Positively Correlated Elements on D-poset

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Abstract. In this paper, a dependence is proposed to be defined on d-poset. It has been called positively correlated elements, denoted by R, and we have defined it with respect to special quantum logic functions that have been defined on d-poset too. Several examples have been illustrated to explain various situations of the constructed dependence when elements are compatible and noncompatible.

Keywords: Probability Space, Quantum Logic, D-poset, Orthoalgebra, Dependences, Lattice.

1. Introduction

In this paper, an attempt to define a dependence on d-poset in association with some special maps called quantum logic maps, [8]. Firstly, we have reconstructed them on d-poset. Then, a definition of positively correlated elements has been illustrated on d-poset via them. Its essential construction has been examined in order to gain any possible new properties. In fact, there are various properties have been presented. Also, several examples, and tables have been illustrated to explain the property of compatibility and non-compatibility, which represent essential facts in many quantum mechanics studies.
A main objective of this study associate quantum logic concepts with some statistical concepts like dependences, and some statistical algebraic maps.
Historically, there are many studies related to this topic. In (1983), Weber has presented a study that associate t-norms with fuzzy systems. Also, in (1991), Gupta has established the basic concepts of t-norms and fuzzy inference. Other concepts related to some statistical concepts, and quantum logic studies have been introduced by Nanasiova in (2003).

Recently, there are several studies that aim to develop the mathematical representations of some maps, operators, and theories via quantum logic studies by generalizing some basic concepts on modern algebraic like d-poset. Honestly, one can refer to many names that have impact in such notions, see [1-2, 6-7].

Finally, we refer to the organization of this study:
In the next section, we present the most basic concepts that can be used to construct reasonable modifications of each structure on d-poset. In section three, we illustrate the main notion that connect a generalization of dependences to the quantum logic maps on d-poset. We present several explanatory examples to each situation related to compatible, or non-compatible elements.

2. Basic notions

The following basic notions are related to d-poset, orthoalgebras, state, and quantum logic maps. Also, some definitions of correlation coefficient have been recalled.

Definition 2.1 [6] A D-poset, or a difference poset, is a partially ordered set L with a partial ordering $\leq$, greatest element 1, and partial binary operation $\ominus: L \times L \to L$, called a difference, such that, for $u, v, w \in L$, $v \ominus u$ is defined if and only if $u \leq v$. Then the following axioms hold for all $u, v, w \in L$:

1. $v \ominus u \leq v$.
2. $v \ominus (v \ominus u) = u$.
3. $u \leq v \leq w \Longrightarrow w \ominus v \leq w \ominus u$ and $(w \ominus u) \ominus (w \ominus v) = v \ominus u$.

Definition 2.2 [3-5] An orthoalgebra is a set L with two particular elements 0,1, and with a partial binary operation $\oplus: L \times L \to L$ such that for all $u, v, w \in L$ we have:

i. If $u \oplus v \in L$, then $v \oplus u \in L$ and $u \oplus v = v \oplus u$ (commutativity).

ii. If $v \oplus w \in L$ and $u \oplus (v \oplus w) \in L$, then $u \oplus v \in L$ and $(u \oplus v) \oplus w \in L$ and, $u \oplus (v \oplus w)$=(u$\oplus$ v) $\oplus$ w (associativity).

iii. For any $u \in L$ there is a unique $v \in L$ such that $u \oplus v$ is defined, and $u \oplus v = 1$ (orthocomplemention).

iv. If $u \oplus u$ is defined, then $u = 0$ (consistency).

Definition 2.3 [8, 2] Let ð be a d-poset. A map $m : L \to [0,1]$ is called state if the following conditions hold:

1. $m(O) = 0$;
2. For all $u, v \in L$, if $u \perp v$, then $m(u \oplus v) = m(u) + m(v)$, where $u \neq v$.

It is well-known that $m(I) = 1$.

Since the main notion that we will construct it later depends on the quantum logic maps on ð, so we review their definitions, respectively as follows:

Definition 2.4 [2] Let ð be a d-poset. A map $p_{dp}: L^2 \to [0,1]$ is called an s-map on d-poset, if it holds the following conditions.

1. $p_{dp}(1,1) = 1$;
2. For all $u, v \in L$, such that $v \ominus u \in L$, and $u \perp v$, then $p_{dp}(u,v) = 0$;
3. For all $u, v \in L$, such that $v \ominus u \in L$, and $u \perp v$, then for any $w \in L$,

$$p_{dp}(u \oplus v, w) = p_{dp}(u,w) + p_{dp}(v,w);$$
$$p_{dp}(w, u \oplus v) = p_{dp}(w,u) + p_{dp}(w,v).$$

Example 2.1 [2] Let $\boldsymbol{p_{dp}}$ be an s-map of compatible elements on a d-poset ð. Then

Table 1. Compatible elements of s-map.

$p_{dp}(.,.)$	O	u	u'	v	v'	I
O	0	0	0	0	0	0
u	0	0.3	0	0.2	0.1	0.3
u'	0	0	0.6	0.4	0.3	0.7
v	0	0.2	0.4	0.6	0	0.6
v'	0	0.1	0.3	0	0.4	0.4
I	0	0.3	0.7	0.6	0.4	1

We can, see that the values of p_{dp} in *Table 1* are compatible, for example $p_{dp}(u,v) = p_{dp}(v,u)$, and etc. Conversely, the following example can be illustrated with an s-map that has non-compatible elements.

Example 2.2 [3] Let p_{dp} be an s-map of non-compatible elements on the d-poset ð. Then

Table 2. Non-compatible elements of s-map.

$p_{dp}(.,.)$	O	u	u'	v	v'	I
O	0	0	0	0	0	0
u	0	0.3	0	0.1	0.2	0.3
u'	0	0	0.7	0.5	0.2	0.7
v	0	0.2	0.4	0.6	0	0.6
v'	0	0.1	0.3	0	0.4	0.4
I	0	0.3	0.7	0.6	0.4	1

Obviously, all the values of p_{dp} support the non-compatibility property. For instance, $p_{dp}(u,v') \neq p_{dp}(v',u)$.

Definition 2.5[2] Let ð be a d-poset. A map $q_{dp}: L^2 \to [0,1]$ is called a j-map on d-poset, if it holds the following conditions.

1. $q_{dp}(O,O) = 0,\ q_{dp}(1,1) = 1$;
2. For all $u, v \in L$, such that $v \ominus u \in L$, then $q_{dp}(u,v) = q_{dp}(u,u) + q_{dp}(v,v)$;
3. For all $u, v \in L$, such that $v \ominus u \in L$, then for any $w \in L$,

$$q_{dp}(u \oplus v, w) = q_{dp}(u,w) + q_{dp}(v,w) - q_{dp}(w,w);$$

$$q_{dp}(w, u \oplus v) = q_{dp}(w, u) + q_{dp}(w, v) - q_{dp}(w, w).$$

Example 2.3 [2] Let q_{dp} be a j-map of compatible elements on the d-poset ð. Then

Table 3. Compatible elements of j-map.

$q_{dp}(.,.)$	O	u	u'	v	v'	I
O	0	0.2	0.8	0.1	0.9	1
u	0.2	0.2	1	0.3	0.9	1
u'	0.8	1	0.8	0.8	1	1
v	0.1	0.3	0.8	0.1	1	1
v'	0.9	0.9	1	1	0.9	1
I	1	1	1	1	1	1

Again, and conversely, a non-compatible case can be shown in the following example.
Example 2.4 [2] Let q be a j-map of non-compatible elements on ð. Then

Table 4. Non-compatible elements of j-map.

$q_{dp}(.,.)$	O	u	u'	v	v'	I
O	0	0.7	0.3	0.66	0.34	1
u	0.7	0.7	1	0.9	0.8	1
u'	0.3	1	0.3	0.76	0.54	1
v	0.66	0.88	0.78	0.66	1	1
v'	0.34	0.82	0.52	1	0.34	1
I	1	1	1	1	1	1

3. Positively Correlated Elements on D-poset

In this part, a notion of positively correlated elements is constructed with respect to a notion of s-map, and j-map on d-poset. Firstly, it is important to illustrate the following essential definition.
Definition 3.1 A function $R_{dp}: L^2 \to [-1, 1]$ which is defined by the following relation
$\forall\, u, v \in L,\ R_{dp} = p_{dp}(u, v) - p_{dp}(u, u)p_{dp}(v, v)$ (1)
is called positively correlated, if $R_{dp}(u, v) > 0$, where p_{dp} is an s-map on ð.
Note that, for all $u, v \in L$, $R_{dp}(u, v) < 0$, then u, v are called negatively correlation. On the other hand, a positively correlated elements R_{dp} has equivalent forms upon the orthocomplement of the elements on d-poset.
Lemma 3.1 let ð be a d-poset, and p_{dp} be an s-map. For all $u, v \in L$, if $u, and\ v$ are a positively correlated, then the following statements are equivalently true.
$R_{dp}(u, v) = R_{dp}(u', v') = -R_{dp}(u, v') = -R_{dp}(u', v), \forall u, v \in L.$
Pr**oof:**
Let $u, v \in D$,$p_{dp}(u, u) = \alpha, p_{dp}(v, v) = \beta$,and let $p_{dp}(u, v) = k \le \min\{\alpha, \beta, 1-\alpha, 1-\beta\}$.
Then $p_{dp}(u, v') = \alpha - k, p_{dp}(u', v) = \beta - k$, and $p_{dp}(u', v') = (1 + k) - (\alpha + \beta)$.
Thus, $R_{dp}(u, v) = p_{dp}(u, v) - p_{dp}(u, u)p_{dp}(v, v) = k - \alpha\beta \ge 0$.
Also $R_{dp}(u', v') = p_{dp}(u', v') - p_{dp}(u', u')p_{dp}(v', v')$
=(1+k)-$(\alpha + \beta)$-$(1 - \alpha) - (1 - \beta)$
$= 1 + k - \alpha - \beta - 1 + \beta + \alpha - \alpha\beta = k - \alpha\beta$.
Hence, $R_{dp}(u', v') = R_{dp}(u, v)$. Let's calculate $R_{dp}(u, v')$
$R_{dp}(u, v') = p_{dp}(u, v') - p_{dp}(u, u)p_{dp}(v', v')$
$= \alpha - k - \alpha(1 - \beta)$
$= \alpha - k - \alpha + \alpha\beta$
$= -(k - \alpha\beta)$.
Hence, $R_{dp}(u, v') = -R_{dp}(u, v)$. Finally, we compute $R_{dp}(u', v)$ we obtain
$R_{dp}(u', v) = p_{dp}(u', v) - p_{dp}(u', u')p_{dp}(v, v)$
$= \beta - k - (1 - \alpha)\beta$

$= \beta - k - \beta + \alpha\beta$
$= -(k - \alpha\beta).$
Therefore, $R_{dp}(u', v) = -R_{dp}(u, v)$.
In order to explain how R_{dp} can be constructed, it is possible to build the following example.
Example 3.1 Recall Example 2.1. Then, the calculations of $R_{dp}(a, b)$, for all $a, b \in D$ are presented in the following table.

Table 5. Values of R_{dp} within compatible elements of s-map on ð.

$\boldsymbol{R_{dp}}(.,.)$	$\boldsymbol{O}$	$\boldsymbol{u}$	$\boldsymbol{u'}$	$\boldsymbol{v}$	$\boldsymbol{v'}$	$\boldsymbol{I}$
$\boldsymbol{O}$	0	0	0	0	0	0
$\boldsymbol{u}$	0	0.18	-0.18	0.02	-0.02	0
$\boldsymbol{u'}$	0	-0.18	0.18	-0.04	0.06	0
$\boldsymbol{v}$	0	0.02	0.04	0.24	-0.24	0
$\boldsymbol{v'}$	0	-0.02	0.06	-0.24	0.24	0
$\boldsymbol{I}$	0	0	0	0	0	0

It is clear that $R_{dp}(u, v) = R_{dp}(u, v)$ because the elements u, and v are compatible. In fact, this is true for all other pairs in *Table 5*.
It can be figured out that the elements in R_{dp} inherits the same property of the elements that have been represented by s-map. This means that when $p_{dp}(u, v) \neq p_{dp}(v, u)$, implies that $R_{dp}(u, v) \neq R_{dp}(v, u)$ (when elements are not compatible), or vice versa.
Corollary 3.1 Let ð be a d-poset. Then $R_{dp}(I, u) = R_{dp}(O, u) = 0$, for all $u \in L$.
Proof
Let $u \in L$. Then
$R_{dp}(I, u) = p_{dp}(I, u) - p(I, I)p(u, u)$ (by equation (1))
But $p_{dp}(I, u) = p_{dp}(u, u)$, and $p_{dp}(I, I) = 1$ (properties of s-map)
Hence $R_{dp}(I, u) = R_{dp}(u, u) - R_{dp}(u, u) = 0$
On the other hand, we have
$R_{dp}(O, u) = p_{dp}(O, u) - p(O, O)p(u, u)$
But $p_{dp}(O, u) = p_{dp}(O, O) = 0 \Rightarrow R_{dp}(O, u) = 0$
This complete the proof.
Moreover, it is possible to show that R_{dp} fulfill the additive property on ð.
Proposition 3.1 Let ð be a d-poset. If $u, v \in L$, such that $v \ominus u \in L$, then for all $w \in L$

$$R_{dp}(u \oplus v, w) = R_{dp}(u, w) + R_{dp}(v, w),$$
$$R_{dp}(w, u \oplus v) = R_{dp}(w, u) + R_{dp}(w, v).$$

Proof
To prove that, $R_{dp}(u \oplus v, w) = R_{dp}(u, w) + R_{dp}(v, w)$.
Let $w \in L$, and $v \ominus u \in L$. Then,
$R_{dp}(u \oplus v, w) = p_{dp}(u \oplus v, w) - p_{dp}(u \oplus v, u \oplus v)p_{dp}(w, w)$
$= p_{dp}(u, w) + p_{dp}(v, w) - \left(p_{dp}(u, u) + p_{dp}(v, v)\right)p_{dp}(w, w)$
$= p_{dp}(u, v) - p_{dp}(u, u)p_{dp}(w, w) + p_{dp}(v, w) - p_{dp}(v, v)p_{dp}(w, w)$
$= R_{dp}(u, w) + R_{dp}(v, w)$
Similarly, $R_{dp}(u \oplus v, w) = R_{dp}(u, w) + R_{dp}(v, w)$.
Example 3.2 Recall Example 2.2. It can be found the calculations of $R_{dp}(a, b)$, for all $a, b \in L$. The results in *Table 6* show the dependence between elements within non-compatibility property.

Table 6. Values of R_{dp} within Non-compatible elements of s-map on ð.

$\boldsymbol{R_{dp}}(.,.)$	$\boldsymbol{O}$	$\boldsymbol{u}$	$\boldsymbol{u'}$	$\boldsymbol{v}$	$\boldsymbol{v'}$	$\boldsymbol{I}$
$\boldsymbol{O}$	0	0	0	0	0	0
$\boldsymbol{u}$	0	0.21	0.21	0.02	-0.02	0
$\boldsymbol{u'}$	0	-0.21	-0.21	-0.02	0.02	0

$\boldsymbol{v}$	0	-0.08	0.08	0.24	-0.24	0
$\boldsymbol{v'}$	0	-0.08	0.08	-0.24	0.24	0
$\boldsymbol{I}$	0	0	0	0	0	0

For instance, it is clear that $R_{dp}(u,v) \neq R_{dp}(v,u)$, which means that elements over R_{dp} are non-compatible. Another basic notion is a notion of positively correlated elements with respect to j-map on d-poset.

Definition 3.2 A function $R^*{}_{dp}: L^2 \to [-1,1]$ which is defined by the following relation $\forall\, u,v \in D$, $R^*{}_{dp} = q_{dp}(u,v) - q_{dp}(u,u)q_{dp}(v,v)$, is called positively correlated, if $R^*{}_{dp}(u,v) > 0$, where q_{dp} is a j-map on ð.

Example 3.3 Recall Example 2.3. The calculations of $R^*{}_{dp}(a,b)$ are presented in *Table 7*.

Table 7. Values of $R^*{}_{dp}$ within compatible elements of j-map on ð.

$R^*{}_{dp}(.,.)$	$\boldsymbol{O}$	$\boldsymbol{u}$	$\boldsymbol{u'}$	$\boldsymbol{v}$	$\boldsymbol{v'}$	$\boldsymbol{I}$
$\boldsymbol{O}$	0	0.2	0.8	0.1	0.9	1
$\boldsymbol{u}$	0.2	0.16	0.84	0.28	0.72	0.8
$\boldsymbol{u'}$	0.8	0.84	0.16	0.72	0.28	0.2
$\boldsymbol{v}$	0.1	0.28	0.72	0.9	0.91	0.9
$\boldsymbol{v'}$	0.9	0.72	0.28	0.91	0.9	0.1
$\boldsymbol{I}$	1	0.8	0.2	0.9	0.1	0

Example 3.4 Recall Example 2.4. Positively correlated $R^*{}_{dp}(a,b)$ with respect to non-compatible elements can be seen in the following table.

Table 8. Values of $R^*{}_{dp}$ within non-compatible elements of j-map on ð.

$R^*{}_{dp}(.,.)$	$\boldsymbol{O}$	$\boldsymbol{u}$	$\boldsymbol{u'}$	$\boldsymbol{v}$	$\boldsymbol{v'}$	$\boldsymbol{I}$
$\boldsymbol{O}$	0	0.7	0.3	0.66	0.34	1
$\boldsymbol{u}$	0.7	0.21	0.79	0.483	0.562	0.3
$\boldsymbol{u'}$	0.3	0.79	0.21	0.562	0.34	0.7
$\boldsymbol{v}$	0.66	0.418	0.582	0.2244	0.7756	0.34
$\boldsymbol{v'}$	0.34	0.582	0.282	0.7756	0.2244	0.66
$\boldsymbol{I}$	1	0.3	0.7	0.34	0.66	0

Finally, we illustrate a definition of a type of correlation coefficient via the dependence R_{dp} on ð.

Definition 3.3 Let ð be a d-poset, and R_{dp} be a positive or negative dependence. A map $r_{dp}: L^2 \to [0,1]$, such that

$$r_{dp}(u,v) = \frac{R_{dp}(u,v)}{\sqrt{R_{dp}(u,u)R_{dp}(v,v)}},\ R_{dp}(u,u), R_{dp}(v,v) \neq 0,$$

respectively. Then r_{dp} is called a correlation coefficient on ð.

Example 3.5 Recall Example 3.1. The calculations of r_{dp} via compatible elements can be shown in *Table 9* in the following way.

Table 9. Values of r_{dp} within Compatible Elements of s-map on ð.

$r_{dp}(.,.)$	$\boldsymbol{u}$	$\boldsymbol{u'}$	$\boldsymbol{v}$	$\boldsymbol{v'}$
$\boldsymbol{u}$	1	-1	0.096	-0.096
$\boldsymbol{u'}$	-1	1	0.115	0.288
$\boldsymbol{v}$	0.096	0.115	1	0.489
$\boldsymbol{v'}$	-0.096	0.288	0.489	1

Example 3.6 Recall Example 3.2. The calculations of r_{dp} can be shown within non-compatibility property as we can see in *table 10*.

Table 10. Values of r_{dp} within Non-compatible Elements of s-map on ð.

$r_{dp}(.,.)$	$\boldsymbol{u}$	$\boldsymbol{u'}$	$\boldsymbol{v}$	$\boldsymbol{v'}$
$\boldsymbol{u}$	1	-1	0.089	-0.089
$\boldsymbol{u'}$	-1	1	-0.089	0.089
$\boldsymbol{v}$	-0.356	0.356	1	-1
$\boldsymbol{v'}$	0.356	-0.356	-1	1

Proposition 3.2 A correlation coefficient r_{dp} on ð fulfills the following statements.

1. $\forall\, u \in D,\, r_{dp}(u,u) = r_{dp}(u',u') = 1$.
2. $\forall\, u \in D,\, r_{dp}(u,u') = r_{dp}(u',u) = -1$.
3. $u \leftrightarrow v$, then $r_{dp}(u,v) = r_{dp}(v,u)$.

Proof

1. Given that $r_{dp}(u,u) = \frac{R_{dp}(u,u)}{\sqrt{R_{dp}(u,u)^2}} = \frac{R_{dp}(u,u)}{R_{dp}(u,u)} = 1$.

Similarly, $r_{dp}(u',u') = \frac{R_{dp}(u',u')}{R_{dp}(u',u')} = 1$.

2. Since $r_{dp}(u,u') = \frac{R_{dp}(u,u')}{\sqrt{R_{dp}(u,u)R_{dp}(u',u')}}$, but by **Lemma 3.1**

Since $(R_{dp}(u,u) = R_{dp}(u',u') = -R_{dp}(u,u') = R_{dp}(u',u))$.

Therefore, $r_{dp}(u,u') = \frac{-R_{dp}(u,u)}{\sqrt{R_{dp}(u,u)R_{dp}(u,u)}} = \frac{-R_{dp}(u,u)}{R_{dp}(u,u)} = -1$.

3. Let $u \leftrightarrow v$, let $r_{dp}(u,v) = \frac{R_{dp}(u,v)}{\sqrt{R_{dp}(u,u)R_{dp}(v,v)}}$, if $u \leftrightarrow v$, then $R_{dp}(u,v) = R_{dp}(v,u)$

Therefore, $r_{dp}(u,v) = \frac{R_{dp}(v,u)}{\sqrt{R_{dp}(v,v)R_{dp}(u,u)}} = r_{dp}(v,u)$.

4 Conclusion

The structures of the modified dependences on d-poset have smooth representations as functions that can be used to deal with compatibility property. The results of the positively correlated elements are different when we calculate them within s-map to those whom calculated within j-map on d-poset. There are several properties of R_{dp}, R^*_{dp}, or even r_{dp} that might differ from the properties of classical dependences.
is an equivalent to the classical dependences. But, the main
difference between the positively correlated elements can have differ values of the commutative
elements which is not possible to happen in classical situation. This fact follows from the
compatibility and non-compatibility of elements on d-poset. A very important fact that has been concluded is related to the correlation coefficient r_{dp} on d-poset when elements are compatible or non-compatible and this can be useful to test some applications that concern with symmetry and asymmetry properties.

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