Basic Construction of Fibrewise Group

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Abstract: The purpose of this paper is to introduce the concepts of fibrewise group, fibrewise subgroup and fibrewise homomorphism. Also we give several results concerning it.
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1. Introduction and Preliminaries

The aim of this work is to promote the fibrewise versions in algebra, particularly in group theory.

To being with we work in the category of fibrewise sets over a given set, called the base set. If the base set is denoted by B then a fibrewise set over B consists of a set X together with a function $p: X \to B$, called the projection. For each point b of B the fibre over b is the subset $X_b = p^{-1}(b)$ of X; fibres may be empty since we do not require p to be surjective, also for each subset B' of B we regard$X_{B'} = p^{-1}(B')$ as a fibrewise set over $B'$with the projection determined by p, the alternative notation X/B' is sometimes convenient.

Let X be a fibrewise set over B with the projection p, then X' is a fibrewise set over B with projection $p\alpha$ for each set $X'$and function$\alpha: X' \to X$; in particular $X'$is

a fibrewise set over B with projection p/X' for each subset $X'$of X. Also X' is a fibrewise set over B' with projection βp for each set $B'$and function$\beta: B \to B'$; in particular X is a fibrewise set over B' with projection given by p for each superset $B'$of B.

We regard the Cartesian product B×T, for any set T, as a fibrewise set over B using the first projection.

Definition 1.1.[2] Let X and Y are fibrewise sets over B, with projections $p: X \to B$and $q: Y \to B$, respectively, a function $\varphi: X \to Y$ is said to be fibrewise function if $qo\varphi = p$, in other words if $\varphi(X_b) \subset Y_b$for each point b of B.

Note that a fibrewise function $\varphi: X \to Y$ over B determines by restriction, a fibrewise function $\varphi_{B'}: X_{B'} \to Y_{B'}$ over B' for each subsetB' of B.

Given an indexed family $\{X_r\}$ of fibrewise sets over B the fibrewise product $\mathbf{\Pi}_B X_r$ is defined, as a fibrewise set over$B$, and comes equipped with the family of fibrewise projections $\pi_r: \mathbf{\Pi}_B X_r \to X_r$. Specifically the fibrewise product is defined as the subset of the ordinary product $\mathbf{\Pi}\, X_r$ in which the fibres are the products of the corresponding fibers of thef actors X_r.

2. Fibrewise Group

Definition 2.1. Let B be a group, a fibrewise group over B is a fibrewise set G with any binary operation makes G a group such that the projection $p: G \to B$ is homomorphism.

Definition 2.2. Let G be a fibrewise group over B. Then any subgroup H of G is a fibrewise group over B with projection $p_{/H}: H \to B$, we call this group a fibrewise subgroup of G over B.

Examples.2.3.

1. Let $(IR^{+},.)$ be a multiplicative group, consisting of positive real's. Then the additive group of real numbers $(IR,+)$ is a fibrewise group over $(IR^{+},.)$, with Projection $p: (IR,+) \to (IR^{+},.)$ defined by $p(x) = e^{x}$.
2. The multiplicative group of non -zero Complex numbers $(\mathbb{C}^{*},.)$ is fibrewise group over the group $(IR^{+},.)$, with projection $q: (\mathbb{C}^{*},.) \to (IR^{+},.)$ defined by $q\,(z) = \sqrt{a^2+b^2}$, Where$z = a + ib$.
3. The group $(IR^{*},.)$ is fibrewise subgroup of the group $(\mathbb{C}^{*},.)$ over the group $(IR^{+},.)$, where ($(IR^{*},.)$ is multiplicative group ofnon-zero real numbers.

Remark.2.4. Let G be a fibrewise group over B, G_b it is not necessary a subgroup of G, where $b \in B$.

Using the properties of homomorphisms, we get the following result.

Theorem.2.5. Let G be a fibrewise group over B, Then:

1. The fibre of the identity e_B of B, G_{e_B} is fibrewise subgroup of the fibrewise group G.
2. If B' is subgroup of a group$B$, then the set $G_{B'} = p(B')$ is fibrewise subgroup of the fibrewise group G.
3. If $g \in G_b$, $b \in B$ then $g^{-1} \in G_{b^{-1}}$.

Definition 2.6. Let G and K be two fibrewise groups over B. Then any homomorphism $\varphi: G \to K$ is called a fibrewise homomorphism if $\varphi$is a fibrewise map.

Definition 2.7. A bijective fibrewise homomorphism is called a fibrewise isomorphism.

Theorem 2.8. Let G be a fibrewise group over B and H be a fibrewise normal subgroup of G. Then G/H is fibrewise group over B, with projection $q: G/H \to B$ such that $q\pi = p$.

$$\begin{tikzcd} G \arrow[rr, "\pi"] \arrow[dr, "P"'] & & G/H \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

Proof:

Let $x, y \in G$, Then $xH, yH \in G/H$

$\Rightarrow q(xHyH) = q\big(\pi(x)\pi(y)\big) = q\pi(xy)$

$= p(xy) = p(x)p(y) = q\pi(x)q\pi(y) = q(xH)q(yH)$.

This implies q is homomorphism and hence G/H is fibrewise group over B.

Lemma 2.9. Let $\varphi: G \to K$ be a fibrewise function, where G and K are fibrewise groups over B, with projections p, q respectively. Then $\varphi(\ker(p)) \subseteq \ker(\mathrm{q})$.

Proof:

Let $x \in \ker(p) \Rightarrow p(x) = e_B \Rightarrow q\varphi(x) = e_B$

$\Rightarrow \varphi(x) \in \ker(\mathrm{q})$

Theorem 2.10. Let $\varphi: G \to K$ be a fibrewise function, where G and K are fibrewise groups over B, with projectionsp, qrespectively. Then:

1. If q is injective then φ is a fibrewise homomorphism, and consequently:

i) $\varphi(e_G) = e_K$, where e_G , e_K denotes the identities of$G$, K respective.

ii) $\varphi(\ker(p)) = e_K$.

iii) If H is is fibrewise subgroup over B of G then $\varphi(H)$ is fibrewise subgroup over B of K.

iv) If H' fibrewise subgroup of K. Then $\varphi^{-1}(H')$ is fibrewise subgroup of G.

v) If H fibrewise normal subgroup of G, then $\varphi(H)$ is fibrewise normal subgroup of K.

2. If p is bijective and q is injective then if G abelian then K is abelian.

3. If q is bijective and p is surjective then if G is cyclic then K is cyclic.

4. If p, q are bijective then φ is fibrewise isomorphism.

Proof:

1. Let $x, y \in \mathrm{G}$ then $\mathrm{p}(xy) = p(x)p(y) \Rightarrow q\varphi(xy) = q\varphi(x)q\varphi(y)$

2. $= q\big(\varphi(x)\varphi(y)\big) \Rightarrow \varphi(xy) = \varphi(x)\varphi(y)$, then φ is a fibrewise homomorphism and hence the proofs i), ii), iii), iv), v) are direct.

3. Let $x, y \in K \Rightarrow q(x), q(y) \in B$

4. $\Rightarrow p^{-1}q(x), p^{-1}q(y) \in G$, G is abelian, then $p^{-1}q(x)p^{-1}q(y) = p^{-1}q(y)p^{-1}q(x) \Rightarrow p^{-1}q(xy)$

5. $= p^{-1}q(yx) \Rightarrow q(xy) = q(yx)$

6. $\Rightarrow xy = yx \Rightarrow K$ is abelian

7. Let G be a fibrewise cyclic group with generator a, let$\varphi(a) = t$, then

8. $\forall k \in K \exists\, b \in B: q(k) = b, \exists g\, G: P(g) = b$, since P, q are surjective. Therefor $q(k) = b = P(g) = q\varphi(g)$, since q is injective, we have $k = \varphi(g)$,

9. $g \in G$ since G is cyclic then there exist natural number $\exists m \in IN: g = a^m$ and $k = \varphi(g) = \varphi(a^m) = q^{-1}P(a^m) = \big(q^{-1}P(a)\big)^m = (\varphi(a))^m = t^m$. We obtain K is generated by element t

10. $\Rightarrow$ K is cyclic.

11. Obvious.

3. FibrewiseDirect Product of FibrewiseGroups

Definition.3.1. Let G and K be fibrewise groups over B , with projections p, q respectively. The fibrewise direct product of G and K, is the fibrewise product $G \times_B K = \{(g, k) :: p(g) = q(k),\ \text{for } g \in G, k \in K\}$we define the operation o on $G \times_B K$ by

$$(g_1, k_1)\mathrm{o}(g_2, k_2) = (g_1 * g_2, k_1 \diamond k_2)$$

where the operations $*$ and $\diamond$ aredefined on G and K respectively. We shall occasionally write $(g_1, k_1)(g_2, k_2) = (g_1 g_2, k_1 k_2)$.

Lemma.3.2. The fibrewise direct product $G \times_B K$ of two fibrewise groups G and K over B is group and if B abelian group, then $G \times_B K$ is fibrewise group over B with projection $\psi: G \times_B K \to B$, defined by $\psi\big((g, k)\big) = p(g)q(k)$.

Proof:

Let G and K be fibrewise groups over B, with projections p, q respectively let $g_1, g_2 \in G$, $k_1, k_2 \in K$

First we prove that the fibrewise direct product $G \times_B K$ is group.

1. To prove that o on $G \times_B K$ is a binary operation, let $(g_1, k_1), (g_2, k_2) \in G \times_B K \Rightarrow p(g_1) = q(k_1)$ and

2. $p(g_2) = q(k_2) \Rightarrow e_B = (p(g_1))^{-1}q(k_1) = p(g_2)(q(k_2))^{-1}$

3. $\Rightarrow p(g_1)(p(g_1))^{-1}q(k_1) = p(g_1)\, p(g_2)(q(k_2))^{-1}$

4. $\Rightarrow q(k_1) = p(g_1)\, p(g_2)(q(k_2))^{-1}$

5. $\Rightarrow q(k_1)q(k_2) = p(g_1)\, p(g_2)(q(k_2))^{-1}\, q(k_2)$

6. $\Rightarrow q(k_1)q(k_2) = p(g_1)p(g_2)$

7. $\Rightarrow q(k_1 k_2) = p(g_1 g_2)$

8. $\Rightarrow (g_1 g_2, k_1 k_2) \in G \times_B K$ and $(g_1, k_1)(g_2, k_2) = (g_1 g_2, k_1 k_2)$.

9. The identity element of $G \times_B K$ is

10. (e_G, e_K) since $p(e_G) = e_B = q(e_K)$.

11. For any $g \in G$, $k \in K$ we have the $((g, k))^{-1} = (g^{-1}, k^{-1})$

12. now, $(g, k) \in G \times_B K$

13. $\Rightarrow p(g) = q(k)$

14. $\Rightarrow (p(g))^{-1} = (q(k))^{-1}$

15. $\Rightarrow p(g^{-1}) = q(k^{-1})$

16. $\Rightarrow (g^{-1}, k^{-1}) \in G \times_B K$.

17. Obviously $G \times_B K$ is associative.

18. Second, we prove that $G \times_B K$ is fibrewise group over B. Since B is abelian group then $\psi\big((g_1, k_1)\,(g_2, k_2)\big) = \psi\big((g_1 g_2, k_1 k_2)\big)$

19. $= p(g_1 g_2) q(k_1 k_2) = p(g_1)p(g_2)q(k_1)q(k_2) = p(g_1)q(k_1)p(g_2)q(k_2)$

20. $= \psi((g_1, k_1))\,\psi((g_2, k_2))$. Hence ψ is homomorphism and $G \times_B K$ is fibrewise group.

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