Gamma^* AG}}(t-\tau)\,[B_2 u(\tau) + \mathcal{A}_{21} y(\zeta,\tau) - \mathcal{H}_{\Gamma^* AG}\mathcal{A}_{11} w_1(.,\tau) - \mathcal{H}_{\Gamma^* AG} B_1 u(\tau)]d\tau \quad (20)$$

and we can get

$$\int_0^t S_{\mathcal{H}_{\Gamma^* AG}}(t-\tau)\mathcal{H}_{\Gamma^* AG}\frac{\partial w_1}{\partial t}(\zeta,\tau)d\tau = \mathcal{H}_{\Gamma^* AG} w_1(.,t) - S_{\mathcal{H}_{\Gamma^* AG}}(t)\mathcal{H}_{\Gamma^* AG} w_{1_0}(.)$$
$$+(\mathcal{A}_{22} - \mathcal{H}_{\Gamma^* AG}\mathcal{A}_{12})\int_0^t S_{\mathcal{H}_{\Gamma^* AG}}(t-\tau)\mathcal{H}_{\Gamma^* AG} w_1(.,\tau)\,d\tau \quad (21)$$

Using Bochner integrability properties and closeness of $(\mathcal{A}_{22} - \mathcal{H}_{\Gamma^* AG}\mathcal{A}_{12})$, the equation (21) becomes

$$\int_0^t S_{\mathcal{H}_{\Gamma^* AG}}(t-\tau)\mathcal{H}_{\Gamma^* AG}\frac{\partial w_1}{\partial t}(\zeta,\tau)d\tau = \mathcal{H}_{\Gamma^* AG} w_1(.,t) - S_{\mathcal{H}_{\Gamma^* AG}}(t)\mathcal{H}_{\Gamma^* AG} w_{1_0}(.)$$
$$+ \int_0^t S_{\mathcal{H}_{\Gamma^* AG}}(t-\tau)(\mathcal{A}_{22} - \mathcal{H}_{\Gamma^* AG}\mathcal{A}_{12})\mathcal{H}_{\Gamma^* AG} w_1(\zeta,\tau)d\tau \quad (22)$$

Substituting (22) into (20), we have

$$\hat{z}(.,t) = S_{\mathcal{H}_{\Gamma^* AG}}(t)\hat{z}_0(\zeta) - S_{\mathcal{H}_{\Gamma^* AG}}(t)\mathcal{H}_{\Gamma^* AG} w_{1_0}(.) + \mathcal{H}_{\Gamma^* AG} w_1(.,t)$$
$$+ \int_0^t S_{\mathcal{H}_{\Gamma^* AG}}(t-\tau)\,[\mathcal{A}_{22}\mathcal{H}_{\Gamma^* AG} - \mathcal{H}_{\Gamma^* AG}\mathcal{A}_{12}\mathcal{H}_{\Gamma^* AG} - \mathcal{H}_{\Gamma^* AG}\mathcal{A}_{11} + \mathcal{A}_{21}]$$
$$w_1(.,\tau)d\tau + \int_0^t S_{\mathcal{H}_{\Gamma^* AG}}(t-\tau)\,[B_2 - \mathcal{H}_{\Gamma^* AG} B_1]u(\tau)d\tau. \quad (23)$$

Setting $\varphi(.,t) = \hat{z}(.,t) - \mathcal{H}_{\Gamma^* AG} y(.,t)$, with $\varphi_0(.,0) = \hat{z}_0(.) - \mathcal{H}_{\Gamma^* AG} w_{1_0}(.)$, where $y_0(.) = w_{1_0}(.)$. Now, assume that $(\mathcal{A}_{22}\mathcal{H}_{\Gamma^* AG} - \mathcal{H}_{\Gamma^* AG}\mathcal{A}_{12}\mathcal{H}_{\Gamma^* AG} - \mathcal{H}_{\Gamma^* AG}\mathcal{A}_{11} + \mathcal{A}_{21})$ and $(B_2 - \mathcal{H}_{\Gamma^* AG} B_1)$ are bounded operators, the equation (23) can be differentiated to yield the following system

$$\begin{cases} \dfrac{\partial \varphi}{\partial t}(\zeta,t) = (\mathcal{A}_{22} - \mathcal{H}_{\Gamma^* AGRO}\mathcal{A}_{12})\varphi\,(\zeta,t) + (\mathcal{A}_{22}\mathcal{H}_{\Gamma^* AGRO} - \mathcal{H}_{\Gamma^* AGRO}\mathcal{A}_{12}\mathcal{H}_{\Gamma^* AGRO} \\ -\mathcal{H}_{\Gamma^* AGRO}\mathcal{A}_{11} + \mathcal{A}_{21})y(.,t) + (B_2 - \mathcal{H}_{\Gamma^* AGRO} B_1)u(t) & \Pi \\ \varphi\,(\zeta,0) = \varphi_0(\zeta) & \Omega \\ \varphi\,(\mu,t) = 0 & \Xi \end{cases}$$

and therefore

$$\frac{\partial z}{\partial t}(\zeta,t) - \frac{\partial w_2}{\partial t}(\zeta,t) = \varphi(\zeta,t) + \mathcal{H}_{\Gamma^* AG} y(\zeta,t) - w_2(\zeta,t)$$

$$= (\mathcal{A}_{22}\hat{z}(\zeta,t) + B_2u(t) + \mathcal{A}_{21}y(.,t) + \mathcal{H}_{\Gamma^*AG}(\tilde{y}(\zeta,t)$$
$$-\mathcal{A}_{12}\hat{z})(\zeta,t) - \mathcal{A}_{21}w_1(\zeta,t) - \mathcal{A}_{22}w_2(\zeta,t) - B_2u(t)$$
$$= (\mathcal{A}_{22} - \mathcal{H}_{\Gamma^*AG}\mathcal{A}_{12})(\hat{z}(\zeta,t) - w_2(\zeta,t)) \quad (24)$$

From the equation

$$\|w_{\Gamma^*AG}S_{\mathcal{H}_{\Gamma^*AG}}(t)w_{2_0}(.)\|_{(H^{1/2}(\Gamma^*))^n} \leq M_{\mathcal{H}_{\Gamma^*AG}}e^{-\alpha_{\mathcal{H}_{\Gamma^*AG}}(t)}$$

we obtain

$$\|\hat{z}(.,t) - w_2(.,t)\|_{(H^{1/2}(\Gamma^*))^n} \leq \|\chi_{\Gamma^*AG}S_{\mathcal{H}_{\Gamma^*AG}}(t)\|_{(H^{1/2}(\Gamma^*))^n}\|\hat{z}(.,0) - w_2(.,0)\|_{(H^{1/2}(\Gamma^*))^n}$$

$$\leq M_{\mathcal{H}_{\Gamma^*AG}}e^{-\alpha_{\mathcal{H}_{\Gamma^*AG}}(t)}\|\hat{z}(.,0) - w_2(.,0)\|_{(H^{1/2}(\Gamma^*))^n} \to 0 \text{ } as \text{ } t \to \infty \quad (25)$$

where the component $\hat{z}(\zeta,t)$ is asymptotic gradient estimator of w_2. Then, we have the system (14) is Γ^*AGRO-observable for the systems (9)-(10).■

From the previous theorem 3.12, we can deduce the following definition which characterizes another new strategic sensor:

Definition 3.13: A sensor is Γ^*AGRO-strategic sensor if the corresponding system is Γ^*AGRO-observable.

4. Application to asymptotic Γ^*AGRO-observer in exchange system

Consider the case of two-phase exchange systems described by the following coupled parabolic equations as in [8]

$$\begin{cases} \frac{\partial w_1}{\partial t}(\zeta_1,\zeta_2,t) = \alpha\frac{\partial^2 w_1}{\partial\zeta^2}(\zeta_1,\zeta_2,t) + \beta\big(w_1(\zeta_1,\zeta_2,t) - w_2(\zeta_1,\zeta_2,t)\big) & \Pi \\ \frac{\partial w_2}{\partial t}(\zeta_1,\zeta_2,t) = \gamma\frac{\partial^2 w_2}{\partial\zeta^2}(\zeta_1,\zeta_2,t) + \beta\big(w_2(\zeta_1,\zeta_2,t) - w_1(\zeta_1,\zeta_2,t)\big) & \Pi \\ w_1(\zeta_1,\zeta_2,0) = w_{1_0}(\zeta_1,\zeta_2),\ w_2(\zeta_1,\zeta_2,0) = w_{2_0}(\zeta_1,\zeta_2) & \Omega \\ w_1(\mu_1,\mu_2,t) = 0,\ w_2(\mu_1,\mu_2,t) = 0 & \Xi \end{cases} \quad (26)$$

and consider $\Omega = (0,1)\times(0,1)$ with subregion $\Gamma^* = (\alpha_1,\beta_1)\times(\alpha_2,\beta_2) \subset \partial\Omega$. Suppose that it is possible to measure the states $w_1(.,t)$, by using q zone sensor $(D_i,f_i)_{i\leq 1\leq q}$. The output function (2) is given by

$$y(.,t) = Cw_1(.,t) = \left[\int_{D_1} w_1(\zeta_1,\zeta_2,t)f_1(\zeta_1,\zeta_2)d\zeta_1 d\zeta_2, \ldots, \int_{D_q} w_1(\zeta_1,\zeta_2,t)f_q(\zeta_1,\zeta_2)d\zeta_1 d\zeta_2\right]^{tr}$$

Now, the problem is to estimate exponentially $w_2(\zeta_1,\zeta_2,t)$. Consider now

$$\frac{\partial w}{\partial t} = \begin{bmatrix} \frac{\partial w_1}{\partial t} \\ \frac{\partial w_2}{\partial t} \end{bmatrix} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad (27)$$

where

$$\mathcal{A}_{11} = \alpha\frac{\partial^2 w_1}{\partial\xi^2}(\zeta_1,\zeta_2,t) + \beta,\ \mathcal{A}_{22} = \gamma\frac{\partial^2 w_2}{\partial\zeta^2}(\zeta_1,\zeta_2,t) + \beta$$

and

$$\mathcal{A}_{12} = \mathcal{A}_{21} = -\beta I.$$

From theorem 3.12, we can construct regional boundary asymptotic gradient reduced order estimator for system (26) if the sensors $(D_i, f_i)_{i\le 1\le q}$ is $\Gamma^* AG$-strategic for the unstable part of the subsystem

$$\begin{cases} \frac{\partial w_1}{\partial t}(\zeta_1,\zeta_2,t) = \gamma \frac{\partial^2 w_1}{\partial \zeta^2}(\zeta_1,\zeta_2,t) + \beta(w_1(\zeta_1,\zeta_2,t) \\ -w_2(\zeta_1,\zeta_2,t)) & \Pi \\ w_1(\zeta_1,\zeta_2,0) = w_{1_0}(\zeta_1,\zeta_2) & \Omega \\ w_1(\mu_1,\mu_2,t) = 0 & \Xi \end{cases} \quad (28)$$

where $\gamma = 0.1$ and $\beta = 1$. If we choose the sensor $(D_i, f_i)_{i\le 1\le q}$ such that

$$y(t) = [\int_{D_1} w_1(\zeta_1,\zeta_2,t) f_1(\zeta_1,\zeta_2) d\zeta_1 d\zeta_2, \dots, \int_{D_q} w_1(\zeta_1,\zeta_2,t) f_q(\zeta_1,\zeta_2) d\zeta_1 d\zeta_2]^{tr} \neq 0,$$

then, there exists $\mathcal{H}_{\Gamma^* AGRO} \in L\left(\mathbb{R}^q, H^{1/2}(\Gamma^*)\right)$ such that the operator $(\mathcal{A}_{22} - H_{\Gamma^* AGRO}\mathcal{A}_{12})$ generates a strongly continuous semi-group stable on a Hilbert space $H^{1/2}(\Gamma^*)$. Thus we have

$$\lim_{n\to\infty} \|\hat{z}((.,t) + \mathcal{H}_{\Gamma^* AGRO} w_1(.,t)) - w_2(.,t)\|_{H^{1/2}(\Gamma^*)} = 0,$$

Where

$$\begin{cases} \frac{\partial \hat{z}}{\partial t}(\zeta_1,\zeta_2,t) = \gamma \frac{\partial^2 \hat{z}}{\partial \zeta^2}(\zeta_1,\zeta_2,t) + \beta((1+\mathcal{H}_{\Gamma^* AGRO})\hat{z}(\zeta_1,\zeta_2,t) \\ \qquad +(\gamma - \alpha\mathcal{H}_{\Gamma^* AGRO})\frac{\partial^2 w_1}{\partial \zeta^2}(\zeta_1,\zeta_2,t) + \beta\left(\mathcal{H}^2_{\Gamma^* AGRO} - 1\right)(\zeta_1,\zeta_2,t)) & \Pi \\ \hat{z}(\zeta_1,\zeta_2,0) = \hat{z}_0(\zeta_1,\zeta_2) & \Omega \\ \hat{z}(\mu_1,\mu_2,t) = 0 & \Xi \end{cases} \quad (29)$$

In this section, we give the specific results related to some examples of sensor locations and we apply these results to different situations of the domain, which usually follow from symmetry considerations. We consider the two-dimensional system defined on $\Omega = (0,1)\times(0,1)$ with the case of system described by the following equations:

$$\begin{cases} \frac{\partial w_2}{\partial t}(\zeta_1,\zeta_2,t) = \gamma \frac{\partial^2 w_2}{\partial \zeta^2}(\zeta_1,\zeta_2,t) + \beta w_2(\zeta_1,\zeta_2,t) - \beta w_1(\zeta_1,\zeta_2,t) & \Pi \\ w_2(\zeta_1,\zeta_2,0) = w_{2_0}(\zeta_1,\zeta_2) & \Omega \\ w_2(\mu_1,\mu_2,t) = 0 & \Xi \end{cases} \quad (30)$$

with the output function

$$y(t) = C w_1(.,t) \quad (31)$$

Let $\Gamma^* = \Pi_{i=1}^2(\alpha_i,\beta_i) = (\alpha_1,\beta_1)\times(\alpha_2,\beta_2)$, in this case the eigenfunctions and eigenvalues for the dynamic system (30) are given by

$$\varphi_{ij}(\zeta_1,\zeta_2) = \frac{2}{\sqrt{(\beta_1-\alpha_1)(\beta_2-\alpha_2)}} \cos i\pi\left(\frac{\zeta_1-\alpha_1}{\beta_1-\alpha_1}\right) \cos j\pi\left(\frac{\zeta_2-\alpha_2}{\beta_2-\alpha_2}\right) \quad (32)$$

and

$$\lambda_{ij} = -\left(\frac{i^2}{(\beta_1-\alpha_1)^2} + \frac{j^2}{(\beta_2-\alpha_2)^2}\right)\pi^2, \quad i,j \ge 1 \quad (33)$$

We examine the two cases illustrated in (Figures 3 and 4).

4.1 Internal rectangular zone sensor

To discuss this case, suppose the system (30)-(31) where the sensor supports D_i are located in Ω as in (Figure 3). The output function can be written by the form

$$y(t) = \int_{D_1} w_1(\zeta_1, \zeta_2, t) f_1(\zeta_1, \zeta_2) d\zeta_1 d\zeta_2, \tag{34}$$

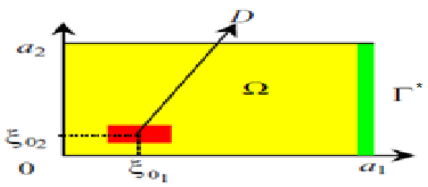

Fig. 3: Regions Ω, Γ^* and location D of zone sensor**.**

Then, the sensor $(D_i, f_i)_{i\leq 1\leq q}$ may be sufficient for Γ^*AGRO-observer, and there exists $\mathcal{H}_{\Gamma^*A_{GRO}} \in L(\mathbb{R}^q, H^{1/2}(\Gamma^*)$ such that the operator $(\mathcal{A}_{22} - \mathcal{H}_{\Gamma^*AGRO}\mathcal{A}_{12})$ generates a strongly continuous semi-group stable on a Hilbert space $H^{1/2}(\Gamma^*)$. Thus we have

$$\lim_{t\to\infty} \|(\hat{z}(\zeta_1, \zeta_2, t) + \mathcal{H}_{\Gamma^*AGRO} w_2(\zeta_1, \zeta_2, t)) - w_1(\zeta_1, \zeta_2, t)\|_{H^{1/2}(\Gamma^*)} = 0,$$

Where

$$\begin{cases} \frac{\partial \hat{z}}{\partial t}(\zeta_1, \zeta_2, t) = \gamma \frac{\partial^2 \hat{z}}{\partial \zeta^2}(\zeta_1, \zeta_2, t) + \beta((1 + \mathcal{H}_{\Gamma^*AGRO})\hat{z}(\zeta_1, \zeta_2, t) & \\ \quad + (\gamma - \alpha \mathcal{H}_{\Gamma^*AGRO}) \frac{\partial w_2}{\partial \zeta^2}(\zeta_1, \zeta_2, t) + \beta(\mathcal{H}^2_{\Gamma^*AGRO} - 1)(\zeta_1, \zeta_2, t)) & \Pi \\ \hat{z}(\zeta_1, \zeta_2, 0) = \hat{z}_0(\zeta_1, \zeta_2) & \Omega \\ \hat{z}(\zeta_1, \zeta_2, t) = 0 & \Xi \end{cases} \tag{35}$$

If

$$D_i = \Pi_{i=1}^2 [\zeta_{0_i} - l_i, \zeta_{0_i} + l_i], \text{with } [\zeta_{0_i} - \alpha_i / \zeta_{0_i} - \beta_i] \in Q.$$

Then, we have the following result.

Proposition 4.1: Let f_i are symmetric about line $w_{0_i} = \zeta_{0_i}$ and the sensors $(D_i, f_i)_{i\leq 1\leq q}$ are not strategic for the systems (30)-(31), and then these systems are not Γ^*AGRO-observable by the Γ^*AGRO –estimator systems (35). If for any $i_0 \in 1 \leq i \leq 2, j_0 \in 1 \leq i \leq q$ such that $\frac{i_0(\zeta_{0_1} - \alpha_1)}{\beta_1 - \alpha_1}, \frac{j_0(\zeta_{0_2} - \alpha_2)}{\beta_2 - \alpha_2} \in Q$.

Proof: Suppose that $i_0 = 1,\ and\ (\beta_1 - \alpha_1)/(\beta_2 - \alpha_2) \in \mathbb{Q}$, then there exists $j_0 \geq 1$ such that

$$j_0\,(\zeta_{0_1} - \alpha_1)/(\beta_1 - \alpha_1) = 0.$$

But

$$y(t) = \langle f_1, \varphi_{i_0} j_0 \rangle =$$
$$\left(\frac{4}{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)}\right)^{1/2} \int_{\alpha_2 - l_2}^{\alpha_2 + l_2} \int_{\alpha_1 - l_1}^{\alpha_1 + l_1} f_1(\zeta_1, \zeta_2) Sin\left[\frac{j_0 \pi(\zeta_{0_1} - \alpha_1)}{(\beta_1 - \alpha_1)}\right] Sin\left[\frac{j_0 \pi(\zeta_{0_2} - \alpha_2)}{(\beta_2 - \alpha_2)}\right] d\zeta_1 d\zeta_2 = 0$$

4.2 Internal pointwise sensor

Consider the case of pointwise sensor located inside of Ω. The system (30) augmented with the following output function:

$$y(t) = \int_{\Omega\backslash\omega} w_2(\zeta_1,\zeta_2,t)\,\delta(\zeta_1 - b_1,\zeta_2 - b_2)d\zeta_1 d\zeta_2 \quad (36)$$

where $b = (b_1, b_2) \in \Omega$ as in (Figure 4) is the location of pointwise sensor with $(b_1 - \alpha_1) / (\beta_1 - \alpha_1)$ and $(b_2 - \alpha_2) / (\beta_2 - \alpha_2) \in Q$.

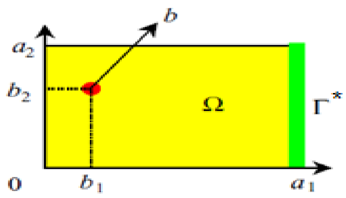

Fig.4: Region Ω, Γ^* and location D of pointwise sensor**.**

Then, the sensor (b, δ_b) may be sufficient for Γ^*AGRO-observability [9], and there exists $\mathcal{H}_{\Gamma^*AGRO} \in L(\mathbb{R}^q, H^{1/2}(\Gamma^*))$ such that the operator $(\mathcal{A}_{22} - \mathcal{H}_{\Gamma^*AGRO}\mathcal{A}_{12})$ generates a strongly continuous semi $-$ group stable on a Hilbert space $H^{1/2}(\Gamma^*)$. Thus we have

$$\lim_{t\to\infty}\left\|(\hat{z}(\zeta_1,\zeta_2,t) + \mathcal{H}_{\Gamma^*_{AGRO}} w_2(\zeta_1,\zeta_2,t)) - w_1(\zeta_1,\zeta_2,t)\right\|_{H^{1/2}(\Gamma^*)} = 0,$$

where

$$\begin{cases} \frac{\partial \hat{z}}{\partial t}(\zeta_1,\zeta_2,t) = \gamma \frac{\partial^2 \hat{z}}{\partial \zeta^2}(\zeta_1,\zeta_2,t) + \beta((1 + \mathcal{H}_{\Gamma^*AGRO})\hat{z}(\zeta_1,\zeta_2,t) & \\ \quad + (\gamma - \alpha\mathcal{H}_{\Gamma^*AGRO})\frac{\partial w_2}{\partial \zeta^2}(\zeta_1,\zeta_2,t) + \beta(\mathcal{H}^2_{\Gamma^*AGRO} - 1)(\zeta_1,\zeta_2,t)) & \Pi \\ \hat{z}(\zeta_1,\zeta_2,0) = \hat{z}_0(\zeta_1,\zeta_2) & \Omega \\ \hat{z}(\zeta_1,\zeta_2,t) = 0 & \Xi \end{cases} \quad (37)$$

Thus, we have the following result.

Corollary 4.2: the systems (30)-(36), are not Γ^*AGRO-observable by the $\Gamma^*AGRO -$ estimator systems (37). If for any $i_0 \in 1 \le i \le 2, j_0 \in 1 \le i \le q$ such that $\frac{i_0(b_1-\alpha_1)}{\beta_1-\alpha_1}, \frac{j_0(b_2-\alpha_2)}{\beta_2-\alpha_2} \in Q$

Proof: Suppose that $i_0 = 1,\ and\ (\beta_1 - \alpha_1)/(\beta_2 - \alpha_2) \in \mathbb{Q}$, then there exists $j_0 \ge 1$ such that Sin $j_0\,(b_1 - \alpha_1)/(\beta_1 - \alpha_1) = 0$. But

$$y(t) = \langle \delta_b, \varphi_{i_0} j_0 \rangle = \left(\frac{4}{(\beta_1-\alpha_1)(\beta_2-\alpha_2)}\right)^{1/2} \int_{\Omega/\omega} \delta_b(b_1,b_2) Sin\left[\frac{j_0\pi(b_1-\alpha_1)}{(\beta_1-\alpha_1)}\right] Sin\left[\frac{j_0\pi(b_2-\alpha_2)}{(\beta_2-\alpha_2)}\right] d\zeta_1 d\zeta_2 = 0$$

Remark 4.3: These results can be extended to the following:

(1) Case of Neumann or mixed boundary conditions [20].

(2) Case of boundary (pointwise, zone) sensors as in [37].

5. Conclusion

The concept have been studied in this paper is related to the $\Gamma AGRO$-observer in connection with sensors structure for a class of distributed parameter systems. More precisely, we have been given a sufficient condition for the existing of an $\Gamma AGRO$-observer which allows to estimate the gradient state in a subregion Γ. For future work, one can be extension these result to the problem of regional boundary asymptotic gradient full order observer in connection with the sensors structures.

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6. References

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