Republic of Iraq
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College of Science

Application of the Characteristic-Free Resolution of Weyl Module to the Lascoux Resolution in the Case of Partition (8,7,3)

A Thesis
Submitted to the Council of College of Science, Mustansiriyah University as a Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Mathematics

By

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بسم اللهِ الرَّحْمَنِ الرَّحِيمِ

اقْرَأْ بِاسْمِ رَبِّكَ الَّذِي خَلَقَ (١)

خَلَقَ الْأِنْسَانَ مِنْ عَلَقٍ (٢)

اقْرَأْ وَرَبُّكَ الْأَكْرَمُ (٣)

الَّذِي عَلَّمَ بِالْقَلَمِ (٤)

عَلَّمَ الْأِنْسَانَ مَا لَمْ يَعْلَمْ (٥)

سورة العلق

صدق اللّه العظيم

الآية (١ -٥)

الإهــداء

إلى الحي الّذي عند ربه يرزق ...

إلى الغائب عن الدنيا الحاضر بقلبي ...

إلى من أفتقده بكل لحظه في حياتي ...

تمنيت أن يكون معي يشاركني فرحتي ...

(أبي رحمه الله)

إلى الشمعة التي تضيء لي الدنيا حناناً ...

إلى صاحبة القلب الدافىء ...

إلى روحي ونور عيوني ...

حفظك لي ربي وأطال بعمرك ...

(أمي الغالية)

إلى ينبوع الحنان الذي لا ينضب ...

إلى أرض العطاء التي لا تجدب ...

الورود الزاهية في بستان دنيتي ...

أدامكم لي ربي سَنَدي في حياتي وحفظكم لي

(أخواتي و أخي)

نيران

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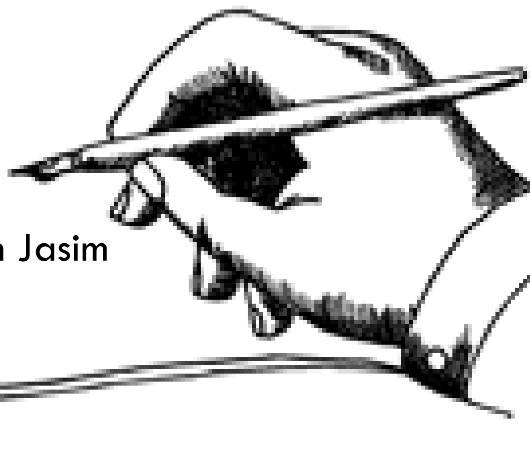


List of Symbols

Symbol	Meaning
m	Multiplication map
Δ	Diagonalization map
η	Unit map
ε	Counit map
$\mathcal{F}$	Free module
$\mathcal{D}_n\mathcal{F}$	The divided power algebra of the field $\mathcal{F}$
$\Lambda_n(\mathcal{F})$	The exterior algebra of the field $\mathcal{F}$
$\mathcal{S}_n(\mathcal{F})$	The symmetric algebra of the field $\mathcal{F}$
$GL_n(\mathcal{F})$	General linear group of degree n over the field $\mathcal{F}$
$d_\alpha(\mathcal{F})$	Schur map
$d'_\alpha(\mathcal{F})$	Weyl map
$\mathcal{L}_\alpha(\mathcal{F})$	Schur module
$\mathcal{K}_\alpha(\mathcal{F})$	Weyl module
$\square_{\lambda/\mu}$	Box map
$\mathrm{Tab}_{\lambda/\mu}$	The set of all tableaux of the shape λ/μ
$\mathcal{P}^+$	Positive place alphabet
$\mathcal{P}^-$	Negative place alphabet
$\blacksquare$	End of the proof

Abstract

Let $\mathcal{R}$ be a commutative ring with identity, $\mathcal{F}$ be a free $\mathcal{R}$-module and $\mathcal{D}_n\mathcal{F}$ be the divided power algebra of degree n.

By employing the technicality of Bar-complex and letter place algebra with Capelli identities, Buchsbaum surveys the resolution of Weyl module and shows that the large class of $GL_n(\mathcal{F})$-modules is defined among all the Weyl modules $\mathcal{K}_{\lambda/\mu}(\mathcal{F})$; where λ/μ is the skew-partition and $\mathcal{K}_{\lambda/\mu}(\mathcal{F})$ is the image of Weyl map $d'_{\lambda/\mu}(\mathcal{F})$.

In this thesis we discuss an application of the resolution of two-rowed Weyl module in the case of partition (8,7) to find the terms of this resolution and prove its exactness. Also as an application of the resolution of three-rowed Weyl module we find the terms of characteristic-free resolution in the case of partition (8,7,3), the terms of Lascoux complex for the same partition and diagrams of complex of Lascoux also for the same partition. As a generalization to the same techniques used by Buchsbaum we find the reduction from the characteristic-free resolution of Weyl module to characteristic-zero resolution (Lascoux resolution) with using the boundary maps which are used in the characteristic-zero in the case of partition (8,7,3). Eventually, by employing mapping Cone and its diagrams we survey the characteristic-zero resolution (Lascoux resolution) and prove that it is exact without depending on the boundary maps for the same partition.

Contents

Introduction

Let R be a field of characteristic-zero and $\mathcal{F}$ is an R-vector space of dimension n. The set of all irreducible polynomial representations of general linear group $GL_n(\mathcal{F})$ of degree n is described by the Schur module $\{\mathcal{L}_\lambda(\mathcal{F})\}$; where λ runs over all partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$. There are a number of classical formulas that express the formal character of the representation $\mathcal{L}_\lambda(\mathcal{F})$ in terms of standard symmetric polynomials. Such formulas are also valid for the more general representation modules $\{\mathcal{L}_{\lambda/\mu}(\mathcal{F})\}$ associated to skew partition λ/μ; where $\mu \subseteq \lambda$.

The Giambelli's identity $[G]$ and the Jacobi-Trudi identity $[\mathcal{J}-\mathcal{T}]$ formulas are studied in this context which are described in [11] as follows:

$$[G]: \mathcal{S}_{\lambda/\mu}(\mathcal{X}) = \det\left(e_{\lambda_i-\mu_j+j-i}(\mathcal{X})\right)$$

$$[\mathcal{J}-\mathcal{T}] : \mathcal{S}_{\lambda/\mu}(\mathcal{X}) = \det\left(h_{\tilde{\lambda}_i-\tilde{\mu}_j+j-i}(\mathcal{X})\right);$$

where

$\mathcal{X} = (x_1, x_2, \dots, x_n)$ is a set of variables;

$e_r(\mathcal{X})$ is the r^th elementary symmetric polynomial function defined by

$$e_r(\mathcal{X}) = \sum_{1\le i_1<\cdots<i_r\le n} x_{i_1}x_{i_2}\dots x_{i_r};$$

$h_r(\mathcal{X})$ is the r^th complete symmetric polynomial function defined by

$$h_r(\mathcal{X}) = \sum_{i_1+\cdots+i_n=r} x_1^{i_1}x_2^{i_2}\dots x_n^{i_n};$$

$\tilde{\lambda}/\tilde{\mu}$ is the skew partition dual to λ/μ;

$\mathcal{S}_{\lambda/\mu}(\mathcal{X})$ is the formal character of $\{\mathcal{L}_{\lambda/\mu}(\mathcal{F})\}$.

The classical formulas $[G]$ and $[\mathcal{J}-\mathcal{T}]$ above express the formal character $\mathcal{S}_{\lambda/\mu}(\mathcal{X})$ of $\mathcal{L}_{\lambda/\mu}(\mathcal{F})$ in terms of the formal characters $\mathcal{S}_{(r)}(\mathcal{X}) = e_r(\mathcal{X})$ of the fundamental representations $\Lambda^r\mathcal{F}$ and in term of the formal characters $\mathcal{S}_{\underbrace{(1,\dots,1)}_{r}}(\mathcal{X}) = h_r(\mathcal{X})$ of the fundamental representations $\mathcal{S}_r(\mathcal{F})$ respectively.

In the Grothendieck ring $\mathcal{K}[GL_n(\mathcal{F})]$ of $GL_n(\mathcal{F})$-modules, the above identities can be replaced by

$$[G]:[\mathcal{L}_{\lambda/\mu}(\mathcal{F})] = \det\left([\Lambda^{\lambda_i-\mu_j+j-i}\mathcal{F}]\right)$$

$$[\mathcal{J}-\mathcal{T}] : [\mathcal{L}_{\lambda/\mu}(\mathcal{F})] = \det\left([\mathcal{S}_{\tilde{\lambda}_i-\tilde{\mu}_j+j-i}\mathcal{F}]\right)$$

The author in [27] translates the expansion of the classical Giambelli determinantal expression $[G]$ into resolution $\mathcal{B}_\bullet$ in characteristic-zero of $\mathcal{L}_\lambda(\mathcal{F})$. In particular, the author asserts that the formula $[G]$ may be realized in characteristic-zero as the "Euler-Poincare" characteristic of the complex $\mathcal{B}_\bullet$ in the ring $\mathcal{K}[GL_n(\mathcal{F})]$.

The precise definitions of the boundary maps are given in [4]; where it is proved (always in characteristic-zero) that the complex $\mathcal{B}_\bullet$ is exact.

To be more explicit using the same notation as in [8], let

$$\mathcal{M}_{i,j} = \Lambda^{\lambda_i-\mu_j+j-i}\mathcal{F}$$

Then, we have:

$[\mathcal{L}_{\lambda/\mu}(\mathcal{F})] = \sum_{\sigma\in S_k}(-1)^{sgn\ \sigma}[\mathcal{M}_{1,\sigma(1)}\otimes\mathcal{M}_{2,\sigma(2)}\otimes\ldots\otimes\mathcal{M}_{k,\sigma(k)}]$

$= \sum_{\ell=0}^{\binom{k}{2}}(-1)^{\ell}[\mathcal{B}_\ell]$;

where

S_k is the symmetric group,

$\ell = \ell(\sigma)$ is the length of the permutation σ,

$\mathcal{B}_\ell = \sum_{\ell=\ell(\sigma)}\mathcal{M}_{1,\sigma(1)}\otimes\mathcal{M}_{2,\sigma(2)}\otimes\ldots\otimes\mathcal{M}_{k,\sigma(k)}$, and

$GL_n(\mathcal{F})$-equivariant boundary maps of the complex

$$\mathcal{B}_\bullet : 0 \longrightarrow \mathcal{B}_{\binom{k}{2}} \xrightarrow{\partial_{\binom{k}{2}}} \ldots \longrightarrow \mathcal{B}_1 \xrightarrow{\partial_1} \mathcal{B}_0 \longrightarrow \mathcal{L}_{\lambda/\mu}(\mathcal{F}) \longrightarrow 0$$

are described in [4].

Note that the terms of the resolution $\mathcal{B}_\bullet$ of $\mathcal{L}_{\lambda/\mu}(\mathcal{F})$ are direct sums of tensor products of the fundamental representations of $GL_n(\mathcal{F})$ and clearly, the exactness of $\mathcal{B}_\bullet$ implies the identity $[G]$.

From now on, let $\mathcal{F}$ be a free module of finite rank over a commutative ring $\mathcal{R}$. In [2] a large class of $GL_n(\mathcal{F})$-modules is defined among them, all the co-Schure module $\mathcal{K}_{\lambda/\mu}(\mathcal{F})$ (Weyl module). Schure and Weyl modules are universally free and there is a natural map of $\mathcal{K}_{\tilde{\lambda}/\tilde{\mu}}(\mathcal{F})$ into $\mathcal{L}_{\lambda/\mu}(\mathcal{F})$. When $\mathcal{R}$ contains the field of rationales $\mathbb{Q}$ this map is an isomorphism. In particular, the identities $[G]$ and $[\mathcal{J}-\mathcal{T}]$, which hold in general in the ring $\mathcal{K}[GL_n(\mathcal{F})]$ take the following form:

$$[G]: [\mathcal{L}_{\lambda/\mu}(\mathcal{F})] = \det\left([\Lambda^{\lambda_i-\mu_j+j-i}(\mathcal{F})]\right)$$

$$[\mathcal{J}-\mathcal{T}] : [\mathcal{K}_{\lambda/\mu}(\mathcal{F})] = \det\left([\mathcal{D}_{\lambda_i-\mu_j+j-i}(\mathcal{F})]\right);$$

where $\mathcal{D}_r(\mathcal{F})$ stands for the divided powers of $\mathcal{F}$.

Notice that, in characteristic-zero since $\mathcal{K}_{\tilde{\lambda}/\tilde{\mu}}(\mathcal{F}) \xrightarrow{\approx} \mathcal{L}_{\lambda/\mu}(\mathcal{F})$ and $\mathcal{D}_r(\mathcal{F}) \approx \mathcal{S}_r(\mathcal{F})$, we get the identities $[G]$ = $[\mathcal{J}-\mathcal{T}]$ of the classical case as described before i.e

$$[\mathcal{K}_{\tilde{\lambda}/\tilde{\mu}}(\mathcal{F})] = \det\left([\mathcal{S}_{\tilde{\lambda}_i-\tilde{\mu}_j+j-i}\mathcal{F}]\right) = \det\left([\mathcal{D}_{\tilde{\lambda}_i-\tilde{\mu}_j+j-i}\mathcal{F}]\right)$$

In general, it is not true that the complex $\mathcal{B}_\bullet$ exact but it can be enlarged to a complex $\widetilde{\mathcal{B}}_\bullet$. More precisely $\mathcal{L}_{\lambda/\mu}(\mathcal{F})(\mathcal{K}_{\lambda/\mu}(\mathcal{F}))$ has a finite resolution $\widetilde{\mathcal{B}}_\bullet$ whose terms are direct sums of the tensor product of exterior of $\mathcal{F}$.

The Authors in [3], [4], [5] and [6] have described the resolutions $\widetilde{\mathcal{B}}_\bullet$ of Weyl and Schure modules by writing down explicit projective resolutions of the two-rowed modules. The existence proof of resolution for the similar problem with an arbitrary number of rows the authors in [7] gave that. While the existence of resolution of Weyl modules whose terms are direct sums of tensor products of divided powers proved by the authors in [19]. By using the duality between Schur and Weyl module one can also solve Schur modules using tensor products of exterior powers.

It is important to point out that, there was not explicit description of these finite resolutions, except for shapes of length two, and a class of shapes of length three.

Using the letter place algebra notation is to modify the standard kind of maps which is used in the above cited papers to place polarizations operators (derivations). The advantage is to replace the arithmetic Koszul complex by an appropriate Bar complex. It simplifies strongly the description of the terms of the resolution. Also, it is clear that for the two-rowed case it is possible to write a splitting homotopy for the resolution by using the letter place approach and by reformulating the resolutions involved in terms of Bar complex.

In [13] and [15] the authors have studied clearly in details the terms of the resolutions for all shapes in the so called class of "almost skew shapes". This characterization is largely located on the "Bar complex" framework, but a total characterization of the boundary map is still an open problem.

The author in [20] presents the skeleton in the resolution of skew-shapes. Especially the terms of Lascoux resolution can be recovered within the formulas approaching in [15] and [16]. Over and above the application of the outcomes aforesaid above, the author in [18] illustrated that by employing the letter place methods and place polarization in a symmetric way.

The authors in [17] studied the corresponding of Weyl module to the partition (2,2,2), the relationship between the resolution of $\mathcal{K}_{(2,2,2)}\mathcal{F}$ in the characteristic-free module and in the Lascoux mode. By this comparison, the characteristic-free boundary maps are modified to obtain the obvious maps of the Lascoux case.

Haytham R. Hassan generalize the techniques in [17] for the partitions (3,3,3), (4,4,3) in [23] and [25] respectively, also he studied in [24] the resolution of Wely module in the case of two-rowed skew-shape (p+t,q)/(t,0) and the complex of Lascoux in partition (4,4,4) in [26]. While the authors Alaa O. Azziz in [9], Nora T. Abdul Razak in [1], Mais M. Mohmmed in [28] and Najah M. Mustafa in [29] used the same technique in [21] and [23] for the partitions (3,3,2), (6,5,3), (6,6,3) and (7,6,3) respectively.

This thesis consists of three chapters. In chapter one, we review some definitions, remarks, theorems and examples to illustrate the concepts Hoph algebra, Schure functors, letter place algebra and differential Bar complex. In chapter two, we exhibit the resolution of two-rowed and three rowed Weyl module and discuss an application of the resolution of two-rowed Weyl module in the case of partition (8,7) and find the terms of this resolution and prove its exactness. In chapter three, we study in detail an application of the resolution of three-rowed Weyl module for the case of the partition (8,7,3) we find respectively the terms of characteristic-free resolution, the terms of Lascoux complex, diagrams of the complex of Lascoux, reduction from the characteristic-free resolution of Weyl module to characteristic-zero resolution (Lascoux resolution) with using the boundary maps which are used in the characteristic-zero. Finally, by employing mapping Cone and its diagrams we survey the characteristic-zero resolution (Lascoux resolution) and prove it to be exact without depending on the boundary maps for the same partition.

Chapter one

Preliminaries

Introduction

This chapter consists of four sections, the Hopf algebra with examples illustrated in the first section, while the Schur functors presented in the second section. Some definitions and examples about the letter place algebra exhibit in the third section. Finally the concept differential Bar complex with some definitions and example given in the last section.

1.1 Hopf algebras

Definition (1.1.1): [10]

Given a commutative ring $\mathcal{R}$ with identity, an **$\mathcal{R}$-algebra** is an $\mathcal{R}$-module $\mathcal{A}$ endowed with two $\mathcal{R}$-morphisms

$$m_{\mathcal{A}}: \mathcal{A} \otimes_{\mathcal{R}} \mathcal{A} \longrightarrow \mathcal{A} \quad \text{(multiplication)} \quad , \quad u_{\mathcal{A}}: \mathcal{R} \longrightarrow \mathcal{A} \text{ (unit)},$$

such that the following diagrams are commute

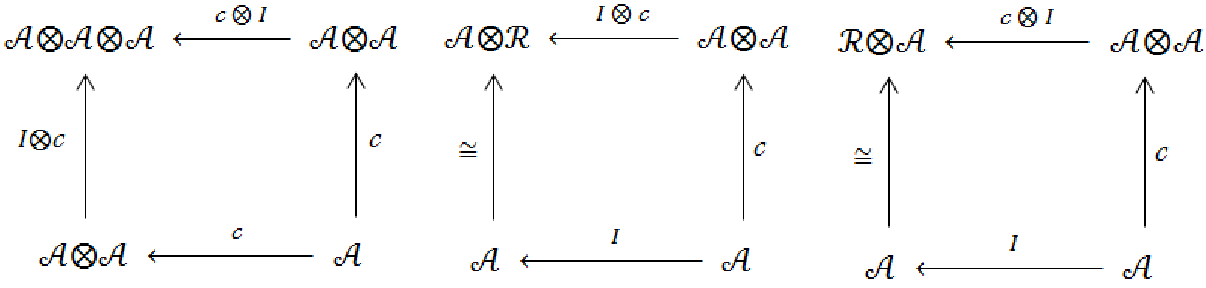
$$\begin{array}{ccc} \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} & \xrightarrow{m\otimes I} & \mathcal{A}\otimes\mathcal{A} \\ {\scriptstyle I\otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ \mathcal{A}\otimes\mathcal{A} & \xrightarrow{m} & \mathcal{A} \end{array} \qquad \begin{array}{ccc} \mathcal{A}\otimes\mathcal{R} & \xrightarrow{I\otimes u} & \mathcal{A}\otimes\mathcal{A} \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle m} \\ \mathcal{A} & \xrightarrow{I} & \mathcal{A} \end{array} \qquad \begin{array}{ccc} \mathcal{R}\otimes\mathcal{A} & \xrightarrow{u\otimes I} & \mathcal{A}\otimes\mathcal{A} \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle m} \\ \mathcal{A} & \xrightarrow{I} & \mathcal{A} \end{array}$$

Definition (1.1.2): [10]

Given a commutative ring $\mathcal{R}$ with identity, an **$\mathcal{R}$-co-algebra** is an $\mathcal{R}$-module $\mathcal{A}$ endowed with two $\mathcal{R}$-morphisms

$$c_{\mathcal{A}}: \mathcal{A} \longrightarrow \mathcal{A} \otimes_{\mathcal{R}} \mathcal{A} \quad \text{(co-multiplication)} \quad , \quad \varepsilon_{\mathcal{A}}: \mathcal{A} \longrightarrow \mathcal{R} \text{ (co-unit)},$$

such that the following diagrams are commute



Definition (1.1.3): [10]

A **graded ring** is a ring $\mathcal{S}$ together with a set of subgroups $\mathcal{S}_d$, $d \geq 0$ such that $\mathcal{S} = \bigoplus_{d\geq 0} \mathcal{S}_d$ as an abelian group, and $st \in \mathcal{S}_{d+e}$ for all $s \in \mathcal{S}_d$, $t \in \mathcal{S}_e$.

Definition (1.1.4): [10]

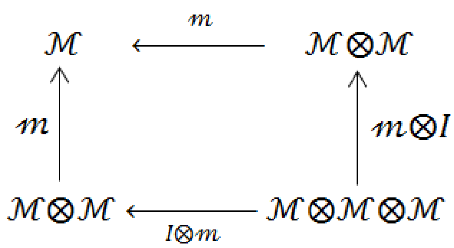
If $\mathcal{S}$ is a graded ring then a **graded $\mathcal{S}$-module** is an $\mathcal{S}$-module $\mathcal{M}$ together with a set of subgroups $\mathcal{M}_n$, $n \in \mathbb{Z}$ such that $\mathcal{M} = \bigoplus_{n\in\mathbb{Z}} \mathcal{M}_n$ as an abelian group, and $sm \in \mathcal{S}_{n+d}$ for all $s \in \mathcal{S}_d$, $m \in \mathcal{M}_n$.

Definition (1.1.5): [21]

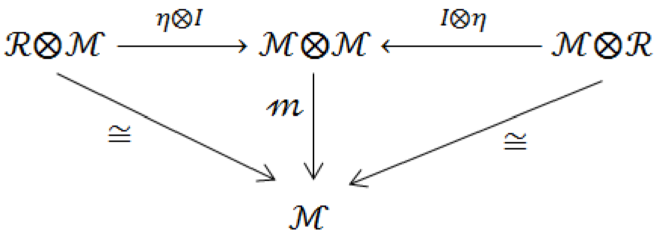
Let $\mathcal{R}$ be a commutative ring. A **graded $\mathcal{R}$-algebra** is a graded $\mathcal{R}$-module $\mathcal{M} = \bigoplus_{i\geq 0} \mathcal{M}_i$ together with a "multiplication" (homogenous)

$$m: \mathcal{M}\otimes\mathcal{M} \longrightarrow \mathcal{M} \quad \text{and} \quad \text{a unit } \eta: \mathcal{R} \longrightarrow \mathcal{M},$$

such that the following diagrams are commute



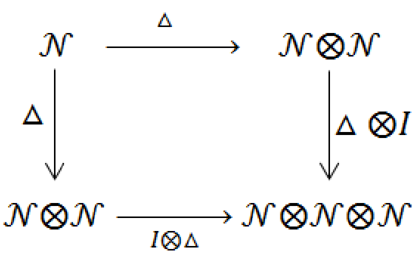
(the associative law)



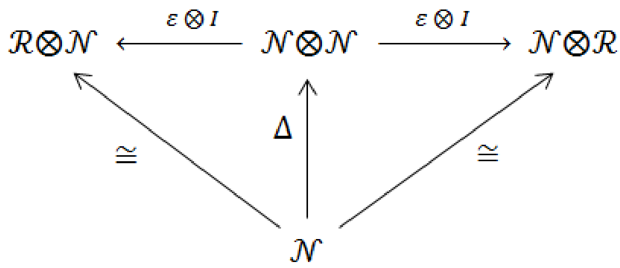
(The unitary property)

Definition (1.1.6): [21]

A **graded $\mathcal{R}$-co-algebra** is a graded $\mathcal{R}$-module $\mathcal{N} = \bigoplus_{i\geq 0} \mathcal{N}_i$ together with "diagonalization" or homogeneous co-multiplication $\Delta: \mathcal{N} \longrightarrow \mathcal{N} \otimes \mathcal{N}$ and a linear map co-unit $\varepsilon: \mathcal{N} \longrightarrow \mathcal{R}$ such that the following diagrams are commute



(The co-associative law)



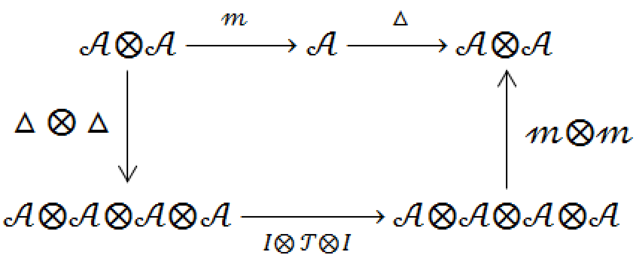
(The co-unitary property)

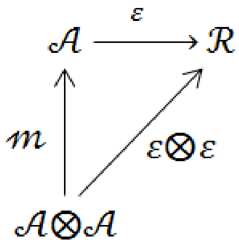
Definition (1.1.7): [21]

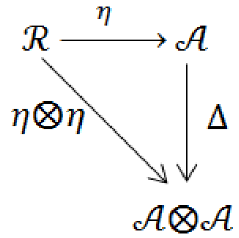
A **graded $\mathcal{R}$-Hopf algebra** is a graded $\mathcal{R}$-module $\mathcal{A}$ together with a multiplication $m: \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$, co-multiplication $\Delta: \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$ and a unit $\eta: \mathcal{R} \longrightarrow \mathcal{A}$ and a co-unit $\varepsilon : \mathcal{A} \longrightarrow \mathcal{R}$ satisfying these properties:

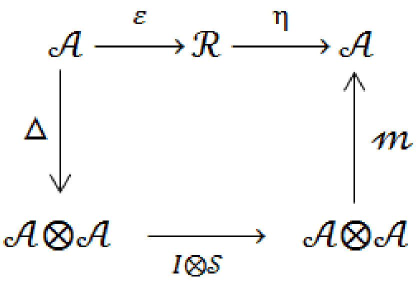
(1) $(\mathcal{A}, m, \eta)$ is a graded $\mathcal{R}$-algebra, $(\mathcal{A}, \Delta, \varepsilon)$ is a graded $\mathcal{R}$-co-algebra, $\varepsilon : \mathcal{A} \longrightarrow \mathcal{R}$ is a map of $\mathcal{R}$-algebras, $\eta : \mathcal{R} \longrightarrow \mathcal{A}$ is a map of $\mathcal{R}$ -co-algebras.

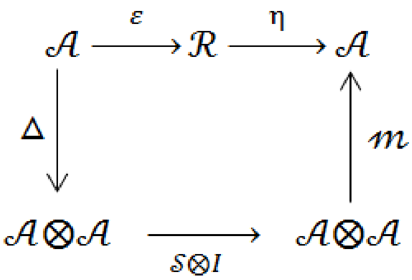
(2) The following diagrams are commute









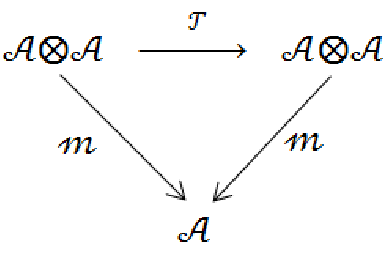


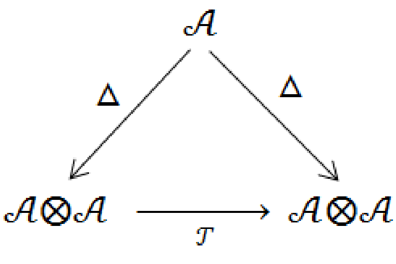
Where $\mathcal{T}: \mathcal{A}\otimes\mathcal{A} \longrightarrow \mathcal{A}\otimes\mathcal{A}$ is the twisting morphism which is defined by

$$\mathcal{T}(a\otimes b) = (-1)^{ij} b\otimes a, \text{ for } a \in \mathcal{A}_i, b \in \mathcal{A}_j,$$

and $\mathcal{S} : \mathcal{A} \longrightarrow \mathcal{A}$ is $\mathcal{R}$-linear (the unique "antipode" map).

If the following two diagrams commute, we say that $\mathcal{A}$ is a **commutative graded $\mathcal{R}$-Hopf algebra**.





In our work, we will presume that $\mathcal{A}$ is **connected** (i.e. $\mathcal{A}_0 = \mathcal{R}$) and for every $i, \mathcal{A}_i$ is finitely generated free $\mathcal{R}$-module.

Now, we exhibit major examples of Hopf algebras.

Example (1.1.8): [2] **(The exterior algebra)**

The exterior algebra of finitely generated free $\mathcal{R}$-module $\mathcal{F}$ is the free graded commutative $\mathcal{R}$-algebra generated by an element of $\mathcal{F}$ in degree one and is denoted by $\Lambda\mathcal{F} = \sum_{r\geq 0}\Lambda^r\mathcal{F}$. It is constructed as the quotient $\mathcal{T}(\mathcal{F})/\mathcal{J}$; where $\mathcal{T}(\mathcal{F}) = \sum_{r\geq 0}\mathcal{T}_r(\mathcal{F})$ is the tensor algebra on $\mathcal{F}$ and $\mathcal{J}$=$\sum_{r\geq 0}\mathcal{J}_r$ is the two-sided homogeneous ideal of T($\mathcal{F}$) generated by elements of the type $x\otimes x$; where $x \in \mathcal{F}$ and the r-th degree component $\Lambda^r\mathcal{F}$ is $\mathcal{T}_r(\mathcal{F})/\mathcal{J}_r$. Since $\Lambda^1\mathcal{F} = \mathcal{F}$ then the canonical projection $\mathcal{T}_r(\mathcal{F}) \longrightarrow \Lambda^r\mathcal{F}$ can be viewed as the component $\mathcal{F}\otimes\mathcal{F}\otimes\mathcal{F}\otimes\ldots\otimes\mathcal{F} \longrightarrow \Lambda^r\mathcal{F}$ of r-fold multiplication in $\Lambda\mathcal{F}$. The diagonal map $\mathcal{F} \longrightarrow \mathcal{F}\otimes\mathcal{F}$ which defined by $x \longrightarrow (x,x)$ induces an $\mathcal{R}$-algebra map $\Lambda\mathcal{F} \longrightarrow \Lambda(\mathcal{F}\oplus\mathcal{F}) \cong \Lambda\mathcal{F}\otimes\Lambda\mathcal{F}$ which is the co-multiplication Δ of Hopf algebra $\Lambda\mathcal{F}$ with the co-unit being the projection $\Lambda\mathcal{F} \longrightarrow \mathcal{R}$ into degree 0.

Example (1.1.9): [2] **(The symmetric algebra)**

The symmetric algebra of finitely generated free $\mathcal{R}$-module $\mathcal{F}$ is the free graded commutative $\mathcal{R}$-algebra generated by elements of $\mathcal{F}$ in degree 2 and it is denoted by $\mathcal{SF} = \sum_{r\geq 0}\mathcal{S}_r\mathcal{F}$; where we write $\mathcal{S}_r\mathcal{F}$ for the elements of degree $2r$. $\mathcal{SF}$ is constructed as the quotient $\mathcal{T}(\mathcal{F})/\mathcal{L}$; where $\mathcal{L}$ is the two sided homogeneous ideal of the tensor algebra $\mathcal{T}(\mathcal{F})$ generated by elements of the form $x_1\otimes x_2 - x_2\otimes x_1$; where $x_1, x_2 \in \mathcal{F}$. Since $\mathcal{S}_1\mathcal{F} = \mathcal{F}$, then the canonical projection $\mathcal{T}_r(\mathcal{F}) \longrightarrow \mathcal{S}_r\mathcal{F}$ is the component $\mathcal{F}\otimes\mathcal{F}\otimes\mathcal{F}\otimes\ldots\otimes\mathcal{F} \longrightarrow \mathcal{S}_r\mathcal{F}$ of r-fold multiplication in $\mathcal{SF}$. The diagonal map $\mathcal{F} \longrightarrow \mathcal{F}\oplus\mathcal{F}$ induces an R-algebra map $\mathcal{SF} \longrightarrow \mathcal{S}(\mathcal{F}\oplus\mathcal{F}) \cong \mathcal{SF}\otimes\mathcal{SF}$, which is the co-multiplication of the Hopf algebra SF with co-unit being the projection $\mathcal{SF} \longrightarrow \mathcal{R}$ into degree 0.

If $x \in \mathcal{F}$, $\Delta(x) = x\otimes 1 + 1\otimes x$ and since Δ is algebra map, then we have

$\Delta\left(x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_t^{\alpha_t}\right) = \sum_{0\leq\beta_i\leq\alpha_i}\binom{\alpha}{\beta}x_1^{\beta_1}x_2^{\beta_2}\ldots x_t^{\beta_t}\otimes x_1^{\alpha_1-\beta_1}x_2^{\alpha_2-\beta_2}\ldots x_t^{\alpha_t-\beta_t}$;where

$\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2}\ldots\binom{\alpha_n}{\beta_n}$ and $\binom{\alpha_i}{\beta_i} = \frac{\alpha_i!}{\beta_i!(\alpha_i-\beta_i)!}$

Example (1.1.10): [2] **(The divided power algebra)**

The divided power algebra $\mathcal{DF} = \sum_{i\geq 0} \mathcal{D}_i\mathcal{F}$ can be defined as the graded commutative algebra generated by element $x^{(i)}$ in degree $2i$; where $x \in \mathcal{F}$ and i is a non-negative integer, satisfying the following conditions:

(1) $\mathcal{D}_0\mathcal{F} = \mathcal{R}, \quad \mathcal{D}_1\mathcal{F} = \mathcal{F}$

(2) $x^{(0)} = 1, \quad x^{(1)} = x \qquad ; \forall\, x^{(i)} \in \mathcal{D}_i$ and $x \in \mathcal{F}$.

(3) $x^{(p)}x^{(q)} = \binom{p+q}{q} x^{(p+q)} \qquad ; \forall\, x \in \mathcal{F}$.

(4) $(x+y)^{(p)} = \sum_{k=0}^{p} x^{(p-k)}y^{(k)} \qquad ; \forall\, x, y \in \mathcal{F}$.

(5) $(xy)^{(p)} = x^{(p)}y^{(p)} \qquad ; \forall\, x, y \in \mathcal{F}$.

(6) $(x^{(p)})^{(q)} = \frac{(pq)!}{q!\,p^{q}!} x^{(pq)}$

As with symmetric algebra, we write $\mathcal{D}_i\mathcal{F}$ for the elements of degree $2i$. If $\xi_1, \xi_2, \dots, \xi_n$ is a basis for $\mathcal{F}$ then the set

$$\{\xi_1^{(\alpha_1)}, \xi_2^{(\alpha_2)}, \dots, \xi_n^{(\alpha_n)} \,\big|\, \alpha_1 + \alpha_2 + \dots + \alpha_n = p\},$$

is the basis for $\mathcal{D}_p\mathcal{F}$ and it is dual to the basis

$$\{x_1^{\alpha_1}, x_2^{\alpha_2}, \dots, x_n^{\alpha_n} \,|\, \alpha_1 + \alpha_2 + \dots + \alpha_n = p\},$$

of $\mathcal{S}_p(\mathcal{F}^*)$; where $x_1, x_2, \dots, x_n$ is the basis of $\mathcal{F}^*$ dual to $\xi_1, \xi_2, \dots, \xi_n$.

$\mathcal{DF}$ has a graded $\mathcal{R}$-Hopf algebra structure as the graded dual of $\mathcal{S}(\mathcal{F}^*)$, with $\Delta_{\mathcal{DF}}(\mathcal{F}) = x \otimes 1 + 1 \otimes x$ for all $x \in \mathcal{F}$. And with $m_{\mathcal{SF}^*}: \mathcal{SF}^* \otimes \mathcal{SF}^* \longrightarrow \mathcal{SF}^*$ is a map of co-algebras, $\Delta_{\mathcal{DF}}: \Delta_{\mathcal{DF}} \longrightarrow \Delta_{\mathcal{DF}}$ is a map of algebras.

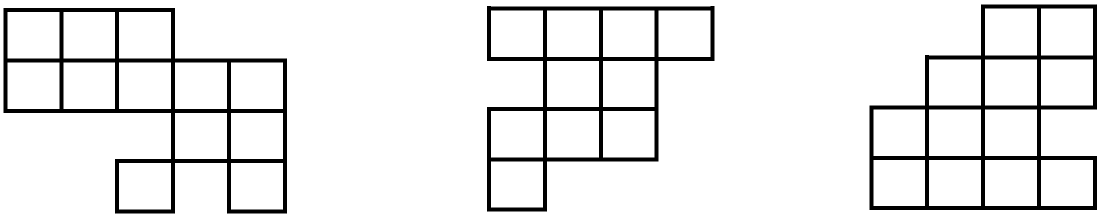
It follows that

$$\Delta_{\mathcal{DF}}\left(f_1^{(\alpha_1)} f_2^{(\alpha_2)} \dots f_t^{(\alpha_t)}\right) = \sum_{0\leq\beta_i\leq\alpha_i} f_1^{(\beta_1)} f_2^{(\beta_2)} \dots f_t^{(\beta_t)} \otimes f_1^{(\alpha_1-\beta_1)} f_2^{(\alpha_2-\beta_2)} \dots f_t^{(\alpha_t-\beta_t)}.$$

The component $\mathcal{D}_r\mathcal{F} \longrightarrow \mathcal{F} \otimes \dots \otimes \mathcal{F}$ of r-fold diagonalization is a split monomorphism (over $\mathcal{R}$) and its image is the module of symmetric r-tensors.

1.2 Schur functors

This section exhibit the definitions of the Schur and Weyl modules as in [2], [3] and [7]; where the authors give a structure that associates a GL_m-representation to any "generalized" shape like



Moreover, we will work over a commutative ring $\mathcal{R}$ with identity and letters $\mathcal{F}$, G etc. will mention to finitely generated free $\mathcal{R}$-modules.

As in section one, the notation $\Lambda^{k}\mathcal{F}, \mathcal{S}_{k}\mathcal{F}$ and $\mathcal{D}_{k}\mathcal{F}$ will mean the k^{th} exterior, symmetric and divided powers of $\mathcal{F}$.

If $a = (a_1, a_2, \dots, a_n)$ is a sequence of integers, then

$\Lambda_a\mathcal{F} = \Lambda^{a_1}\mathcal{F} \otimes \Lambda^{a_2}\mathcal{F} \otimes \dots \otimes \Lambda^{a_n}\mathcal{F}$

$\mathcal{S}_a\mathcal{F} = \mathcal{S}_{a_1}\mathcal{F} \otimes \mathcal{S}_{a_2}\mathcal{F} \otimes \dots \otimes \mathcal{S}_{a_n}\mathcal{F}$

$\mathcal{D}_a\mathcal{F} = \mathcal{D}_{a_1}\mathcal{F} \otimes \mathcal{D}_{a_2}\mathcal{F} \otimes \dots \otimes \mathcal{D}_{a_n}\mathcal{F}$

If $a = \sum a_i$, there is no confusion about what is meant by the diagonalizations

$\Lambda_a\mathcal{F} \longrightarrow \Lambda^{a_1}\mathcal{F} \otimes \Lambda^{a_2}\mathcal{F} \otimes \dots \otimes \Lambda^{a_n}\mathcal{F}$

$\mathcal{S}_a\mathcal{F} \longrightarrow \mathcal{S}_{a_1}\mathcal{F} \otimes \mathcal{S}_{a_2}\mathcal{F} \otimes \dots \otimes \mathcal{S}_{a_n}\mathcal{F}$

$\mathcal{D}_a\mathcal{F} \longrightarrow \mathcal{D}_{a_1}\mathcal{F} \otimes \mathcal{D}_{a_2}\mathcal{F} \otimes \dots \otimes \mathcal{D}_{a_n}\mathcal{F}$

We need all the following definitions which are appearing in [2].

Definitions (1.2.1):

- A **partition** of length $n = \ell(\lambda)$ is a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of non - negative integers in non-increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$.

- The **weight of a partition**, or more generally of any finite sequence λ of non-negative integers is the sum of all the terms of λ and is denoted by $|\lambda|$ i.e.

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

It is often convenient not to distinguish between $(\lambda_1, \lambda_2, \dots, \lambda_n)$ and $(\lambda_1, \lambda_2, \dots, \lambda_n, 0)$ for this purpose we let $\mathcal{N}^\infty$ denote the set of all infinite sequences of non negative integers containing only a finite number of non-zero terms. Given any finite sequence $(\lambda_1, \lambda_2, \dots, \lambda_n)$ we can think of it as a sequence $(\lambda_1, \lambda_2, \dots, \lambda_n, 0,0, \dots)$ in $\mathcal{N}^\infty$ by extension with zeroes.

- A **relative sequence** is a pair (λ, μ) of sequences in $\mathcal{N}^\infty$ such that $\mu \leq \lambda$ means that $\mu_i \leq \lambda_i$ for all $i \geq 1$. We shall use the notation λ/μ to represent relative sequences.

- If both λ and μ are partitions, then the relative sequence λ/μ will be called a **skew partition.** It is natural to think of a sequence λ in $\mathcal{N}^\infty$ as relative sequence $\lambda/(0)$ by talking the zero sequence (0) = (0,0,…) as the second part of the pair.

- Suppose that $\lambda/\mu = (\lambda_1, \lambda_2, \dots, \lambda_n)/(\mu_1, \mu_2, \dots, \mu_n)$ is a skew partition. The **diagram** $\Delta_{\lambda/\mu}$ of λ/μ is defined to be the set of all ordered pairs (i, j) of integers satisfying the inequalities $1 \leq i \leq n$ and $\mu_i < j \leq \lambda_i$ jointly.

- The **shape matrix** of λ/μ is $n \times t$ matrix $\alpha = (\alpha_{ij})$ defined by the rule

$$\alpha_{ij} = \begin{cases} 1 & \text{if} \quad \mu_i < j \leq \lambda_i\,, \\ 0 & \text{otherwise} \end{cases} \quad ;$$

where we take $t = \lambda_1$.

For any partition or sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ as an infinite sequence $(\lambda_1, \lambda_2, \dots, \lambda_n, 0,0, \dots)$ with finite support, it may be convenient to think of $n \times t$ shape matrix $\alpha = (\alpha_{ij})$ as an infinite matrix

$$\alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1t} & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nt} & 0 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

of zeros and ones with finite support.

- If α is the shape matrix of a relative sequence λ/μ then the **support** of α is exactly the diagram of λ/μ .

- The **weight of a shape matrix** $\alpha = (\alpha_{ij})$ is defined to be the sum of all the entries $(\alpha_{ij})$of α and is denoted by $|\alpha|$.

  If $\alpha = \lambda/\mu$ is the shape matrix associated with a relative sequence, then clearly $|\alpha| = |\lambda| - |\mu|$.

- If $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathcal{N}^\infty$ is a partition, then its **conjugate** (or transpose) is defined to be the partition $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \dots)$; where $\tilde{\lambda}_j$ is the number of terms of λ which are greater than or equal to j.

  Similarly, if $\alpha = \left(\alpha_{ij}\right)$ is a shape matrix, $\tilde{\alpha} = (\tilde{\alpha}_{ij})$ is defined to be the transpose of α by taking $(\alpha_{ij}) = (\tilde{\alpha}_{ji})$.

Notice that if α is the shape matrix of a relative sequence λ/μ, then $\alpha_i = \lambda_i - \mu_i$ for all i.

- If λ/μ is a skew partition, then $\tilde{\alpha}_j = \tilde{\lambda}_j - \tilde{\mu}_j$ for all j. To a finite shape matrix $\alpha = (\alpha_{ij})$, with $i = 1,2,\dots,n$, $j = 1,2,\dots,t$, there is associate between the sequence $a = (a_1, a_2, \dots, a_n)$ of row sums of α; where $a_i = \sum_{j=1}^{t} \alpha_{ij}$ and the sequence $b = (b_1, b_2, \dots, b_t)$ of column sums of α; where $b_j = \sum_{i=1}^{n} \alpha_{ij}$.

To each shape matrix α and to each free module $\mathcal{F}$, there are associated two maps

$d_\alpha(\mathcal{F}): \Lambda_a \mathcal{F} \to \mathcal{S}_b \mathcal{F}$ (Schur map)

$d'_\alpha(\mathcal{F}): \mathcal{D}_a \mathcal{F} \to \Lambda_b \mathcal{F}$ (Weyl map),

whose images will be called respectively **Schur modules** and **Weyl modules** denoted by $\mathcal{L}_\alpha(\mathcal{F})$ and $\mathcal{K}_\alpha(\mathcal{F})$ respectively.

$d_\alpha(\mathcal{F})$ and $d'_\alpha(\mathcal{F})$ are defined as follows:

Consider first the map $u = \Delta \otimes \Delta \otimes \dots \otimes \Delta$

$$\Lambda_a \mathcal{F} \xrightarrow{u} (\Lambda^{a_{11}}\mathcal{F} \otimes \Lambda^{a_{12}}\mathcal{F} \otimes \dots \otimes \Lambda^{a_{1t}}\mathcal{F}) \otimes \dots \otimes (\Lambda^{a_{n1}}\mathcal{F} \otimes \Lambda^{a_{n2}}\mathcal{F} \otimes \dots \otimes \Lambda^{a_{nt}}\mathcal{F}); \qquad \dots(*)$$

where each $\Lambda^{a_i}\mathcal{F}$ maps by appropriate diagonalization Δ into $\Lambda^{a_{i1}}\mathcal{F} \otimes \Lambda^{a_{i2}}\mathcal{F} \otimes \dots \otimes \Lambda^{a_{it}}\mathcal{F}$.

By rearranging terms of $(*)$, we have an isomorphism

$$(\Lambda^{a_{11}}\mathcal{F} \otimes \Lambda^{a_{12}}\mathcal{F} \otimes \dots \otimes \Lambda^{a_{1t}}\mathcal{F}) \otimes \dots \otimes (\Lambda^{a_{n1}}\mathcal{F} \otimes \Lambda^{a_{n2}}\mathcal{F} \otimes \dots \otimes \Lambda^{a_{nt}}\mathcal{F})$$

$$\xrightarrow{\theta} (\Lambda^{a_{11}}\mathcal{F} \otimes \Lambda^{a_{21}}\mathcal{F} \otimes \dots \otimes \Lambda^{a_{n1}}\mathcal{F}) \otimes \dots \otimes (\Lambda^{a_{1t}}\mathcal{F} \otimes \Lambda^{a_{2t}}\mathcal{F} \otimes \dots \otimes \Lambda^{a_{nt}}\mathcal{F})$$

$$=$$

$$(\mathcal{S}_{a_{11}}\mathcal{F} \otimes \mathcal{S}_{a_{21}}\mathcal{F} \otimes \mathcal{S}_{a_{n1}}\mathcal{F}) \otimes \dots \otimes (\mathcal{S}_{a_{1t}}\mathcal{F} \otimes \mathcal{S}_{a_{2t}}\mathcal{F} \otimes \dots \otimes \mathcal{S}_{a_{nt}}\mathcal{F})$$

Finally, by multiplication in the symmetric algebra $\mathcal{S}\mathcal{F}$, for each factor above, we have the map

$$\mathcal{S}_{a_{1j}}\mathcal{F} \otimes \mathcal{S}_{a_{2j}}\mathcal{F} \otimes \ldots \otimes \mathcal{S}_{a_{nj}}\mathcal{F} \xrightarrow{m} \mathcal{S}_{b_j}\mathcal{F}$$

so that one obtains the composite map

$$\Lambda_a F$$

$$\xrightarrow{u} (\Lambda^{a_{11}}\mathcal{F} \otimes \Lambda^{a_{12}}\mathcal{F} \otimes \ldots \otimes \Lambda^{a_{1t}}\mathcal{F}) \otimes \ldots \otimes (\Lambda^{a_{n1}}\mathcal{F} \otimes \Lambda^{a_{n2}}\mathcal{F} \otimes \ldots \otimes \Lambda^{a_{nt}}\mathcal{F})$$

$$\xrightarrow{\theta} (\mathcal{S}_{a_{11}}\mathcal{F} \otimes \mathcal{S}_{a_{21}}\mathcal{F} \otimes \mathcal{S}_{a_{n1}}\mathcal{F}) \otimes \ldots \otimes (\mathcal{S}_{a_{1t}}\mathcal{F} \otimes \mathcal{S}_{a_{2t}}\mathcal{F} \otimes \ldots \otimes \mathcal{S}_{a_{nt}}\mathcal{F})$$

$$\xrightarrow{v} (\mathcal{S}_{b1}\mathcal{F} \otimes \mathcal{S}_{b2}\mathcal{F} \otimes \mathcal{S}_{bt}\mathcal{F}) = \mathcal{S}_b\mathcal{F} \ ;$$

where $v = m_1 \otimes m_2 \otimes \ldots \otimes m_t$.

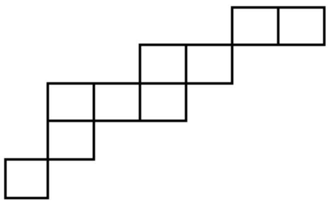
The following example clarifies the above definitions.

Example (1.2.2):

Let λ/μ =(7,5,4,2,1)/(5,3,1,1,0), then we have:

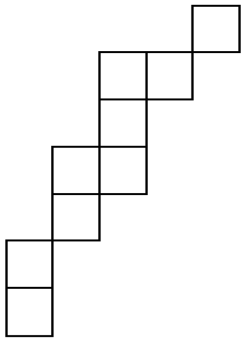
1. The shape matrix of $\lambda/\mu = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

2. The diagram of λ/μ is



3. The shape matrix of $\tilde{\lambda}/\tilde{\mu} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$

4. The diagram of $\tilde{\lambda}/\tilde{\mu}$ is



5. The sequence of row sums of λ/μ is (2,2,3,1,1) and the sequence of column sums of λ/μ is (1,2,1,2,1,1,1).

6. The sequence of row sums of $\tilde{\lambda}/\tilde{\mu}$ is (1,2,1,2,1,1,1) and the sequence of column sums of $\tilde{\lambda}/\tilde{\mu}$ is (2,2,3,1,1).

Definition (1.2.3): [2]

The Schur map $d_\alpha(\mathcal{F})$ associated to the shape matrix α and the free module F is the following composite map:

$$d_\alpha(\mathcal{F}) = v \circ \theta \circ u$$

Similar diagonalization, rearrangement, identification and multiplication maps, give the definition of the Weyl map $d'_\alpha(\mathcal{F})$ as the following composition map

$$\mathcal{D}_{\boldsymbol{\alpha}}\mathcal{F} \xrightarrow{u'} \left(\mathcal{D}_{a_{11}}\mathcal{F}\otimes\mathcal{D}_{a_{12}}\mathcal{F}\otimes \dots \otimes\mathcal{D}_{a_{1t}}\mathcal{F}\right)\otimes \dots \otimes(\mathcal{D}_{a_{n1}}\mathcal{F}\otimes D_{a_{n2}}\mathcal{F}\otimes \dots \otimes\mathcal{D}_{a_{nt}}\mathcal{F})$$

$$\xrightarrow{\theta'} (\Lambda^{a_{11}}\mathcal{F}\otimes\Lambda^{a_{21}}\mathcal{F}\otimes \dots \otimes\Lambda^{a_{n1}}\mathcal{F})\otimes \dots \otimes(\Lambda^{a_{1t}}\mathcal{F}\otimes\Lambda^{a_{2t}}\mathcal{F}\otimes \dots \otimes\Lambda^{a_{n\,t}}\mathcal{F})$$

$$\xrightarrow{v'} (\Lambda^{a_{b1}}\mathcal{F}\otimes\Lambda^{a_{b2}}\mathcal{F}\otimes \dots \otimes\Lambda^{a_{bt}}\mathcal{F}) = \Lambda^{b}\mathcal{F}.$$

Such that $d'_\alpha(\mathcal{F}) = v' \circ \theta' \circ u'$

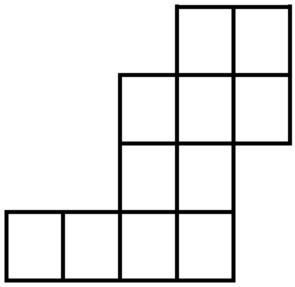
In our work, we will be dealing only with two types of shape matrices which are partition and skew-partition.

Example (1.2.4):

Let $\lambda = (5,5,4,4)$, μ =(3,2,2,0) then we have

1. The shape matrix of $\lambda/\mu = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$

2. The diagram of λ/μ is



3. $d_\lambda(\mathcal{F})\colon \Lambda^2\mathcal{F}\otimes\Lambda^3\mathcal{F}\otimes\Lambda^2\mathcal{F}\otimes\Lambda^4\mathcal{F} \longrightarrow \mathcal{S}_1\mathcal{F}\otimes\mathcal{S}_1\mathcal{F}\otimes\mathcal{S}_3\mathcal{F}\otimes\mathcal{S}_4\mathcal{F}\otimes\mathcal{S}_2\mathcal{F}$

4. $d'_\lambda(\mathcal{F})\colon \mathcal{D}_2\mathcal{F}\otimes\mathcal{D}_3\mathcal{F}\otimes\mathcal{D}_2\mathcal{F}\otimes\mathcal{D}_4\mathcal{F} \longrightarrow \Lambda^1\mathcal{F}\otimes\Lambda^1\mathcal{F}\otimes\Lambda^3\mathcal{F}\otimes\Lambda^4\mathcal{F}\otimes\Lambda^2\mathcal{F}$

Definition (1.2.5): [2]

Let λ/μ be a skew-partition and let $p_i = \lambda_i - \mu_i$, for $i = 1,2,\dots,n$, $n = \ell(\lambda)$ and let $t_i = \mu_i - \mu_{i+1} + 1$ for $i = 1,2,\dots,n-1$; for each $i \le n-1$ and $\ell \ge 0$, we have the following map:

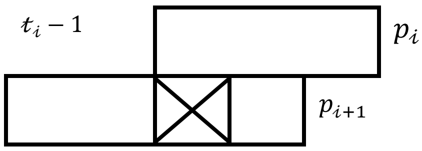
$$\Lambda^{p_1}\mathcal{F} \otimes \Lambda^{p_2}\mathcal{F} \otimes \dots \otimes \Lambda^{p_i+t_i+\ell}\mathcal{F} \otimes \Lambda^{p_{i+1}-t_i-\ell}\mathcal{F} \otimes \dots \otimes \Lambda^{p_n}\mathcal{F}$$

$$\longrightarrow \Lambda^{p_1}\mathcal{F} \otimes \Lambda^{p_2}\mathcal{F} \otimes \dots \otimes \Lambda^{p_n}\mathcal{F}$$

Defined by diagonalizing $\Lambda^{p_i+t_i+\ell}\mathcal{F}$ into $\Lambda^{p_i}\mathcal{F}\otimes\Lambda^{t_i+\ell}\mathcal{F}$ and then multiplying $\Lambda^{t_i+\ell}\mathcal{F}\otimes\Lambda^{p_{i+1}-t_i-\ell}\mathcal{F}$ into $\Lambda^{p_{i+1}}\mathcal{F}$.

We denoted this map by $\square_i^\ell$ and let

$$\square_{\lambda_i/\mu_i} = \sum_{\ell=0}^{p_{i+1}-t_i} \square_i^\ell$$



The map

$\square_{\lambda/\mu}: \sum_{i,\ell} \Lambda^{p_1} \mathcal{F} \otimes \Lambda^{p_2} \mathcal{F} \otimes \ldots \otimes \Lambda^{p_i+t_i+\ell} \mathcal{F} \otimes \Lambda^{p_{i+1}-t_i-\ell} \mathcal{F} \otimes \ldots \otimes \Lambda^{p_n} \mathcal{F}$

$\longrightarrow \Lambda^{p_1} \mathcal{F} \otimes \Lambda^{p_2} \mathcal{F} \otimes \ldots \otimes \Lambda^{p_n} \mathcal{F}$

is defined by

$$\square_{\lambda/\mu} = \sum_{i=1}^{n-1} \square_{\lambda_i/\mu_i} \qquad \ldots(**)$$

The authors in [2] shown that $d_{\lambda/\mu}(\mathcal{F}) \circ \square_{\lambda/\mu} = 0$.

In particular, it follows that there exists a natural map

$$\theta_{\lambda/\mu}: \bar{\mathcal{L}}_{\lambda/\mu}(\mathcal{F}) = \operatorname{co}\ker \square_{\lambda/\mu} \longrightarrow \mathcal{L}_{\lambda/\mu}(\mathcal{F})$$

The exact same structure of maps can be made if we replace all exterior powers by divided powers. In particular, there exists a natural map

$$\theta'_{\lambda/\mu}: \overline{\mathcal{K}}_{\lambda/\mu}(\mathcal{F}) = \operatorname{co}\ker \square'_{\lambda/\mu} \longrightarrow \mathcal{K}_{\lambda/\mu}(\mathcal{F});$$

where

$\square'_{\lambda/\mu} = \sum_{i=1}^{n-1} \square'_{\lambda_i/\mu_i} \quad , \quad \square'_{\lambda_i/\mu_i} = \sum_{\ell=0}^{p_{i+1}-t_i-1} \square'^{\ell}_{i}$

$\square'^{\ell}_{i} : \mathcal{D}_{p_1} \mathcal{F} \otimes \mathcal{D}_{p_2} \mathcal{F} \otimes \ldots \otimes \mathcal{D}_{p_i+t_i+\ell} \mathcal{F} \otimes \mathcal{D}_{p_{i+1}-t_i-\ell} \mathcal{F} \otimes \ldots \otimes \mathcal{D}_{p_n} \mathcal{F}$

$\longrightarrow \mathcal{D}_{p_1} \mathcal{F} \otimes \mathcal{D}_{p_2} \mathcal{F} \otimes \ldots \otimes \mathcal{D}_{p_n} \mathcal{F}$

Theorem (1.2.6): [2]

For any skew-partition λ/μ, the module $\mathcal{L}_{\lambda/\mu}(\mathcal{F})$ ($\mathcal{K}_{\lambda/\mu}(\mathcal{F})$) is free and the morphism $\theta_{\lambda/\mu}$ ($\theta'_{\lambda/\mu}$) is an isomorphism. In particular, it follows that $\mathcal{L}_{\lambda/\mu}(\mathcal{F})$ ($\mathcal{K}_{\lambda/\mu}(\mathcal{F})$) is universally free module.

To describe a basis for $\mathcal{L}_{\lambda/\mu}(\mathcal{F})(\mathcal{K}_{\lambda/\mu}(\mathcal{F}))$ in terms of an explicit basis for $\mathcal{F}$ one needs the notation of tableaux.

First notice that if $\mathcal{S} = \{f_1, f_2, \dots, f_m\}$ is a basis for the module $\mathcal{F}$ and $\mathrm{I} = \{1 < i_1 < i_2 < \cdots < i_s < m\}$ is a strictly increasing subset of $\{1,2,\dots,m\}$ then $f_{\mathrm{I}} = f_{i_1, i_2, \dots, i_s} = f_{i_1} \wedge f_{i_2} \wedge \dots \wedge f_{i_s} \in \Lambda^s \mathcal{F}$.

In particular the elements
$f_{\mathrm{I}_1} \otimes f_{\mathrm{I}_2} \otimes \dots \otimes f_{\mathrm{I}_n} \in \Lambda^{s_1}\mathcal{F} \otimes \Lambda^{s_2}\mathcal{F} \otimes \dots \otimes \Lambda^{s_n}\mathcal{F}$,
form a basis of $\Lambda_{\lambda/\mu}(\mathcal{F})$; where I_i is a strictly increasing subset of $\{1,2,\dots,m\}$ having $\mathcal{S}_i$ elements.

From the above theorem we have the following remarks:

Remark (1.2.7): [2]

The elements $d_{\lambda/\mu}(f_{\mathrm{I}_1} \otimes f_{\mathrm{I}_2} \otimes \dots \otimes f_{\mathrm{I}_m}) \in \mathcal{L}_{\lambda/\mu}(\mathcal{F})$, are a set of generators for $\mathcal{L}_{\lambda/\mu}(\mathcal{F})$.

Now if $\mathcal{J}$ is any non-decreasing sequence $1 \le j_1 \le j_2 \le \cdots \le j_s \le m$ of integers, grouping these integers into distinct clumps:
$1 \le j_1 = j_2 = \cdots = j_{t_1} < j_{t_1+1} = \cdots = j_{t_2} < j_{t_2+1} = \cdots = j_{t_l} < j_{t_l+1} = \cdots = j_s \le m$
One obtains a basis element of $\mathcal{D}_s\mathcal{F}$ by setting
$f_{\mathcal{J}} = j_{j_1}^{(t_1)} j_{j_2}^{(t_2-t_1)} \dots j_{j_s}^{(t_s-t_1)} \in \mathcal{D}_s\mathcal{F}$

In particular, the elements $f_{j_1} \otimes f_{j_2} \otimes \dots \otimes f_{j_n} \in \mathcal{D}_{s_1}\mathcal{F} \otimes \mathcal{D}_{s_2}\mathcal{F} \otimes \dots \otimes \mathcal{D}_{s_n}\mathcal{F}$ forms a basis of $\mathcal{D}_{\lambda/\mu}\mathcal{F}$; where $\mathcal{J}_k$ is any non-decreasing subset of $\{1,2,\dots,m\}$ having s_k elements.

Remark (1.2.8): [2]

The elements $d'_{\lambda/\mu}(f_{\mathcal{J}_1} \otimes f_{\mathcal{J}_2} \otimes \dots \otimes f_{\mathcal{J}_n}) \in \mathcal{K}_{\lambda/\mu}\mathcal{F}$, are a set of generators for $\mathcal{K}_{\lambda/\mu}\mathcal{F}$.

Definition (1.2.9): [2]

Let $\mathcal{S} = \{g_1, g_2, \dots, g_m\}$ be a totally ordered basis for the free module $\mathcal{F}$ and let λ/μ be a skew-partition with diagram $\Delta_{\lambda/\mu}$. A **tableau** of shape λ/μ with values in $\mathcal{S}$ is a function $\mathcal{T}$ from $\Delta_{\lambda/\mu}$ to $\mathcal{S}$. The set of all such tableaux is denoted by $\mathrm{Tab}_{\lambda/\mu}(\mathcal{S})$.

Notice that a tableau $\mathcal{T} \in \mathrm{Tab}_{\lambda/\mu}(\mathcal{S})$ can be thought of as the diagram $\Delta_{\lambda/\mu}$ filled in with basic elements, conversely, any $\mathcal{T} \in \mathrm{Tab}_{\lambda/\mu}(\mathcal{S})$ gives an element in $\Lambda_{\lambda/\mu}(\mathcal{F})(\mathcal{D}_{\lambda/\mu}\mathcal{F})$, which is not necessarily a basis element of $\Lambda_{\lambda/\mu}(\mathcal{F})(\mathcal{D}_{\lambda/\mu}\mathcal{F})$.

This lead to define a tableau $\mathcal{T} \in \mathrm{Tab}_{\lambda/\mu}(\mathcal{S})$ to be **row-standard (co-row-standard)** if in each row of the diagram, the basis entries are strictly increasing (non-decreasing) from left to right. $\mathcal{T}$ is said to be **column-standard (co-column-standard)** if in each column of the diagram, the basis entries are non-decreasing (strictly increasing) from top to bottom. $\mathcal{T}$ is said to be **standard (co-standard)** if it is both row and column-standard (co-row and co-column-standard).

The following example illustrates the above definition.

Example (1.2.10):

If $h = h_2 \wedge h_4 \wedge h_6 \wedge h_7 \otimes h_1 \wedge h_3 \wedge h_5 \otimes h_2 \wedge h_3 \in \Lambda_{(7,5,3)/(3,2,1)}\mathcal{F}$, and $\tilde{h} = h_1 \cdot h_3^{(2)} \cdot h_7 \otimes h_2^{(2)} \cdot h_4 \otimes h_1 \cdot h_3 \in \mathcal{D}_{(7,5,3)/(3,2,1)}\mathcal{F}$, then

$\mathcal{T}_{h}$:

		h_2	h_4	h_6	h_7
	h_1	h_3	h_5		
h_2	h_3				

$\mathcal{T}_{\tilde{h}}$:

		h_1	h_3	h_3	h_7
	h_2	h_2	h_4		
h_1	h_3				

Thus $\mathcal{T}_{h}$ and $\mathcal{T}_{\tilde{h}}$ are standard.

The following theorem depicts a basis for $\mathcal{L}_{\lambda/\mu}(\mathcal{F})\mathcal{K}_{\lambda/\mu}(\mathcal{F})$ in terms of tableaux.

Theorem (1.2.11): [2]

If $\mathcal{S} = \{f_1, \dots, f_m\}$ is a basis for $\mathcal{F}$, then $\{d_{\lambda/\mu}(\mathcal{F})(\mathcal{T})/\mathcal{T}$ is a standard tableau in $\mathcal{S}$ of shape $\lambda/\mu\}$ is a basis for $\mathcal{L}_{\lambda/\mu}(\mathcal{F})$. For Weyl modules, we have $\{d'_{\lambda/\mu}(\mathcal{F})(\mathcal{T})/(\mathcal{T})$ is a co-standard tableau in $\mathcal{S}$ of shape $\lambda/\mu\}$ is a basis of $\mathcal{K}_{\lambda/\mu}\mathcal{F}$.

1.3 Letter place algebra

This section is a survey of the notion of the principal tools we need to translate into letter-place language, the description of the Weyl (Schur) maps $d'_{\alpha}(d_{\alpha})$ and of the "box maps" $\square'_{\alpha}(\square_{\alpha})$ pointed at in the survey section 1.2, [2]. For a complete treatment of the letter-place algebra, we will refer to [20]; where multi-signed, alphabets and places are treated in a uniform and general set-up. In our context, we will describe the basic elements of a given tensor product

$\mathcal{D}_{\beta_1}\mathcal{F} \otimes \mathcal{D}_{\beta_2}\mathcal{F} \otimes \dots \otimes \mathcal{D}_{\beta_n}\mathcal{F}$ using the **positive letters alphabet** $L = \{\ell_1, \ell_2, \dots, \ell_m\} = \mathcal{S}$ (recall that $\mathcal{S} = \{f_1, f_2, \dots, f_m\}$ is a totally ordered basis for the module $\mathcal{F}$). Also, in order to keep track of the position i in the above tensor product, the totally ordered set $\mathcal{P}^+ = \{1,2, \dots, i, \dots, n\}$ of places is considered as a positive place alphabet.

For example:

An element $w = w_1 \otimes w_2 \otimes \dots \otimes w_n \in \mathcal{D}_{\beta_1}\mathcal{F} \otimes \mathcal{D}_{\beta_2}\mathcal{F} \otimes \dots \otimes \mathcal{D}_{\beta_n}\mathcal{F}$ would be written in letter-place algebra as

$\left(w_1 \middle| 1^{(\beta_1)}\right)\left(w_2 \middle| 2^{(\beta_2)}\right) \cdots \left(w_n \middle| n^{(\beta_n)}\right) \in \mathcal{D}_{\beta_1}\mathcal{F} \otimes \mathcal{D}_{\beta_2}\mathcal{F} \otimes \cdots \otimes \mathcal{D}_{\beta_n}\mathcal{F}$,

to indicate that w is the tensor product of a basis element w_1 in degree β_1 in the first factor, w_2 in degree β_2 in the second factor and so on w_n of degree β_n in the last factor, [12] .

Adopting the double tableau notation as in [14], we will also write

$$w = \left(\begin{matrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{matrix} \middle| \begin{matrix} 1^{(\beta_1)} \\ 2^{(\beta_2)} \\ \vdots \\ n^{(\beta_n)} \end{matrix} \right) \in \mathcal{D}_{\beta_1}\mathcal{F} \otimes \mathcal{D}_{\beta_2}\mathcal{F} \otimes \dots \otimes \mathcal{D}_{\beta_n}\mathcal{F}$$

Moreover, the following symbols will be often used

$$w' = \left(v \middle| 1^{(r)} 2^{(s)}\right) = \sum_{(v)} v_{(1)} \otimes v_{(2)} \in \mathcal{D}_r\mathcal{F} \otimes \mathcal{D}_s\mathcal{F}, \qquad \dots(1.3.1)$$

where $v \in \mathcal{D}_{r+s}\mathcal{F}$ and $\Delta_{(r+s)}: \mathcal{D}_{r+s}\mathcal{F} \to \mathcal{D}_r\mathcal{F} \otimes \mathcal{D}_s\mathcal{F}$ is the appropriate degree diagonalization map (Sweedler notation for the co-product applied to v), and

$$w'' = \left(\begin{matrix} v \\ v' \end{matrix} \middle| \begin{matrix} 1^{(s)} 2^{(k)} \\ 2^{(q-k)} \end{matrix} \right) = \left(v \middle| 1^{(s)} 2^{(k)}\right)\left(v' \middle| 2^{(q-k)}\right). \qquad \dots(1.3.2)$$

At this point in order to clarify the letter-place conventions and calculations, we first give a brief summary of letter-place set up, [17].

Given two free Z-modules $\mathcal{L}$ and $\mathcal{P}^{+}$ one can construct a bilinear pairing (or Laplace pairing) (|) of the divided power algebras $\mathcal{D}(\mathcal{L})$ and $\mathcal{D}(\mathcal{P}^{+})$ into $\mathcal{D}(\mathcal{L}\otimes\mathcal{P}^{+})$.

We follow the definitions given in [22] which are properly applied to a direct sum of free modules $\mathcal{P} = \mathcal{P}^{+}\oplus\mathcal{P}^{-}$ (positively and negatively signed places).
In particular, we have specialized above to the case $\mathcal{P}^{-} = 0$, so we let $\mathcal{P} = \mathcal{P}^{+}$.

Notice that in general, in this theory, we have also positive and negative letters, i.e $\mathcal{L} = \mathcal{L}^{+} \oplus \mathcal{L}^{-}$. In our case (of divided powers), we have $\mathcal{L}^{-} = 0$ and $\mathcal{L} = \mathcal{L}^{+}$; in terms of bases, for $\mathcal{L}^{-} = \mathcal{P}^{-} = 0$, (|) generalizes the permanent.

We identify the basis $\{\ell\otimes p \mid \ell \in\mathcal{L}, p \in \mathcal{P}\}$ of $\mathcal{L} \oplus \mathcal{P}$ with the set $\{(\ell \mid p) \mid \ell \in \mathcal{L}, p \in \mathcal{P}\}$ of "letter-places". The algebra $\mathcal{D}(\mathcal{L} \oplus \mathcal{P})$ can now be identified with the commutative associative algebra $\mathcal{D}([\mathcal{L} \mid \mathcal{P}])$ generated by all $(\ell|p)$ and satisfying the relations:

$$b^{0} = 1,\ b^{(i)}b^{(j)} = \binom{i+j}{j} b^{(i+j)}, \text{ for all } b = (\ell|p). \qquad \ldots(1.3.3)$$

For $\ell_1, \ell_2, \ldots, \ell_k \in \mathcal{L}$ and $p_1, p_2, \ldots, p_n \in \mathcal{P}$, we have:

$$(\ell_1, \ell_2, \ldots, \ell_k | p_1, p_2, \ldots, p_n) = \begin{cases} \sum_{\sigma\in S_k} \left(\ell_{\sigma(1)} \middle| p_1\right)\left(\ell_{\sigma(2)} \middle| p_2\right)\cdots\left(\ell_{\sigma(k)} \middle| p_k\right) & ; \text{if } n = k \\ 0 & ; \text{otherwise} \end{cases}$$
$$\ldots(1.3.4)$$

In our case, as above, we will generally by using a positive letter alphabet $\mathcal{L} = \mathcal{S} = \{f_1, f_2, \ldots, f_m\}$ i.e. for us $\mathcal{L} = \mathcal{F}$ and a positive place alphabet $\mathcal{P} = \{1, 2, \ldots, n\}$ which corresponds to a fixed choice of a basis of the positive places module $\mathcal{P}$.

We recall the following expansion properties of the bi-product $(\,|\,)$

$\left(\ell^{(k)}\middle|p^{(k)}\right) = (\ell|p)^{(k)}$, for $\ell \in \mathcal{L}, p \in \mathcal{P}^+$

$(w|u\,u') = \sum_{(w)}\left(w_{(1)}\middle|u\right)\left(w_{(2)}\middle|u'\right)$

$(ww'|u) = \sum_{(u)}\left(w\middle|u_{(1)}\right)\left(w'\middle|u_{(2)}\right)$;

where

$w = \ell^{(\alpha)} = \ell_1^{(\alpha_1)}\ell_2^{(\alpha_2)} \dots \ell_m^{(\alpha_m)}$, $w' = \ell^{(\alpha')} = \ell_1^{(\alpha'_1)}\ell_2^{(\alpha'_2)} \dots \ell_m^{(\alpha'_m)}$

$u = p^{(\beta)} = 1^{(\beta_1)}2^{(\beta_2)} \dots n^{(\beta_n)}$, $u' = p^{(\beta')} = 1^{(\beta'_1)}2^{(\beta'_2)} \dots n^{(\beta'_n)}$,

$\sum_{(w)} w_{(1)} \otimes w_{(2)}$ and $\sum_{(u)} u_{(1)} \otimes u_{(2)}$,

are the Sweedler notations for the co-product Δ in the appropriate degrees applied to w and u respectively.

Notice finally that in general we have the following rule:

$$(w|u) = \sum_{(\ell^{(\alpha)})}\left(w_{(1)}\middle|1^{(\beta_1)}\right)\left(w_{(2)}\middle|2^{(\beta_2)}\right) \dots \left(w_{(n)}\middle|n^{(\beta_n)}\right)$$
$$= \sum_{(p^{(\beta)})}\left(\ell_1^{(\alpha_1)}\middle|u_{(1)}\right)\left(\ell_2^{(\alpha_2)}\middle|u_{(2)}\right) \dots \left(\ell_m^{(\alpha_m)}\middle|u_{(m)}\right)$$

Notice that since $\mathcal{L} = \mathcal{L}^+$ and $\mathcal{P} = \mathcal{P}^+$ are totally ordered sets, we can talk not only about "double tableaux" as in (1.3.1), (1.3.2) but also about **double standard tableaux**. In particularly given basis words $w_1, w_2, \dots, w_s$ in $\mathcal{D}([\mathcal{L}])$ and $u_1, u_2, \dots, u_s$ in $\mathcal{D}([\mathcal{P}])$ we have the tableaux:

$$(\mathcal{T}|\mathcal{T}') = \left(\begin{array}{c|c} w_1 & u_1 \\ w_2 & u_2 \\ \vdots & \vdots \\ w_s & u_s \end{array}\right) = (\omega_1|u_1)(\omega_2|u_2) \dots (\omega_s|u_s). \qquad \dots(1.3.5)$$

Recall that any basis word $w_i \in \mathcal{D}_{\lambda_i}([\mathcal{L}])$; for $i = 1,2, \dots, s$ is uniquely defined by a non-decreasing subsequence $\mathcal{J}_i: 1 \le j_{i_1} \le j_{i_2} \le \dots \le j_{i_{\lambda_i}} \le m$.

Similar statement holds for

$u_i = 1^{(b_{i1})}2^{b_{i2}} \dots n^{(b_{i\,n})} \in \mathcal{D}_{\lambda_i}([\mathcal{P}]), \lambda_i = \sum_{j=1}^{n} b_{i\,j}$.

In particular (1.3.5) also write as following:

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_s \end{matrix}\left|\begin{matrix} 1^{(b_{11})}2^{(b_{12})} \dots n^{(b_{1n})} \\ 1^{(b_{21})}2^{(b_{22})} \dots n^{(b_{2n})} \\ \vdots \\ 1^{(b_{s1})}2^{(b_{s2})} \dots n^{(b_{sn})} \end{pmatrix} \in \mathcal{D}_{\beta_1}\mathcal{F} \otimes \mathcal{D}_{\beta_2}\mathcal{F} \otimes \dots \otimes \mathcal{D}_{\beta_n}\mathcal{F}\ , \qquad \dots(1.3.6)$$

where $\beta_j = \sum_{i=1}^{s} b_{i\,j}$.

Example (1.3.1):

Let $w = \ell_1\ell_2\ell_3\ell_4^{(3)}\ell_5$, $u = 1$ and $u' = 2^{(6)}$, from (1.3.3) and (1.3.4) we have

$$(w|u\,u') = (\ell_1|1)\left(\ell_2\ell_3\ell_4^{(3)}\ell_5\middle|2^{(6)}\right) + (\ell_2|1)\left(\ell_1\ell_3\ell_4^{(3)}\ell_5\middle|2^{(6)}\right) + (\ell_3|1)\left(\ell_1\ell_2\ell_4^{(3)}\ell_5\middle|2^{(6)}\right) + (\ell_4|1)\left(\ell_1\ell_2\ell_3\ell_4^{(2)}\ell_5\middle|2^{(6)}\right) + (\ell_5|1)\left(\ell_1\ell_2\ell_3\ell_4^{(3)}\middle|2^{(6)}\right)$$

$$= (\ell_1|1)\left[(\ell_2|2)\left(\ell_3\ell_4^{(3)}\ell_5\middle|2^{(5)}\right)\right] + (\ell_2|1)\left[(\ell_1|2)\left(\ell_3\ell_4^{(3)}\ell_5\middle|2^{(5)}\right)\right] + (\ell_3|1)\left[(\ell_1|2)\left(\ell_2\ell_4^{(3)}\ell_5\middle|2^{(5)}\right)\right] + (\ell_4|1)\left[(\ell_1|2)\left(\ell_2\ell_3\ell_4^{(2)}\ell_5\middle|2^{(5)}\right)\right] + (\ell_5|1)\left[(\ell_1|2)\left(\ell_2\ell_3\ell_4^{(3)}\middle|2^{(5)}\right)\right]$$

$$= (\ell_1|1)(\ell_2|2)\left[(\ell_3|2)\left(\ell_4^{(3)}\ell_5\middle|2^{(4)}\right)\right] + (\ell_2|1)(\ell_1|2)\left[(\ell_3|2)\left(\ell_4^{(3)}\ell_5\middle|2^{(4)}\right)\right] + (\ell_3|1)(\ell_1|2)\left[(\ell_2|2)\left(\ell_4^{(3)}\ell_5\middle|2^{(4)}\right)\right] + (\ell_4|1)(\ell_1|2)\left[(\ell_2|2)\left(\ell_3\ell_4^{(2)}\ell_5\middle|2^{(4)}\right)\right] + (\ell_5|1)(\ell_1|2)\left[(\ell_2|2)\left(\ell_3\ell_4^{(3)}\middle|2^{(4)}\right)\right]$$

$$= (\ell_1|1)(\ell_2|2)(\ell_3|2)\left[\left(\ell_4^{(3)}\middle|2^{(3)}\right)(\ell_5|2)\right] + (\ell_2|1)(\ell_1|2)(\ell_3|2)\left[\left(\ell_4^{(3)}\middle|2^{(3)}\right)(\ell_5|2)\right] + (\ell_3|1)(\ell_1|2)(\ell_2|2)\left[\left(\ell_4^{(3)}\middle|2^{(3)}\right)(\ell_5|2)\right] + (\ell_4|1)(\ell_1|2)(\ell_2|2)\left[(\ell_3|2)\left(\ell_4^{(2)}\ell_5\middle|2^{(3)}\right)\right] + (\ell_5|1)(\ell_1|2)(\ell_2|2)\left[(\ell_3|2)\left(\ell_4^{(3)}\middle|2^{(3)}\right)\right]$$

$= (\ell_1|1)(\ell_2|2)(\ell_3|2)(\ell_4|2)^{(3)}(\ell_5|2) + (\ell_2|1)(\ell_1|2)(\ell_3|2)\,(\ell_4|2)^{(3)}(\ell_5|2) + (\ell_3|1)(\ell_1|2)(\ell_2|2)(\ell_4|2)^{(3)}(\ell_5|2) + (\ell_4|1)(\ell_1|2)(\ell_2|2)(\ell_3|2)(\ell_4|2)^{(2)}(\ell_5|2) + (\ell_5|1)(\ell_1|2)(\ell_2|2)(\ell_3|2)(\ell_4|2)^{(3)}$

Example (1.3.2):

If $\mathcal{L} = \{\hbar_1, \hbar_2, \hbar_3, \hbar_4, \hbar_5\}$, $\mathcal{P} = \{1,2,3\}$,

$w_1 = \hbar_1^{(3)}\hbar_4\hbar_5^{(3)}, w_2 = \hbar_1\hbar_2^{(2)}\hbar_3^{(3)}\hbar_4\hbar_5, w_3 = \hbar_2\hbar_3\hbar_5^{(3)}, w_4 = \hbar_1\hbar_2\hbar_4^{(2)}$

$u_1 = 1^{(5)}2^{(2)}, u_2 = 1^{(3)}2^{(3)}3^{(2)}, u_3 = 2^{(3)}3^{(2)}$ and $u_4 = 1^{(2)}3^{(2)}$

Then by stratify (1.3.6) we gain:

$$\left(\begin{array}{c|c} w_1 & u_1 \\ w_2 & u_2 \\ w_3 & u_3 \\ w_4 & u_4 \end{array}\right) = \left(\begin{array}{c|c} \hbar_1\hbar_1\hbar_1\hbar_4\hbar_5\hbar_5\hbar_5 & 1111122 \\ \hbar_1\hbar_2\hbar_2\hbar_3\hbar_3\hbar_3\hbar_4\hbar_5 & 11122233 \\ \hbar_2\hbar_3\hbar_5\hbar_5\hbar_5 & 22233 \\ \hbar_1\hbar_2\hbar_4\hbar_4 & 1133 \end{array}\right) \in \mathcal{D}_9\mathcal{F}\otimes\mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_6\mathcal{F}$$

Definition (1.3.3): [22]

A double tableau $(\mathcal{T}|\mathcal{T}')$ as in (1.3.6); where w_i, u_i are basis words, is called **co-standard** if:

(1) $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s$, i.e. the sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ is a partition.

(2) $\mathcal{T} \in \mathrm{Tab}_\lambda(\mathcal{L})$ and $\mathcal{T}' \in \mathrm{Tab}_\lambda(\mathcal{P})$ are co-row and co-column standard.

Remark (1.3.4): [22]

The set of double tableaux

$$\left\{\begin{array}{l} (\mathcal{T}|\mathcal{T}') = \left(\begin{array}{c|c} w_1 & 1^{(\lambda_1)} \\ w_2 & 2^{(\lambda_2)} \\ \vdots & \vdots \\ w_s & s^{(\lambda_s)} \end{array}\right) \\ \qquad = (w_1\otimes 1\otimes\cdots\otimes 1)(1\otimes w_2\otimes 1\otimes\cdots\otimes 1)\cdots(1\otimes\cdots\otimes 1\otimes w_s) \\ \qquad = w_1\otimes w_2\otimes\cdots\otimes w_s \in \mathcal{D}_{\lambda_1}\mathcal{F}\otimes\mathcal{D}_{\lambda_2}\mathcal{F}\otimes\cdots\otimes\mathcal{D}_{\lambda_s}\mathcal{F}; \\ \text{such that } w_i \in \mathcal{D}_{\lambda_i}\mathcal{F}, \end{array}\right.$$

give a basis for $\mathcal{D}_{\lambda_1}\mathcal{F}\otimes\mathcal{D}_{\lambda_2}\mathcal{F}\otimes \ldots \otimes\mathcal{D}_{\lambda_s}\mathcal{F}$.

The following is a major result in Letter-place algebra:

Theorem (1.3.5): [20]

The set of all co-standard tableaux $(\mathcal{T}|\mathcal{T}') \in \mathcal{D}_{\beta_1}\mathcal{F}\otimes\mathcal{D}_{\beta_2}\mathcal{F}\otimes \ldots \otimes\mathcal{D}_{\beta_n}\mathcal{F}$ as described above form a basis for $\mathcal{D}_{\beta_1}\mathcal{F}\otimes\mathcal{D}_{\beta_2}\mathcal{F}\otimes \ldots \otimes\mathcal{D}_{\beta_n}\mathcal{F}$.

Example (1.3.6): [20]

The list below describes the shapes of all co-standard bi-tableaux in the case $\mathcal{D}_{p}\otimes\mathcal{D}_{q}$, (i.e.$\mathcal{P}^+ = \{1,2\}$).

		p				q	
1	1	⋯	1	2	2	⋯	2

$= \mathcal{T}'_{\lambda_0}$

		p				$q-1$	
1	1	⋯	1	2	2	⋯	2
2							

$= \mathcal{T}'_{\lambda_1}$

⋮

		p				$q-\ell$	
1	1	⋯	1	2	2	⋯	2
2	⋯	2					
	ℓ						

$= \mathcal{T}'_{\lambda_\ell}$

⋮

where $\ell \le p$.

To view how to employ the letter-place language to interpret the Weyl (Schur) map $d'_{\lambda/\mu}(d_{\lambda/\mu})$ and the box maps $\square'_{\lambda/\mu}(\square_{\lambda/\mu})$, we need the following definition:

Definition (1.3.7): [30]

In letter place algebra a linear operator ∂ is a **positive derivation** when

$$\partial(ww') = \partial(w)w' + w\partial(w'),$$

and **negative derivation** when

$$\partial(ww') = \partial(w)w' + (-1)^{|w|}w\partial(w')$$

If ∂ is a derivation, we denote by ∂^{k} the k-th iterate of the operator ∂.

If ∂ is a negative derivation, then $\partial^2 = 0$ for $k > 1$.

If ∂ is a positive derivation, one has:

$\partial^{k}(ww') = \sum_{i=0}^{k} \binom{k}{i} \partial^{k}(w)\partial^{k-1}(w')$.

A place polarization written ∂_{ab} (read: replace the letter a by the letter b); where $a, b \in \mathcal{P} = \mathcal{P}^{+} \oplus \mathcal{P}^{-}$ in uniquely defined by the following conditions:

1. $\partial_{ba}(a) = b$;
2. $\partial_{ba}(c) = 0$; if $c \neq a$;
3. $\partial_{ba}\left(a^{(k)}\right) = b.a^{(k-1)}$ if a is a positive letter;
4. When a and b are both of the same sign, ∂_{ab} is a positive derivation, and when exactly one of the letters a and b is negative, ∂_{ab} is a negative derivation,
5. When both a and b are positive, the following conditions uniquely define the k-th divided power $\partial_{ba}^{(k)}$ of the polarization ∂_{ba}:

   $\partial_{ba}^{(k)}\left(a^{(i)}\right) = a^{(i-k)}b^{(k)}$ if $i \geq k$

   $\partial_{ba}^{(k)}\left(a^{(i)}\right) = 0$ if $i < k$

   $\partial_{ba}^{(k)}(ww') = \sum_{i=0}^{k} \partial_{ba}^{(i)}(w)\partial_{ba}^{(k-i)}(w')$
6. The sequence $\partial_{ba}^{(k)}$, $k = 1,2,\ldots$ is the unique sequence of linear operations satisfying the following conditions:

   $\partial_{ba}^{(1)} = \partial_{ba}$

   $\partial_{ba}^{(i)}\partial_{ba}^{(j)} = \binom{i+j}{i} \partial_{ba}^{(i+j)}$, for $i, j = 1,2,3,\ldots$

Example (1.3.8): [23]

Let $\mathcal{P}^+ = \{1,2\}$, $\mathcal{P}^- = \mathcal{L}^- = 0$, then for $v \in \mathcal{D}_{a+b}\mathcal{F}$, the diagonalization map

$$\Delta: \mathcal{D}_{a+b}\mathcal{F} \otimes \mathcal{D}_0\mathcal{F} \to \mathcal{D}_a\mathcal{F} \otimes \mathcal{D}_b\mathcal{F}$$

$$v \otimes I \mapsto \textstyle\sum_{(v)} v_{(1)} \otimes v_{(2)} = \Delta_{(a,b)}(v),$$

can be written in letter place notation as:

$$\partial_{21}^{(b)}\left(\left(v \middle| 1^{(a+b)}\right)\right) = \left(v \middle| 1^{(a)} 2^{(b)}\right).$$

Similarly, if $w \in \mathcal{D}_{p+k}\mathcal{F}$, $w' \in \mathcal{D}_{q-k}\mathcal{F}$, then the box map.

$$\Box: \mathcal{D}_{p+k}\mathcal{F} \otimes \mathcal{D}_{q-k}\mathcal{F} \to \mathcal{D}_p\mathcal{F} \otimes \mathcal{D}_q\mathcal{F}$$

$$w \otimes w' \to \textstyle\sum_{(w)} w_{(1)} \otimes w_{(2)} w',$$

in letter place notations becomes

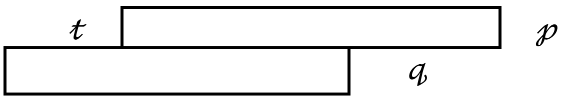
$$\partial_{21}^{(k)} \begin{pmatrix} w \\ w' \end{pmatrix}\!\!\begin{vmatrix} 1^{(p+k)} \\ 2^{(q-k)} \end{vmatrix} = \begin{pmatrix} w \\ w' \end{pmatrix}\!\!\begin{vmatrix} 1^{(p)} 2^{(k)} \\ 2^{(q-k)} \end{vmatrix}.$$

As in [16], if $\mathcal{P}^- = \{1', 2', \dots, n'\}$ for $w \in \mathcal{D}_p\mathcal{F}$, we have:

$$(w | 1'2' \dots n') \cong \textstyle\sum_{(w)} w_{(1)} \otimes \dots \otimes w_{(n)}$$

$$= \textstyle\sum \left(w_{(1)} \middle| 1'\right) \dots \left(w_{(n)} \middle| n'\right) \in \underbrace{\Lambda^1\mathcal{F} \otimes \dots \otimes \Lambda^1\mathcal{F}}_{n-times}$$

Now consider the partition

$\lambda =$ 

And the Weyl map

$$d'_{\lambda/\mu}: \mathcal{D}_p \otimes \mathcal{D}_q \to \underbrace{\Lambda^1 \otimes \dots \otimes \Lambda^1}_{t} \otimes \underbrace{\Lambda^2 \otimes \dots \otimes \Lambda^2}_{q-t} \otimes \underbrace{\Lambda^1 \otimes \dots \otimes \Lambda^1}_{p-q+t}.$$

If we take a double standard tableau, say $\begin{pmatrix} w \\ w' \end{pmatrix}\!\!\begin{vmatrix} 1^{(p)} 2^{(k)} \\ 2^{(q-k)} \end{vmatrix}$, in $\mathcal{D}_p \otimes \mathcal{D}_q$, $d'_{\lambda/\mu}$ can be defined as the composition of place polarizations, from positive places $\{1,2\}$ to negative places $\mathcal{P}^- = \{1', 2', \dots, (p+t)'\}$ as:

$$d'_{\lambda/\mu} = \partial_{q',2} \dots \partial_{1',2} \partial_{(p+t)',1} \dots \partial_{(t+1)',1};$$

where, $\partial_{u'v}$ stands for the place polarization from v to negative place u'.

Example (1.3.9):

Let $\mathcal{P} = \mathcal{P}^+ = \{1,2,3\}$, for $w \in \mathcal{D}_8\mathcal{F}, w' \in \mathcal{D}_5\mathcal{F}$ and $w'' \in \mathcal{D}_3\mathcal{F}$, we have

$$\partial_{21}^{(k)}\left(\begin{matrix} w \\ w' \\ w'' \end{matrix}\left| \begin{matrix} 1^{(8)}2^{(2)}3^{(1)} \\ 2^{(3)}3^{(1)} \\ 3^{(1)} \end{matrix}\right.\right) = \left(\begin{matrix} w \\ w' \\ w'' \end{matrix}\left| \begin{matrix} 1^{(8-k)}2^{(k)}2^{(2)}3^{(1)} \\ 2^{(3)}3^{(1)} \\ 3^{(1)} \end{matrix}\right.\right)$$

$$= \binom{k+2}{k}\left(\begin{matrix} w \\ w' \\ w'' \end{matrix}\left| \begin{matrix} 1^{(8-k)}2^{(k+2)}3^{(1)} \\ 2^{(3)}3^{(1)} \\ 3^{(1)} \end{matrix}\right.\right)$$

Example (1.3.10):

Let $\lambda = (8,7,3)$, then $d'_\lambda\left(\begin{matrix} w \\ w' \\ w'' \end{matrix}\left| \begin{matrix} 1^{(8)} \\ 2^{(7)} \\ 3^{(3)} \end{matrix}\right.\right) = \left(\begin{matrix} w \\ w' \\ w'' \end{matrix}\left| \begin{matrix} 1'2'3'4'5'6'7'8' \\ 1'2'3'4'5'6'7' \\ 1'2'3' \end{matrix}\right.\right)$; where

$$d'_\lambda = \partial_{3'3}\partial_{2'3}\partial_{1'3}\partial_{7'2}\partial_{6'2}\partial_{5'2}\partial_{4'2}\partial_{3'2}\partial_{2'2}\partial_{1'2}\partial_{8'1}\partial_{7'1}\partial_{6'1}\partial_{5'1}\partial_{4'1}\partial_{3'1}\partial_{2'1}\partial_{1'1}.$$

Proposition (1.3.11): [15] **(Capelli identities)**

Let $i, j, k, \ell \in \mathcal{P}^+$, then the divided powers of the place polarizations satisfy the following identities:

(1) If $k \neq j$, then

$$\partial_{ij}^{(r)}\partial_{jk}^{(s)} = \sum_{\alpha\geq 0}\partial_{jk}^{(s-\alpha)}\partial_{ij}^{(r-\alpha)}\partial_{ik}^{(\alpha)}$$

$$\partial_{jk}^{(s)}\partial_{ij}^{(r)} = \sum_{\alpha\geq 0}(-1)^\alpha\partial_{ij}^{(r-\alpha)}\partial_{jk}^{(s-\alpha)}\partial_{ik}^{(\alpha)}$$

(2) If $i \neq k$ and $j \neq \ell$ then

$$\partial_{ik}^{(s)}\partial_{i\ell}^{(r)} = \partial_{i\ell}^{(r)}\partial_{ik}^{(s)}$$

In our work, we need the following Capelli identities relations:

- $\mathcal{Z}_{32}\mathcal{Z}_{31}z(v) = \mathcal{Z}_{32}^{(2)}y\partial_{21}(v) - \mathcal{Z}_{21}x\partial_{32}^{(2)}(v)$
- $\mathcal{Z}_{31}\mathcal{Z}_{21}^{(b)}x(v) = -\mathcal{Z}_{21}^{(b+1)}x\partial_{32}(v) + \mathcal{Z}_{32}y\partial_{21}^{(b+1)}(v)$
- $\mathcal{Z}_{32}\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(b)}x(v) = (b-1)\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(b+1)}x(v) - \mathcal{Z}_{21}x\mathcal{Z}_{32}^{(2)}\mathcal{Z}_{21}^{(b)}x(v)$
- $\mathcal{Z}_{32}^{(2)}\mathcal{Z}_{21}^{(b)}x(v) = \mathcal{Z}_{21}^{(b)}x\partial_{32}^{(2)}(v) + \mathcal{Z}_{21}^{(b-1)}x\partial_{32}\partial_{31}(v) + \mathcal{Z}_{21}^{(b-2)}x\partial_{31}^{(2)}(v)$
- $\mathcal{Z}_{32}^{(2)}\mathcal{Z}_{31}\mathcal{Z}_{21}^{(b)}x(v) = (b-2)\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(b+1)}x(v) - \mathcal{Z}_{21}x\mathcal{Z}_{32}^{(3)}\mathcal{Z}_{21}^{(b)}x(v)$

Remark (1.3.12): [9]

From the equalities in Proposition (1.3.11,(1)), for $r = s = 1, i = 3, j = 2$ and $k = 1$, we have $\partial_{32}\partial_{21} - \partial_{21}\partial_{32} = \partial_{31}$.

1.4 The differential Bar complex

This section illustrates one of the basic construction of homological algebra which is the **Bar resolution** and its generalization which is called **differential Bar complex**, frequently used in the sequel; where we also review the characteristic-free projective resolution of the two-rowed Weyl modules obtained in [13] and [15] by using differential Bar complex technique.

From [15] we will borrow the following complex:

Let Λ be an algebra over the commutative ring $\mathcal{R}$, and $\mathcal{F}$ a Λ-module. The Bar complex is the following:

$$\underbrace{\Lambda\otimes\Lambda\otimes \ldots \otimes\Lambda\otimes\mathcal{F}}_{t-times} \xrightarrow{\eta_t} \underbrace{\Lambda\otimes\Lambda\otimes \ldots \otimes\Lambda\otimes\mathcal{F}}_{t-1-times} \xrightarrow{\eta_{t-1}} \cdots \to \Lambda\otimes\mathcal{F} \xrightarrow{\eta_1} \mathcal{F} \ ;$$

where η_1 is simply the action of Λ on $\mathcal{F}$ and in general

$$\eta_\ell(\lambda_1\otimes\lambda_2\otimes \ldots \otimes\lambda_\ell\otimes f) =$$
$$= \sum_{j=1}^{\ell-1}(-1)^{j-\ell}\lambda_1\otimes\lambda_2\otimes \ldots \otimes\lambda_j\lambda_{j+1}\otimes \ldots \otimes\lambda_\ell\otimes f + (-1)^{\ell-1}\lambda_1\otimes\lambda_2\otimes \ldots \otimes\lambda_{\ell-1}\otimes\lambda_\ell f \quad \ldots(1.4.1)$$

Let $\Lambda(\mathcal{S})$denote the exterior algebra over $\mathbb{Z}$ on a set of free generators $\mathcal{S}$ called the **separators**, let $\mathcal{A}$ be an associative algebra with identity.

The algebra $\Lambda(\mathcal{S})$ has a natural Z_2-grading: if m is the product of an even number of generators, we set $|m| = 0$ otherwise$|m| = 1$.

Definition (1.4.1): [13]

The free product of the algebra $\mathcal{A}$ and the algebra $\Lambda(\mathcal{S})$ will be called the **algebra Bar** on the algebra $\mathcal{A}$ with set separator's $\mathcal{S}$ and denoted by

$\mathrm{Bar}(\mathcal{A};\mathcal{S}) = \widetilde{\Lambda}$. By an element of $\widetilde{\Lambda}$ is a $\mathbb{Z}$-linear combination of elements of the form:

$$\widetilde{\lambda} = w_1 m_1 w_2 m_2 \dots w_k m_k, \qquad \dots(1.4.2)$$

with $w_i \in \mathcal{A}$,m_i non zero monomials in $\Lambda(\mathcal{S})$; notice that we may have $w_i = 1_{\mathcal{A}}$, $m_j \in \Lambda^0(\mathcal{S}) = \mathbb{Z}$, moreover $\widetilde{\Lambda}$ inherits a Z_2-grading defined by:

$$|\widetilde{\lambda}| = 0 \text{ if } |m_1 m_2 \cdots m_k| = 0 \text{ and } |\widetilde{\lambda}| = 1 \text{ if } |m_1 m_2 \cdots m_k| = 1.$$

Now for $\mathcal{T} \subseteq \mathcal{S}$ a T-grading called $\mathrm{Bar}(\mathcal{A};\mathcal{S};\mathcal{T},\bullet)$ of the underlying module of the algebra $\widetilde{\Lambda}$ is obtained by considering all elements $\widetilde{\lambda}$ in (1.4.2) such that m_j are monomials just in $\mathcal{T}$. In particular the submodule $\mathrm{Bar}(\mathcal{A};\mathcal{S};\mathcal{T},i)$ of $\mathcal{T}$-degree i is spanned by all elements $\widetilde{\lambda}$ in (1.4.2) such that i is the total number of occurrences of separators in the set $\mathcal{T}$ appearing in the sequence$(m_1 m_2 \cdots m_k)$.

Recall that for every separator x, there exists a unique anti-derivation ∂_x of algebra $\Lambda(\mathcal{S})$, such that $\partial_x(x) = 1$; where 1 is the identity of the exterior algebra $\Lambda(\mathcal{S})$, and $\partial_x(y) = 0$ for every $y \in \mathcal{S}$; where $y \neq x$. Recall also that $(\partial_x)^{(2)} = 0$ and $\partial_x \partial_y = -\partial_y \partial_x$.

The anti-derivation ∂_x uniquely extends to anti-derivation of Z_2-graded algebra $\widetilde{\Lambda}$, again denoted by ∂_x defined as follows:

Let $\widetilde{\lambda}$ as in (1.4.2), set $\partial_x(x) = 1_{\widetilde{\Lambda}}$, and

$$\partial_x(\widetilde{\lambda}) = w_1 \partial_x(m_1) w_2 m_2 \dots w_k m_{kk} + (-1)^{|m_1|} w_1 m_1 w_2 \partial_x(m_2) \dots w_k m_k + (-1)^{\sum_{i=1}^{k-1}|m|} w_1 m_1 w_2 m_2 \dots w_k \partial_x(m_k),$$

so the anti-derivation ∂_x is well defined on $\widetilde{\Lambda}$ and the properties $(\partial_x)^{(2)} = 0$, $\partial_x\, \partial_y = -\partial_y \partial_x$ still hold.

Definition (1.4.2): [15]

If $\mathcal{T}$ is a non-empty finite subset of $\mathcal{S}$, the operator $\partial_{\mathcal{T}} = \sum_{x\in\mathcal{T}} \partial_x$ is called **the $\mathcal{T}$ -boundary operator**, i.e. we have for $i = 0,1,2,\dots$

$$\cdots \to Bar(\mathcal{A};\mathcal{S};\mathcal{T},i+1) \xrightarrow{\partial_{\mathcal{T},i}} Bar(\mathcal{A};\mathcal{S};\mathcal{T},i) \to \cdots$$

Definition (1.4.3): [13]

Let $\mathcal{M}$ be $\mathcal{A}$-module and let $w(v)$ denoted the action of $w \in \mathcal{A}$ on $v \in \mathcal{M}$. The **free Bar module of the $\mathcal{A}$-module $\mathcal{M}$ with a set of separators $\mathcal{S}$** denoted by $\widetilde{\mathcal{M}} = Bar(\mathcal{M},\mathcal{A};\mathcal{S})$ is the $\widetilde{\Lambda}$-modul $\widetilde{\Lambda}\otimes_{\Lambda}\mathcal{M}$.

Notice that: $\widetilde{\mathcal{M}}$ is spanned by all elements of form

$$\widetilde{m} = w_1 m_1 w_2 m_2 \dots w_k m_k \otimes v = \widetilde{\lambda} \otimes v;$$

where, if $m_k = 1_{\Lambda(\mathcal{S})}$, then

$$\widetilde{m} = w_1 m_1 w_2 m_2 \dots w_{k-1} m_{k-1} \otimes w_k(v)$$

As for the case of ∂_x extending to $\widetilde{\Lambda}$, again we have that ∂_x gives a well-defined anti-derivation on $\widetilde{\mathcal{M}}$, still denoted by ∂_x and defined as follows:

$$\partial_x(\widetilde{m}) = \partial_x(\widetilde{\lambda})\otimes v$$

At this point it is clear that, given $\mathcal{T} \subseteq S$, $\partial_{\mathcal{T}} = \sum_{x\in\mathcal{T}} \partial_x$ as in the above definition, we can also define the complex $Bar(\mathcal{M},\mathcal{A};\mathcal{S},\mathcal{T},\bullet) = \widetilde{\mathcal{M}}_{\bullet}$.

$$\cdots \to \widetilde{\mathcal{M}}_{i+1} \xrightarrow{\partial_{\mathcal{T}}} \widetilde{\mathcal{M}}_i \to \cdots$$

The following example is given in [13] and [15].

Example (1.4.4):

Let $\mathcal{S} = \{x\}$. Then $\widetilde{\mathcal{M}}$ is spanned by all elements of the form

$$\widetilde{m} = w_1 x w_2 x \dots w_i x \otimes v,$$

and the derivation ∂_x is computed as follows:

$$\partial_x(\widetilde{m}) = w_1 w_2 x \dots w_i x \otimes v - w_1 x w_2 w_3 x \dots w_i x \otimes v + \cdots + (-1)^{i-1} w_1 x w_2 x \dots w_{i-1} x \otimes w_i(v)$$

Chapter Two

Public Outcomes of Resolution for Weyl Modules

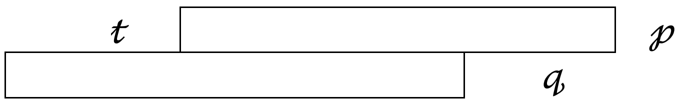
Introduction

This chapter divided into two sections, in the first section we study the resolution of two-rowed Weyl module and discuss an application for it in the case of partition (8,7) and find the terms of this resolution and prove its exactness. However, the resolution of the three rowed Weyl module presented in the second section.

2.1 Resolution for the two-rowed Weyl module

In this section, we will survey the resolution for the two-rowed Weyl module $\mathcal{K}_{\lambda/\mu}\mathcal{F}$ as it is described in [15] and [16]; where

$\lambda/\mu =$



From condition (∗) in the Definitions (1.2.1) and condition (∗∗) in the Definition (1.2.5) we recall that for $\mathcal{K}_{\lambda/\mu}\mathcal{F} = \mathrm{Im}(d'_{\lambda/\mu})$ we have

$$\sum \mathcal{D}_{p+k}\otimes\mathcal{D}_{q-k} \xrightarrow{\ \square\ } \mathcal{D}_{p}\otimes\mathcal{D}_{q} \xrightarrow{d'_{\lambda/\mu}} \mathcal{K}_{\lambda/\mu} \longrightarrow 0 \qquad \ldots(2.1.1)$$

Using letter place notation, so the maps mention in (2.1.1) can be described as follows:

$$\begin{pmatrix} w \\ w' \end{pmatrix}\!\!\left|\begin{matrix} 1^{(p+k)} \\ 2^{(q-k)} \end{matrix}\right) \xrightarrow{\partial_{21}^{(k)}} \begin{pmatrix} w \\ w' \end{pmatrix}\!\!\left|\begin{matrix} 1^{(p)} \ 2^{(k)} \\ 2^{(q-k)} \end{matrix}\right) \to \sum_{w} \begin{pmatrix} w_{(1)} \\ w' w_{(2)} \end{pmatrix}\!\!\left|\begin{matrix} (t+1)'(t+2)' \ldots (p+t)' \\ 1'2'3' \ldots q' \end{matrix}\right);$$

where

$$w\otimes w' \in \mathcal{D}_{p+k}\otimes\mathcal{D}_{q-k} \quad , \quad \square = \sum_{k=t+1}^{q} \partial_{21}^{(k)} ,$$

and

$$d'_{\lambda/\mu} = \partial_{q'2} \ldots \partial_{1'2}\partial_{(p+t)'1} \ldots \partial_{(t+1)'1},$$

is the composition of place polarizations, from positive places {1,2} to negative place {1', 2',…, $(p+t)$'}.

In specific, □ sends an element $x \otimes y$ of $\mathcal{D}_{p+k} \otimes \mathcal{D}_{q-k}$ to $\sum x_p \otimes x'_k\, y$; where $\sum x_p \otimes x'_k$ is the component of the diagonal of x in $\mathcal{D}_p \otimes \mathcal{D}_k$, [2].

Definition (2.1.1):

Let $\mathcal{Z}_{21}$ be the free generator of divided power algebra $\mathcal{D}(\mathcal{Z}_{21})$ in one generator. The divided power element $\mathcal{Z}_{21}^{(k)}$ of degree k of the free generator $\mathcal{Z}_{21}$ acts on $\mathcal{D}_{p+k} \otimes \mathcal{D}_{q-k}$ by place polarization of degree k from place 1 to place 2.

In specific, the (graded) algebra (with identity). $\mathcal{A} = \mathcal{D}(\mathcal{Z}_{21})$ act on the graded module $\mathcal{M} = \mathcal{D}_{p+k} \otimes \mathcal{D}_{q-k} = \sum \mathcal{M}_{q-k}$ (the degree of the second factor determines the grading), [15].

Hence $\mathcal{M}$ is a (graded) left $\mathcal{A}$-module; where for $w = \mathcal{Z}_{21}^{(k)} \in \mathcal{A}$ and $v \in \mathcal{D}_{\beta_1} \otimes \mathcal{D}_{\beta_2}$, so we have:

$$w(v) = \mathcal{Z}_{21}^{(k)}(v) = \partial_{21}^{(k)}(v)$$

If we take (t^{+}) graded strand of degree q,

$$\mathcal{M}_\bullet : 0 \longrightarrow \mathcal{M}_{q-t} \xrightarrow{\partial_S} \dots \longrightarrow \mathcal{M}_e \xrightarrow{\partial_S} \mathcal{M}_1 \xrightarrow{\partial_S} \mathcal{M}_0,$$

of the normalized Bar complex Bar$(\mathcal{M}, \mathcal{A}; \mathcal{S}, \bullet)$; where $\mathcal{S} = \{x\}$ as illustrated in the example (1.4.4)

So $\mathcal{M}_\bullet$ is the following complex

$$\sum_{k_i \geq 0} \mathcal{Z}_{21}^{(t+k_1)} x \mathcal{Z}_{21}^{(k_2)} x \dots \mathcal{Z}_{21}^{(k_e)} x \mathcal{D}_{p+t+|k|} \otimes \mathcal{D}_{q-t-|k|}$$

$$\xrightarrow{d_e} \sum_{k_i \geq 0} \mathcal{Z}_{21}^{(t+k_1)} x \mathcal{Z}_{21}^{(k_2)} x \dots \mathcal{Z}_{21}^{(k_{e-1})} x \mathcal{D}_{p+t+|k|} \otimes \mathcal{D}_{q-t-|k|} \xrightarrow{d_{e-1}} \dots$$

$$\xrightarrow{d_1} \sum_{k_i \geq 0} \mathcal{Z}_{21}^{(t+k)} x \mathcal{D}_{p+t+k} \otimes \mathcal{D}_{q-t-k} \xrightarrow{d_0} \mathcal{D}_p \otimes \mathcal{D}_q \; ; \qquad \dots(2.1.2)$$

where $|k| = \sum k_i$ and d_e is the boundary operator ∂_x.

Notice that (2.1.2) describes a left complex $(\partial_x^2 = 0)$ over the Weyl module in terms of Bar complex and letter-place algebra, moreover, the separator x

disappears between a $\mathcal{Z}_{ab}^{(t)}$ and elements in the tensor product of divided powers, this means $\partial_{ab}^{(t)}$ is applied to that tensor product, [16].

Theorem (2.1.2): [16]

The complex (2.1.2) is a resolution of $\mathcal{K}_{\lambda/\mu}\mathcal{F}$.

Notice that the proof is based on the construction of a contracting homotopy [32] { $\mathcal{S}_i$} which defined as follows:

$$\mathcal{S}_0 : \mathcal{D}_p \otimes \mathcal{D}_q \longrightarrow \sum_{k>0} \mathcal{Z}^{(t+k)} x \mathcal{D}_{p+t+k} \otimes \mathcal{D}_{q-t-k}$$

$$\begin{pmatrix} w \\ w' \end{pmatrix}\!\!\left|\begin{matrix} 1^{(p)} & 2^{(k)} \\ 2^{(q-k)} & \end{matrix}\right) \longrightarrow \begin{cases} 0 & ; \text{ if } \quad k \le t \\ \mathcal{Z}_{21}^{(k)} x \begin{pmatrix} w \\ w' \end{pmatrix}\!\!\left|\begin{matrix} 1^{(p+k)} \\ 2^{(q-k)} \end{matrix}\right) & ; \text{ if } \quad k > t \end{cases}$$

And for the higher dimensions as

$$\mathcal{S}_{\ell-1} : \sum_{k_i>0} \mathcal{Z}_{21}^{(t+k_1)} x \mathcal{Z}_{21}^{(k_2)} x \dots \mathcal{Z}_{21}^{(k_{\ell-1})} x\, \mathcal{D}_{p+t+|k|} \otimes \mathcal{D}_{q-t-|k|}$$
$$\longrightarrow \mathcal{Z}_{21}^{(t+k_1)} x \mathcal{Z}_{21}^{(k_2)} x \dots \mathcal{Z}_{21}^{(k_{\ell-1})} x\, \mathcal{Z}_{21}^{(k_\ell)} x\, \mathcal{D}_{p+t+|k|} \otimes \mathcal{D}_{q-t-|k|}$$

$$\mathcal{Z}_{21}^{(t+k_1)} x \mathcal{Z}_{21}^{(k_2)} x \dots \mathcal{Z}_{21}^{(k_{\ell-1})} x \begin{pmatrix} w \\ w' \end{pmatrix}\!\!\left|\begin{matrix} 1^{(p+t+|k|)} & 2^{(m)} \\ 2^{(q-t-|k|-m)} & \end{matrix}\right)$$

$$\longrightarrow \begin{cases} 0 & ;\text{if} \quad m = 0 \\ \mathcal{Z}_{21}^{(t+k_1)} x \mathcal{Z}_{21}^{(k_2)} x \dots \mathcal{Z}_{21}^{(k_{\ell-1})} x\, \mathcal{Z}_{21}^{(m)} x \begin{pmatrix} w \\ w' \end{pmatrix}\!\!\left|\begin{matrix} 1^{(p+t+|k|+m)} \\ 2^{(q-t-|k|-m)} \end{matrix}\right) & ;\text{if} \quad m > 0 \end{cases}$$

The authors in [15] write the modules of the resolution as $\mathcal{M}_i$ for $i = 0,1,\dots, q - t$, with $\mathcal{M}_0 = \mathcal{D}_p \otimes \mathcal{D}_q$, and

$$\mathcal{M}_i = \mathcal{Z}_{21}^{(t+k_1)} x \mathcal{Z}_{21}^{(k_2)} x \dots \mathcal{Z}_{21}^{(k_i)} x\, \mathcal{D}_{p+t+|k|} \otimes \mathcal{D}_{q-t-|k|} \text{ , for } i \ge 1$$

The following example clarifies the above Theorem.

Example (2.1.3):

Consider the case of the partition (8,7).

The terms of the characteristic-free resolution are

$$\mathcal{M}_0 = \mathcal{D}_8 \otimes \mathcal{D}_7$$

$$\mathcal{M}_1 = \mathcal{Z}_{21}^{(1)} x\, \mathcal{D}_9 \otimes \mathcal{D}_6 \oplus \mathcal{Z}_{21}^{(2)} x\, \mathcal{D}_{10} \otimes \mathcal{D}_5 \oplus \mathcal{Z}_{21}^{(3)} x\, \mathcal{D}_{11} \otimes \mathcal{D}_4 \oplus \mathcal{Z}_{21}^{(4)} x\, \mathcal{D}_{12} \otimes \mathcal{D}_3 \oplus$$
$$\mathcal{Z}_{21}^{(5)} x\, \mathcal{D}_{13} \otimes \mathcal{D}_2 \oplus \mathcal{Z}_{21}^{(6)} x\, \mathcal{D}_{14} \otimes \mathcal{D}_1 \oplus \mathcal{Z}_{21}^{(7)} x\, \mathcal{D}_{15} \otimes \mathcal{D}_0$$

$$\mathcal{M}_2 = \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x\, \mathcal{D}_{10} \otimes \mathcal{D}_5 \oplus \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x\, \mathcal{D}_{11} \otimes \mathcal{D}_4 \oplus \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x\, \mathcal{D}_{11} \otimes \mathcal{D}_4 \oplus$$
$$\mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(2)} x\, \mathcal{D}_{12} \otimes \mathcal{D}_3 \oplus \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(1)} x\, \mathcal{D}_{12} \otimes \mathcal{D}_3 \oplus \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(3)} x\, \mathcal{D}_{12} \otimes \mathcal{D}_3 \oplus$$
$$\mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(1)} x\, \mathcal{D}_{13} \otimes \mathcal{D}_2 \oplus \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(4)} x\, \mathcal{D}_{13} \otimes \mathcal{D}_2 \oplus \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(3)} x\, \mathcal{D}_{13} \otimes \mathcal{D}_2 \oplus$$
$$\mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(2)} x\, \mathcal{D}_{13} \otimes \mathcal{D}_2 \oplus \mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21}^{(1)} x\, \mathcal{D}_{14} \otimes \mathcal{D}_1 \oplus \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(5)} x\, \mathcal{D}_{14} \otimes \mathcal{D}_1 \oplus$$
$$\mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(3)} x\, \mathcal{D}_{14} \otimes \mathcal{D}_1 \oplus \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(4)} x\, \mathcal{D}_{14} \otimes \mathcal{D}_1 \oplus \mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(2)} x\, \mathcal{D}_{14} \otimes \mathcal{D}_1 \oplus$$
$$\mathcal{Z}_{21}^{(6)} x \mathcal{Z}_{21}^{(1)} x\, \mathcal{D}_{15} \otimes \mathcal{D}_0 \oplus \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(6)} x\, \mathcal{D}_{15} \otimes \mathcal{D}_0 \oplus \mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21}^{(2)} x\, \mathcal{D}_{15} \otimes \mathcal{D}_0 \oplus$$
$$\mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(5)} x\, \mathcal{D}_{15} \otimes \mathcal{D}_0 \oplus \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(4)} x\, \mathcal{D}_{15} \otimes \mathcal{D}_0 \oplus \mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(3)} x\, \mathcal{D}_{15} \otimes \mathcal{D}_0$$

$$\mathcal{M}_3 = \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x\, \mathcal{D}_{11} \otimes \mathcal{D}_4 \oplus \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x\, \mathcal{D}_{12} \otimes \mathcal{D}_3 \oplus$$
$$\mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x\, \mathcal{D}_{12} \otimes \mathcal{D}_3 \oplus \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x\, \mathcal{D}_{12} \otimes \mathcal{D}_3 \oplus$$
$$\mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x\, \mathcal{D}_{13} \otimes \mathcal{D}_2 \oplus \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(1)} x\, \mathcal{D}_{13} \otimes \mathcal{D}_2 \oplus$$
$$\mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(3)} x\, \mathcal{D}_{13} \otimes \mathcal{D}_2 \oplus \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x\, \mathcal{D}_{13} \otimes \mathcal{D}_2 \oplus$$
$$\mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x\, \mathcal{D}_{13} \otimes \mathcal{D}_2 \oplus \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(2)} x\, \mathcal{D}_{13} \otimes \mathcal{D}_2 \oplus$$
$$\mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x\, \mathcal{D}_{14} \otimes \mathcal{D}_1 \oplus \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(1)} x\, \mathcal{D}_{14} \otimes \mathcal{D}_1 \oplus$$
$$\mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(4)} x\, \mathcal{D}_{14} \otimes \mathcal{D}_1 \oplus \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x\, \mathcal{D}_{14} \otimes \mathcal{D}_1 \oplus$$
$$\mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x\, \mathcal{D}_{14} \otimes \mathcal{D}_1 \oplus \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(1)} x\, \mathcal{D}_{14} \otimes \mathcal{D}_1 \oplus$$
$$\mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(3)} x\, \mathcal{D}_{14} \otimes \mathcal{D}_1 \oplus \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(3)} x\, \mathcal{D}_{14} \otimes \mathcal{D}_1 \oplus$$
$$\mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(2)} x\, \mathcal{D}_{14} \otimes \mathcal{D}_1 \oplus \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(2)} x\, \mathcal{D}_{14} \otimes \mathcal{D}_1 \oplus$$

$$\mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus \mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}^{(1)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus$$
$$\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(5)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus \mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(3)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus$$
$$\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(2)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus \mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus$$
$$\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(1)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus \mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(3)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus$$
$$\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(3)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus \mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(1)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus$$
$$\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(4)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus \mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(1)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus$$
$$\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(2)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus \mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(2)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus$$
$$\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(4)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0$$

$$\mathcal{M}_4 = \mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\ \mathcal{D}_{12}\otimes\mathcal{D}_3 \oplus \mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\ \mathcal{D}_{13}\otimes\mathcal{D}_2 \oplus$$
$$\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\ \mathcal{D}_{13}\otimes\mathcal{D}_2 \oplus \mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(1)}x\ \mathcal{D}_{13}\otimes\mathcal{D}_2 \oplus$$
$$\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(2)}x\ \mathcal{D}_{13}\otimes\mathcal{D}_2 \oplus \mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\ \mathcal{D}_{14}\otimes\mathcal{D}_1 \oplus$$
$$\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\ \mathcal{D}_{14}\otimes\mathcal{D}_1 \oplus \mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(1)}x\ \mathcal{D}_{14}\otimes\mathcal{D}_1 \oplus$$
$$\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(3)}x\ \mathcal{D}_{14}\otimes\mathcal{D}_1 \oplus \mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\ \mathcal{D}_{14}\otimes\mathcal{D}_1 \oplus$$
$$\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(1)}x\ \mathcal{D}_{14}\otimes\mathcal{D}_1 \oplus \mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(2)}x\ \mathcal{D}_{14}\otimes\mathcal{D}_1 \oplus$$
$$\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(1)}x\ \mathcal{D}_{14}\otimes\mathcal{D}_1 \oplus \mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x\ \mathcal{D}_{14}\otimes\mathcal{D}_1 \oplus$$
$$\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(2)}x\ \mathcal{D}_{14}\otimes\mathcal{D}_1 \oplus \mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus$$
$$\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus \mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(1)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus$$
$$\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(4)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus \mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus$$
$$\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(1)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus \mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(3)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus$$
$$\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus \mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(1)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus$$
$$\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(2)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus \mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(3)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus$$
$$\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(2)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus \mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(3)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus$$
$$\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(2)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus \mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(1)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus$$
$$\mathcal{Z}_{21}^{(1)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(1)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus \mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(1)}x\ \mathcal{D}_{15}\otimes\mathcal{D}_0 \oplus$$

$\mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x\ \mathcal{D}_{15} \otimes \mathcal{D}_0 \oplus \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(2)} x\ \mathcal{D}_{15} \otimes \mathcal{D}_0 \oplus$
$\mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(2)} x\ \mathcal{D}_{15} \otimes \mathcal{D}_0$

$\mathcal{M}_5 = \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x\ \mathcal{D}_{13} \otimes \mathcal{D}_2 \oplus$
$\mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x\ \mathcal{D}_{14} \otimes \mathcal{D}_1 \oplus$
$\mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x\ \mathcal{D}_{14} \otimes \mathcal{D}_1 \oplus$
$\mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x\ \mathcal{D}_{14} \otimes \mathcal{D}_1 \oplus$
$\mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x\ \mathcal{D}_{14} \otimes \mathcal{D}_1 \oplus$
$\mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x\ \mathcal{D}_{14} \otimes \mathcal{D}_1 \oplus$
$\mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x\ \mathcal{D}_{15} \otimes \mathcal{D}_0 \oplus$
$\mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x\ \mathcal{D}_{15} \otimes \mathcal{D}_0 \oplus$
$\mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x\ \mathcal{D}_{15} \otimes \mathcal{D}_0 \oplus$
$\mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(1)} x\ \mathcal{D}_{15} \otimes \mathcal{D}_0 \oplus$
$\mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(3)} x\ \mathcal{D}_{15} \otimes \mathcal{D}_0 \oplus$
$\mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x\ \mathcal{D}_{15} \otimes \mathcal{D}_0 \oplus$
$\mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x\ \mathcal{D}_{15} \otimes \mathcal{D}_0 \oplus$
$\mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x\ \mathcal{D}_{15} \otimes \mathcal{D}_0 \oplus$
$\mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x\ \mathcal{D}_{15} \otimes \mathcal{D}_0 \oplus$
$\mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x\ \mathcal{D}_{15} \otimes \mathcal{D}_0 \oplus$
$\mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x\ \mathcal{D}_{15} \otimes \mathcal{D}_0 \oplus$
$\mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x\ \mathcal{D}_{15} \otimes \mathcal{D}_0 \oplus$
$\mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x\ \mathcal{D}_{15} \otimes \mathcal{D}_0 \oplus$
$\mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(2)} x\ \mathcal{D}_{15} \otimes \mathcal{D}_0 \oplus$
$\mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x\ \mathcal{D}_{15} \otimes \mathcal{D}_0$

$$\mathcal{M}_6 = \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x\ \mathcal{D}_{14} \otimes \mathcal{D}_1 \oplus$$
$$\mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x\ \mathcal{D}_{15} \otimes \mathcal{D}_0 \oplus$$
$$\mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x\ \mathcal{D}_{15} \otimes \mathcal{D}_0 \oplus$$
$$\mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x\ \mathcal{D}_{15} \otimes \mathcal{D}_0 \oplus$$
$$\mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x\ \mathcal{D}_{15} \otimes \mathcal{D}_0 \oplus$$
$$\mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(1)} x\ \mathcal{D}_{15} \otimes \mathcal{D}_0 \oplus$$
$$\mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(2)} x\ \mathcal{D}_{15} \otimes \mathcal{D}_0$$

$$\mathcal{M}_7 = \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x \mathcal{Z}_{21}^{(1)} x\ \mathcal{D}_{15} \otimes \mathcal{D}_0$$

The homotopies $\{\mathcal{S}_i\}$; where i = 0,1,2, …, 6 are

$$\mathcal{S}_0 : \mathcal{D}_8 \otimes \mathcal{D}_7 \longrightarrow \sum_{k>0} \mathcal{Z}_{21}^{(k)} x\ \mathcal{D}_{8+k} \otimes \mathcal{D}_{7-k}$$

$$\mathcal{S}_0\left(\left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8)} & 2^{(k)} \\ 2^{(7-k)} & \end{matrix}\right)\right) = \begin{cases} 0 & ;\ \text{if} \quad k \le 0 \\ \mathcal{Z}_{21}^{(k)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+k)} \\ 2^{(7-k)} \end{matrix}\right) & ;\ \text{if} \quad k = 1,2,3,4,5,6,7 \end{cases}$$

$$\mathcal{S}_1 : \sum_{k>0} \mathcal{Z}_{21}^{(k)} x\ \mathcal{D}_{8+k} \otimes \mathcal{D}_{7-k} \longrightarrow \mathcal{Z}_{21}^{(k_1)} x \mathcal{Z}_{21}^{(k_2)} x\ \mathcal{D}_{8+k} \otimes \mathcal{D}_{7-k}$$

$$\mathcal{S}_1\left(\mathcal{Z}_{21}^{(k)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+k)} & 2^{(m)} \\ 2^{(7-k-m)} & \end{matrix}\right)\right)$$

$$= \begin{cases} 0 & ;\ \text{if} \quad m = 0 \\ \mathcal{Z}_{21}^{(k)} x \mathcal{Z}_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+k+m)} \\ 2^{(7-k-m)} \end{matrix}\right) & ;\ \text{if} \quad m = 1,2,3,4,5,6 \end{cases}$$

$$\mathcal{S}_2 : \sum_{k_i>0} \mathcal{Z}_{21}^{(k_1)} x \mathcal{Z}_{21}^{(k_2)} x\ \mathcal{D}_{8+|k|} \otimes \mathcal{D}_{7-|k|} \longrightarrow \mathcal{Z}_{21}^{(k_1)} x \mathcal{Z}_{21}^{(k_2)} x \mathcal{Z}_{21}^{(k_3)} x\ \mathcal{D}_{8+|k|} \otimes \mathcal{D}_{7-|k|}$$

$$\mathcal{S}_2\left(Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix}\right)\right)$$

$$=\begin{cases} 0 & ;\ \text{if } m=0 \\ Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) & ;\ \text{if } m=1,2,3,4,5 \end{cases} \text{; where}$$

$|k| = k_1 + k_2.$

$\mathcal{S}_3 : \sum_{k_i>0} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x\, \mathcal{D}_{8+|k|} \otimes \mathcal{D}_{7-|k|}$

$\longrightarrow Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x\, \mathcal{D}_{8+|k|} \otimes \mathcal{D}_{7-|k|}$

$$\mathcal{S}_3\left(Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix}\right)\right)$$

$$=\begin{cases} 0 & ;\ \text{if } m=0 \\ Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) & ;\ \text{if } m=1,2,3,4 \end{cases} \text{; where}$$

$|k| = k_1 + k_2 + k_3.$

$\mathcal{S}_4 : \sum_{k_i>0} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x\, Z_{21}^{(k_4)} x\, \mathcal{D}_{8+|k|} \otimes \mathcal{D}_{7-|k|}$

$\longrightarrow Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x\, Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x\, \mathcal{D}_{8+|k|} \otimes \mathcal{D}_{7-|k|}$

$$\mathcal{S}_4\left(Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x\, Z_{21}^{(k_4)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix}\right)\right)$$

$$=\begin{cases} 0 & ;\ \text{if } m=0 \\ Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x\, Z_{21}^{(k_4)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) & ;\ \text{if } m=1,2,3 \end{cases} \text{; where}$$

$|k| = k_1 + k_2 + k_3 + k_4.$

$$\mathcal{S}_5 : \sum_{k_i > 0} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x\, Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x\, \mathcal{D}_{8+|k|} \otimes \mathcal{D}_{7-|k|}$$

$$\longrightarrow Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x\, Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(k_6)} x\, \mathcal{D}_{8+|k|} \otimes \mathcal{D}_{7-|k|}$$

$$\mathcal{S}_5 \left(Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x\, Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix} \right) \right)$$

$$= \begin{cases} 0 & ; \text{ if } m = 0 \\ Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x\, Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix} \right) & ; \text{ if } m = 1,2 \end{cases} \text{; where}$$

$|k| = k_1 + k_2 + k_3 + k_4 + k_5.$

$$\mathcal{S}_6 : \sum_{k_i > 0} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x\, Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(k_6)} x\, \mathcal{D}_{8+|k|} \otimes \mathcal{D}_{7-|k|}$$

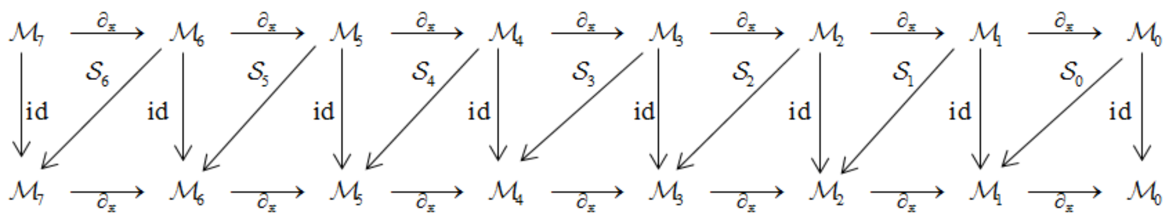
$$\longrightarrow Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x\, Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(k_6)} x Z_{21}^{(k_7)} x\, \mathcal{D}_{8+|k|} \otimes \mathcal{D}_{7-|k|}$$

$$\mathcal{S}_6 \left(Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x\, Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(k_6)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix} \right) \right)$$

$$= \begin{cases} 0 & ; \text{if } m = 0 \\ Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x\, Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(k_6)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix} \right) & ; \text{if } m = 1 \end{cases} \text{; where}$$

$|k| = k_1 + k_2 + k_3 + k_4 + k_5 + k_6.$

So we have the following diagram:-



Now we have

$$\mathcal{S}_0\partial_x\left(Z_{21}^{(k)}x\left(\begin{matrix}w\\w'\end{matrix}\middle|\begin{matrix}1^{(8+k)} & 2^{(m)}\\2^{(7-k-m)} & \end{matrix}\right)\right)=\mathcal{S}_0\partial_{12}^{(k)}\left(\begin{matrix}w\\w'\end{matrix}\middle|\begin{matrix}1^{(8+k)} & 2^{(m)}\\2^{(7-k-m)} & \end{matrix}\right)$$

$$=\binom{k+m}{m}Z_{21}^{(k+m)}x\left(\begin{matrix}w\\w'\end{matrix}\middle|\begin{matrix}1^{(8+k+m)}\\2^{(7-k-m)}\end{matrix}\right),$$

and

$$\partial_x\mathcal{S}_1\left(Z_{21}^{(k)}x\left(\begin{matrix}w\\w'\end{matrix}\middle|\begin{matrix}1^{(8+k)} & 2^{(m)}\\2^{(7-k-m)} & \end{matrix}\right)\right)=\partial_x\left(Z_{21}^{(k)}xZ_{21}^{(m)}x\left(\begin{matrix}w\\w'\end{matrix}\middle|\begin{matrix}1^{(8+k+m)}\\2^{(7-k-m)}\end{matrix}\right)\right)$$

$$=-\binom{k+m}{m}Z_{21}^{(k+m)}x\left(\begin{matrix}w\\w'\end{matrix}\middle|\begin{matrix}1^{(8+k+m)}\\2^{(7-k-m)}\end{matrix}\right)+Z_{21}^{(k)}x\left(\begin{matrix}w\\w'\end{matrix}\middle|\begin{matrix}1^{(8+k)} & 2^{(m)}\\2^{(7-k-m)} & \end{matrix}\right)$$

It is clear that $\mathcal{S}_0\partial_x+\partial_x\mathcal{S}_1=\mathrm{id}_{\mathcal{M}_1}$.

$$\mathcal{S}_1\partial_x\left(Z_{21}^{(k_1)}xZ_{21}^{(k_2)}x\left(\begin{matrix}w\\w'\end{matrix}\middle|\begin{matrix}1^{(8+|k|)} & 2^{(m)}\\2^{(7-|k|-m)} & \end{matrix}\right)\right)$$

$$=\mathcal{S}_1\left[-\binom{|k|}{k_2}Z_{21}^{(|k|)}x\left(\begin{matrix}w\\w'\end{matrix}\middle|\begin{matrix}1^{(8+|k|)} & 2^{(m)}\\2^{(7-|k|-m)} & \end{matrix}\right)+\right.$$

$$\left.Z_{21}^{(k_1)}x\partial_{21}^{(k_2)}\left(\begin{matrix}w\\w'\end{matrix}\middle|\begin{matrix}1^{(8+|k|)} & 2^{(m)}\\2^{(7-|k|-m)} & \end{matrix}\right)\right]$$

$$=-\binom{|k|}{k_2}Z_{21}^{(|k|)}xZ_{21}^{(m)}x\left(\begin{matrix}w\\w'\end{matrix}\middle|\begin{matrix}1^{(8+|k|+m)}\\2^{(7-|k|-m)}\end{matrix}\right)+$$

$$\binom{k_2+m}{m}Z_{21}^{(k_1)}xZ_{21}^{(k_2+m)}x\left(\begin{matrix}w\\w'\end{matrix}\middle|\begin{matrix}1^{(8+|k|+m)}\\2^{(7-|k|-m)}\end{matrix}\right),$$

and

$$\partial_x\mathcal{S}_2\left(Z_{21}^{(k_1)}xZ_{21}^{(k_2)}x\left(\begin{matrix}w\\w'\end{matrix}\middle|\begin{matrix}1^{(8+|k|)} & 2^{(m)}\\2^{(7-|k|-m)} & \end{matrix}\right)\right)=$$

$$\partial_x\left(Z_{21}^{(k_1)}xZ_{21}^{(k_2)}xZ_{21}^{(m)}x\left(\begin{matrix}w\\w'\end{matrix}\middle|\begin{matrix}1^{(8+|k|+m)}\\2^{(7-|k|-m)}\end{matrix}\right)\right)$$

$$=\binom{|k|}{k_2}Z_{21}^{(|k|)}xZ_{21}^{(m)}x\left(\begin{matrix}w\\w'\end{matrix}\middle|\begin{matrix}1^{(8+|k|+m)}\\2^{(7-|k|-m)}\end{matrix}\right)-$$

$$\binom{k_2+m}{m}Z_{21}^{(k_1)}xZ_{21}^{(k_2+m)}x\left(\begin{matrix}w\\w'\end{matrix}\middle|\begin{matrix}1^{(8+|k|+m)}\\2^{(7-|k|-m)}\end{matrix}\right)+$$

$$Z_{21}^{(k_1)}xZ_{21}^{(k_2)}x\left(\begin{matrix}w\\w'\end{matrix}\middle|\begin{matrix}1^{(8+|k|)} & 2^{(m)}\\2^{(7-|k|-m)} & \end{matrix}\right)\text{ ; where }|k|=k_1+k_2.$$

It is clear that $\mathcal{S}_1\partial_x+\partial_x\mathcal{S}_2=\mathrm{id}_{\mathcal{M}_2}$.

$$\mathcal{S}_2\partial_x\left(Z_{21}^{(k_1)}xZ_{21}^{(k_2)}xZ_{21}^{(k_3)}x\begin{pmatrix} w \\ w' \end{pmatrix}\!\!\left|\begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix}\right)\right)$$

$$= \mathcal{S}_2\left[\binom{k_1+k_2}{k_2}Z_{21}^{(k_1+k_2)}xZ_{21}^{(k_3)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix}\right) -\right.$$

$$\binom{k_2+k_3}{k_3}Z_{21}^{(k_1)}xZ_{21}^{(k_2+k_3)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix}\right) +$$

$$\left.Z_{21}^{(k_1)}xZ_{21}^{(k_2)}x\partial_{21}^{(k_3)}\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix}\right)\right]$$

$$= \binom{k_1+k_2}{k_2}Z_{21}^{(k_1+k_2)}xZ_{21}^{(k_3)}xZ_{21}^{(m)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) -$$

$$\binom{k_2+k_3}{k_3}Z_{21}^{(k_1)}xZ_{21}^{(k_2+k_3)}xZ_{21}^{(m)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) +$$

$$\binom{k_3+m}{m}Z_{21}^{(k_1)}xZ_{21}^{(k_2)}xZ_{21}^{(k_3+m)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right),$$

and

$$\partial_x\mathcal{S}_3\left(Z_{21}^{(k_1)}xZ_{21}^{(k_2)}xZ_{21}^{(k_3)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix}\right)\right) =$$

$$\partial_x\left(Z_{21}^{(k_1)}xZ_{21}^{(k_2)}xZ_{21}^{(k_3)}xZ_{21}^{(m)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right)\right)$$

$$= -\binom{k_1+k_2}{k_2}Z_{21}^{(k_1+k_2)}xZ_{21}^{(k_3)}xZ_{21}^{(m)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) +$$

$$\binom{k_2+k_3}{k_3}Z_{21}^{(k_1)}xZ_{21}^{(k_2+k_3)}xZ_{21}^{(m)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) -$$

$$\binom{k_3+m}{m}Z_{21}^{(k_1)}xZ_{21}^{(k_2)}xZ_{21}^{(k_3+m)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) +$$

$$Z_{21}^{(k_1)}xZ_{21}^{(k_2)}xZ_{21}^{(k_3)}x\partial_{21}^{(m)}\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right)$$

$$= -\binom{k_1+k_2}{k_2}Z_{21}^{(k_1+k_2)}xZ_{21}^{(k_3)}xZ_{21}^{(m)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) +$$

$$\binom{k_2+k_3}{k_3}Z_{21}^{(k_1)}xZ_{21}^{(k_2+k_3)}xZ_{21}^{(m)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) -$$

$$\binom{k_3+m}{m}Z_{21}^{(k_1)}xZ_{21}^{(k_2)}xZ_{21}^{(k_3+m)}x\left(\begin{matrix} w \\ w' \end{matrix}\middle|\begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) +$$

$$Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x \begin{pmatrix} w \\ w' \end{pmatrix} \left| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix} \right) \text{; where } |k| = k_1 + k_2 + k_3.$$

It is clear that $\mathcal{S}_2 \partial_x + \partial_x \mathcal{S}_3 = \mathrm{id}_{\mathcal{M}_3}$.

$$\mathcal{S}_3 \partial_x \left(Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix} \right) \right)$$

$$= \mathcal{S}_3 \Bigg[-\binom{k_1+k_2}{k_2} Z_{21}^{(k_1+k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix} \right) +$$

$$\binom{k_2+k_3}{k_3} Z_{21}^{(k_1)} x Z_{21}^{(k_2+k_3)} x Z_{21}^{(k_4)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix} \right) -$$

$$\binom{k_3+k_4}{k_4} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3+k_4)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix} \right) +$$

$$Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x \partial_{21}^{(k_4)} \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix} \right) \Bigg]$$

$$= -\binom{k_1+k_2}{k_2} Z_{21}^{(k_1+k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix} \right) +$$

$$\binom{k_2+k_3}{k_3} Z_{21}^{(k_1)} x Z_{21}^{(k_2+k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix} \right) -$$

$$\binom{k_3+k_4}{k_4} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3+k_4)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix} \right) +$$

$$\binom{k_4+m}{m} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4+m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix} \right),$$

and

$$\partial_x \mathcal{S}_4 \left(Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix} \right) \right) =$$

$$\partial_x \left(Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix} \right) \right)$$

$$= \binom{k_1+k_2}{k_2} Z_{21}^{(k_1+k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix} \right) -$$

$$\binom{k_2+k_3}{k_3} Z_{21}^{(k_1)} x Z_{21}^{(k_2+k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix} \right) +$$

$$\binom{k_3+k_4}{k_4} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3+k_4)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix} \right) -$$

$$\binom{k_4+m}{m} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4+m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) +$$

$$Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x \partial_{21}^{(m)} \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right)$$

$$= \binom{k_1+k_2}{k_2} Z_{21}^{(k_1+k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) -$$

$$\binom{k_2+k_3}{k_3} Z_{21}^{(k_1)} x Z_{21}^{(k_2+k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) +$$

$$\binom{k_3+k_4}{k_4} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3+k_4)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) -$$

$$\binom{k_4+m}{m} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4+m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) +$$

$$Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix}\right) ;$$

Where $|k| = k_1 + k_2 + k_3 + k_4$.

It is clear that $\mathcal{S}_3 \partial_x + \partial_x \mathcal{S}_4 = \mathrm{id}_{\mathcal{M}_4}$.

$$\mathcal{S}_4 \partial_x \left(Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix}\right) \right) =$$

$$\mathcal{S}_4 \left[\binom{k_1+k_2}{k_2} Z_{21}^{(k_1+k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix}\right) - \right.$$

$$\binom{k_2+k_3}{k_3} Z_{21}^{(k_1)} x Z_{21}^{(k_2+k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix}\right) +$$

$$\binom{k_3+k_4}{k_4} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3+k_4)} x Z_{21}^{(k_5)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix}\right) -$$

$$\binom{k_4+k_5}{k_5} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4+k_5)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix}\right) +$$

$$\left. Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x \partial_{21}^{(k_5)} \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix}\right) \right]$$

$$= \binom{k_1+k_2}{k_2} Z_{21}^{(k_1+k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) -$$

$$\binom{k_2+k_3}{k_3} Z_{21}^{(k_1)} x Z_{21}^{(k_2+k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) +$$

$$\binom{k_3+k_4}{k_4} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3+k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) -$$

$$\binom{k_4+k_5}{k_5} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4+k_5)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) +$$

$$\binom{k_5+m}{m} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5+m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right),$$

and

$$\partial_x S_5 \left(Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix}\right)\right) =$$

$$\partial_x \left(Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right)\right)$$

$$= -\binom{k_1+k_2}{k_2} Z_{21}^{(k_1+k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) +$$

$$\binom{k_2+k_3}{k_3} Z_{21}^{(k_1)} x Z_{21}^{(k_2+k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) -$$

$$\binom{k_3+k_4}{k_4} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3+k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) +$$

$$\binom{k_4+k_5}{k_5} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4+k_5)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) -$$

$$\binom{k_5+m}{m} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5+m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) +$$

$$Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x \partial_{21}^{(m)} \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right)$$

$$= -\binom{k_1+k_2}{k_2} Z_{21}^{(k_1+k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) +$$

$$\binom{k_2+k_3}{k_3} Z_{21}^{(k_1)} x Z_{21}^{(k_2+k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) -$$

$$\binom{k_3+k_4}{k_4} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3+k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) +$$

$$\binom{k_4+k_5}{k_5} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4+k_5)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) -$$

$$\binom{k_5+m}{m} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5+m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right) +$$

$$Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix}\right);$$

where $|k| = k_1 + k_2 + k_3 + k_4 + k_5$.

It is clear that $\mathcal{S}_4 \partial_x + \partial_x \mathcal{S}_5 = \mathrm{id}_{\mathcal{M}_5}$.

$$\mathcal{S}_5 \partial_x \left(Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(k_6)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix}\right)\right) =$$

$$\mathcal{S}_5 \left[-\binom{k_1 + k_2}{k_2} Z_{21}^{(k_1+k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(k_6)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix}\right) \right.$$

$$+ \binom{k_2 + k_3}{k_3} Z_{21}^{(k_1)} x Z_{21}^{(k_2+k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(k_6)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix}\right) -$$

$$\binom{k_3 + k_4}{k_4} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3+k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(k_6)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix}\right) +$$

$$\binom{k_4 + k_5}{k_5} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4+k_5)} x Z_{21}^{(k_6)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix}\right) -$$

$$\binom{k_5 + k_6}{k_6} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5+k_6)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix}\right) +$$

$$\left. Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x \partial_{21}^{(k_6)} \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix}\right)\right]$$

$$= -\binom{k_1 + k_2}{k_2} Z_{21}^{(k_1+k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(k_6)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right)$$

$$+ \binom{k_2 + k_3}{k_3} Z_{21}^{(k_1)} x Z_{21}^{(k_2+k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(k_6)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right)$$

$$- \binom{k_3 + k_4}{k_4} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3+k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(k_6)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right)$$

$$+ \binom{k_4 + k_5}{k_5} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4+k_5)} x Z_{21}^{(k_6)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right)$$

$$- \binom{k_5 + k_6}{k_6} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5+k_6)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right)$$

$$+ \binom{k_6 + m}{m} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(k_6+m)} x \left(\begin{matrix} w \\ w' \end{matrix}\middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix}\right)\Bigg],$$

and

$$\partial_x \mathcal{S}_6 \left(Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(k_6)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix} \right) \right) =$$

$$\partial_x \left(Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(k_6)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix} \right) \right)$$

$$= \binom{k_1 + k_2}{k_2} Z_{21}^{(k_1+k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(k_6)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix} \right) -$$

$$\binom{k_2 + k_3}{k_3} Z_{21}^{(k_1)} x Z_{21}^{(k_2+k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(k_6)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix} \right) +$$

$$\binom{k_3 + k_4}{k_4} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3+k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(k_6)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix} \right) -$$

$$\binom{k_4 + k_5}{k_5} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4+k_5)} x Z_{21}^{(k_6)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix} \right) +$$

$$\binom{k_5 + k_6}{k_6} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5+k_6)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix} \right) -$$

$$\binom{k_6 + m}{m} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(k_6+m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix} \right) +$$

$$Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(k_6)} x \partial_{21}^{(m)} \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix} \right)$$

$$= \binom{k_1 + k_2}{k_2} Z_{21}^{(k_1+k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(k_6)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix} \right) -$$

$$\binom{k_2 + k_3}{k_3} Z_{21}^{(k_1)} x Z_{21}^{(k_2+k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(k_6)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix} \right) +$$

$$\binom{k_3 + k_4}{k_4} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3+k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(k_6)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix} \right) -$$

$$\binom{k_4 + k_5}{k_5} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4+k_5)} x Z_{21}^{(k_6)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix} \right) +$$

$$\binom{k_5 + k_6}{k_6} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5+k_6)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix} \right) -$$

$$\binom{k_6 + m}{m} Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(k_6+m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|+m)} \\ 2^{(7-|k|-m)} \end{matrix} \right) +$$

$$Z_{21}^{(k_1)} x Z_{21}^{(k_2)} x Z_{21}^{(k_3)} x Z_{21}^{(k_4)} x Z_{21}^{(k_5)} x Z_{21}^{(k_6)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|k|)} & 2^{(m)} \\ 2^{(7-|k|-m)} & \end{matrix} \right);$$

where $|k| = k_1 + k_2 + k_3 + k_4 + k_5 + k_6$.

It is clear that $\mathcal{S}_5 \partial_x + \partial_x \mathcal{S}_6 = \mathrm{id}_{\mathcal{M}_6}$.

From above we obtain that $\{\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4, \mathcal{S}_5, \mathcal{S}_6\}$ is a contracting homotopy [31] and [32] which implies the complex

$$0 \longrightarrow \mathcal{M}_7 \longrightarrow \mathcal{M}_6 \longrightarrow \mathcal{M}_5 \longrightarrow \mathcal{M}_4 \longrightarrow \mathcal{M}_3 \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_0$$

is exact.

2.2 Resolution for the three-rowed Weyl module

We exhibit the theory of the resolution Res$[p, q, r; t_1, t_2]$ of the Weyl module $[p, q, r; t_1, t_2]$ associated with the three-rowed skew-shape, [15]

$\lambda/\mu =$

The Weyl module $\mathcal{K}_{\lambda/\mu}$ is exhibited by the box map

$$\begin{matrix} \sum_{k>0} \mathcal{D}_{p+t_1+k} \otimes \mathcal{D}_{q-t_1-k} \otimes \mathcal{D}_r \\ \oplus \\ \sum_{e>0} \mathcal{D}_p \otimes \mathcal{D}_{q+t_2+e} \otimes \mathcal{D}_{r-t_2-e} \end{matrix} \xrightarrow{\ \square\ } \mathcal{D}_p \otimes \mathcal{D}_q \otimes \mathcal{D}_r \xrightarrow{d'_{\lambda/\mu}} \mathcal{K}_{\lambda/\mu} \qquad \ldots(2.2.1)$$

As in (2.1.1), the maps

$$\sum_{k>0} \mathcal{D}_{p+t_1+k} \otimes \mathcal{D}_{q-t_1-k} \otimes \mathcal{D}_r \longrightarrow \mathcal{D}_p \otimes \mathcal{D}_q \otimes \mathcal{D}_r\ ,$$

may be explicated as k^{th} divided power of the place polarization from place 1 to place 2 (i.e. $\partial_{21}^{(k)}$), the maps

$$\sum_{e>0} \mathcal{D}_p \otimes \mathcal{D}_{q+t_2+e} \otimes \mathcal{D}_{r-t_2-e} \longrightarrow \mathcal{D}_p \otimes \mathcal{D}_q \otimes \mathcal{D}_r\ ,$$

may be explicated as e^{th} divided power of the place polarization from place 2 to place 3 (i.e. $\partial_{32}^{(e)}$), and as in two-rowed case.

The authors in [15] introduce two generators $\mathcal{Z}_{21}$ and $\mathcal{Z}_{32}$ with their divided powers writing in place of (2.2.1)

$$\begin{array}{l}\sum_{k>0} \mathcal{Z}_{21}^{(t_1+k)} x \mathcal{D}_{p+t_1+k} \otimes \mathcal{D}_{q-t_1-k} \otimes \mathcal{D}_r \\ \qquad\qquad \oplus \qquad\qquad\qquad\qquad \xrightarrow{\square} \mathcal{D}_p \otimes \mathcal{D}_q \otimes \mathcal{D}_r \qquad \ldots(2.2.2) \\ \sum_{e>0} \mathcal{Z}_{32}^{(t_2+e)} y \mathcal{D}_p \otimes \mathcal{D}_{q+t_2+e} \otimes \mathcal{D}_{r-t_2-e} \,;\end{array}$$

where x and y stand for separator variables, and the boundary map is $\partial_x + \partial_y$.

For the case of one-triple overlap the authors in [15] exhibit all specifics of this case which has one triple overlap i.e. $r \leq t_1 + t_2 + 1$.

Theorem (2.2.1): [15]

Let $[p, q, r; t_1, t_2]$ be a Weyl module with $r \leq t_1 + t_2 + 1$ Then the complex $\mathcal{M}_{\bullet}$ is a projective resolution of $\mathcal{K}_{\lambda/\mu}$ when the maps are mentioned by $\partial_{\mathcal{S}} = \partial_x + \partial_y$.

From [15] we recall the following proposition to give the compact form of the terms of $\mathcal{M}_{\bullet}$.

Proposition (2.2.2): [15]

Let Bar $(\mathcal{A}; \mathcal{S})$ be the Bar complex defined in Definition (1.4.3); where $\mathcal{A}$ is the free associative non-commutative algebra generated by the variable $\mathcal{Z}_{mn}$ with $m, n \in \{1,2,3\}$, and $\mathcal{S} = \{x, y, z\}$. For a fixed m, n, σ, and e, the symbol

$$\mathcal{Z}_{mn}^{(\sigma)} \odot \underline{\mathcal{Z}}_{mn}^{(e)}$$

is denoted the homogenous strand of the Bar complex of total degree $\sigma + e$ with an initial term of degree $\geq \sigma$.

For example

$$\mathcal{Z}_{32}^{(\sigma)} \odot \underline{\mathcal{Z}}_{32}^{(2)} : 0 \longrightarrow \mathcal{Z}_{32}^{(\sigma)} y \mathcal{Z}_{32} y \mathcal{Z}_{32} \longrightarrow \begin{matrix} \mathcal{Z}_{32}^{(\sigma+1)} y \mathcal{Z}_{32} \\ \oplus \\ \mathcal{Z}_{32}^{(\sigma)} y \mathcal{Z}_{32}^{(2)} \end{matrix} \longrightarrow \mathcal{Z}_{32}^{(\sigma+2)} \longrightarrow 0$$

$\underline{\mathcal{Z}}_{mn}^{(e)}$, $e > 0$

Equals the homogeneous strand of the normalized Bar complex of degree 1.

As in [15] the terms of complex $\mathcal{M}_{\bullet}$ are described as:

$\text{Res}([p_1, p_2; t_1]) \otimes \mathcal{D}_{p_3} \oplus \sum_{e \geq 0} \mathcal{Z}_{32}^{(t_2+1)} \odot \underline{\mathcal{Z}}_{32}^{(e)} \text{Res}([p_1, p_2 + t_2 + 1 + e; t_1 + t_2 + 1 + e]) \otimes \mathcal{D}_{p_{3-(t_2+1+e)}}$

The following example illustrates the above formulation.

Example (2.2.3): [23]

For the three-rowed partition $(p, q, 1)$, the terms of the resolution are:

$Res([p, q; 0]) \otimes \mathcal{D}_1 \oplus \underline{\mathcal{Z}}_{32} y Res([p, q + 1; 1]) \otimes \mathcal{D}_0$;

where $\underline{\mathcal{Z}}_{32} y$ is the complex

$$0 \longrightarrow \mathcal{Z}_{32} y \longrightarrow \mathcal{Z}_{32} \longrightarrow 0$$

From the terms of $\mathcal{M}_{\bullet}$ the only basically new terms are the terms which have the following formulation:

$$\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(k_1)} x \mathcal{Z}_{21}^{(k_2)} x \ldots \mathcal{Z}_{21}^{(k_{n-1})} x \mathcal{D}_{p+|k|} \otimes D_{q+1-|k|} \otimes D_0$$

By employing the boundary map $\partial_x + \partial_y$ on these terms we get some terms of the formulation:

$$\mathcal{Z}_{32} \mathcal{Z}_{21}^{(k_1)} x \mathcal{Z}_{21}^{(k_2)} x \ldots \mathcal{Z}_{21}^{(k_{n-1})} x \mathcal{D}_{p-|k|} \otimes D_{q+1-|k|} \otimes D_0$$

Then to obtain the resolution

$$\mathcal{M}_{\bullet}: \dots \mathcal{M}_{i} \longrightarrow \mathcal{M}_{i-1} \longrightarrow \dots \longrightarrow \mathcal{M}_{1} \longrightarrow \mathcal{M}_{0}$$

of $\mathcal{K}_{\lambda/\mu}$ recall that the quotient of Bar$(\mathcal{M}, \mathcal{A}; \mathcal{S}) = \overline{\mathcal{M}_{\bullet}}$ module is the following (Capelli identities) relations:

$$\mathcal{Z}_{32}^{(a)} \mathcal{Z}_{21}^{(b)} x = \sum_{k<b} \mathcal{Z}_{21}^{(b-k)} x\, \mathcal{Z}_{32}^{(a-k)} \partial_{31}^{(k)};$$

where the symbol $\partial_{31}^{(k)}$ means the divided power of the usual place polarization.

Resemble in the two-rowed case we write the module of the resolution as $\mathcal{M}_0 = \mathcal{D}_p \otimes D_q \otimes D_r$ and $\mathcal{M}_i$ with $i = 1,2,\dots$ for the terms which have dimension i (i.e. the number of the separators).

For the case of two-triple overlap, we survey it in the next chapter for the case of the partition (8,7,3) which is one of its applications.

Chapter Three

Resolution of Weyl Module in the Case of Partition (8,7,3)

Introduction

In this chapter we survey in specify an application of the resolution of three-rowed Weyl module for the case of the partition (8,7,3); where we find the terms of characteristic-free resolution in the first section, the terms and diagrams of Lascoux complex in the second section, while reduction from the characteristic-free resolution of Weyl module to characteristic-zero resolution (Lascoux resolution) find it in the third section. Eventually, in the last section we employing the mapping Cone and its diagrams to gain the characteristic-zero resolution (Lascoux resolution) and prove it to be exact without depending on the boundary maps.

3.1 The characteristic-free resolution in the case of partition (8,7,3)

We stratify the following formula for the case of partition $(\mathscr{p},\mathscr{q},\mathscr{r})$ to obtain the terms of the resolution for the partition (8,7,3), [15]

$$Res([\mathscr{p},\mathscr{q};0])\otimes\mathcal{D}_{\mathscr{r}}\oplus\sum_{e\geq 0}\underline{\mathcal{Z}}_{32}^{(e+1)}\mathscr{y}\,Res([\mathscr{p},\mathscr{q}+e+1;e+1])\otimes\mathcal{D}_{\mathscr{r}-e-1}\oplus$$
$$\sum_{e_1\geq 0,e_2\geq e_1}\underline{\mathcal{Z}}_{32}^{(e_2+1)}\mathscr{y}\underline{\mathcal{Z}}_{31}^{(e_1+1)}zRes([\mathscr{p}+e_1+1,\mathscr{q}+e_2+1;e_2-e_1])\otimes$$
$$\mathcal{D}_{\mathscr{r}-(e_1+e_2+2)};$$

where $\underline{\mathcal{Z}}_{ab}^{(m)}$ is the pursue Bar complex

$$0\to\underbrace{\mathcal{Z}_{ab}\mathscr{w}\mathcal{Z}_{ab}\mathscr{w}\ldots\mathcal{Z}_{ab}}_{m-times}\to\sum_{k_i\geq 1,\sum k_i=m}\mathcal{Z}_{ab}^{(k_1)}\mathscr{w}\mathcal{Z}_{ab}^{(k_2)}\mathscr{w}\ldots\mathcal{Z}_{ab}^{(k_{m-1})}\to\cdots\to\mathcal{Z}_{ab}^{(m)}\to 0$$

Hence the terms of the resolution for the case for the partition (8,7,3) is

$$Res([8,7;0])\otimes\mathcal{D}_3\oplus\sum_{e\geq 0}\underline{\mathcal{Z}}_{32}^{(e+1)}\mathscr{y}\,Res([8,7+e+1;e+1])\otimes\mathcal{D}_{3-e-1}\oplus$$
$$\sum_{e_1\geq 0,e_2\geq e_1}\underline{\mathcal{Z}}_{32}^{(e_2+1)}\mathscr{y}\underline{\mathcal{Z}}_{31}^{(e_1+1)}zRes([8+e_1+1,7+e_2+1;e_2-e_1])\otimes$$
$$\mathcal{D}_{3-(e_1+e_2+2)} \qquad \ldots(3.1.1)$$

So

$\sum_{e\geq 0} \underline{\mathcal{Z}}_{32}^{(e+1)} y\, Res([8,7+e+1; e+1]) \otimes \mathcal{D}_{3-e-1} =$

$\underline{\mathcal{Z}}_{32} y\, Res([8,8;1]) \otimes \mathcal{D}_2 \oplus \underline{\mathcal{Z}}_{32}^{(2)} y\, Res([8,9;2]) \otimes \mathcal{D}_1 \oplus \underline{\mathcal{Z}}_{32}^{(3)} y\, Res([8,10;3]) \otimes \mathcal{D}_0,$

and

$\sum_{e_1\geq 0, e_2\geq e_1} \underline{\mathcal{Z}}_{32}^{(e_2+1)} y \underline{\mathcal{Z}}_{31}^{(e_1+1)} z Res([8+e_1+1, 7+e_2+1; e_2-e_1]) \otimes$

$\mathcal{D}_{3-(e_1+e_2+2)} = \underline{\mathcal{Z}}_{32} y \underline{\mathcal{Z}}_{31} z Res([9,8;0]) \otimes \mathcal{D}_1 \oplus \underline{\mathcal{Z}}_{32}^{(2)} y \underline{\mathcal{Z}}_{31} z Res([9,9;1]) \otimes \mathcal{D}_0;$

where $\underline{\mathcal{Z}}_{32} y$ is the Bar complex

$$0 \longrightarrow \mathcal{Z}_{32} y \xrightarrow{\partial_y} \mathcal{Z}_{32} \longrightarrow 0$$

$\underline{\mathcal{Z}}_{32}^{(2)} y$ is the Bar complex

$$0 \longrightarrow \mathcal{Z}_{32} y \mathcal{Z}_{32} y \xrightarrow{\partial_y} \mathcal{Z}_{32}^{(2)} y \xrightarrow{\partial_y} \mathcal{Z}_{32}^{(2)} \longrightarrow 0$$

$\underline{\mathcal{Z}}_{32}^{(3)} y$ is the Bar complex

$$0 \longrightarrow \mathcal{Z}_{32} y \mathcal{Z}_{32} y \mathcal{Z}_{32} y \xrightarrow{\partial_y} \begin{matrix} \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{32} y \\ \oplus \\ \mathcal{Z}_{32} y \mathcal{Z}_{32}^{(2)} y \end{matrix} \xrightarrow{\partial_y} \mathcal{Z}_{32}^{(3)} y \xrightarrow{\partial_y} \mathcal{Z}_{32}^{(3)} \longrightarrow 0$$

and $\underline{\mathcal{Z}}_{31} z$ is the Bar complex

$$0 \longrightarrow \mathcal{Z}_{31} z \xrightarrow{\partial_z} \mathcal{Z}_{31} \longrightarrow 0;$$

where x, y and z stand for the separator variables, and the boundary map is $\partial_x + \partial_y + \partial_z$.

Let Bar $(\mathcal{M}, \mathcal{A}; \mathcal{S})$ be the free Bar module on the set $\mathcal{S} = \{x, y, z\}$; where $\mathcal{A}$ is the free associative algebra generated by $\mathcal{Z}_{21}, \mathcal{Z}_{32}$, and $\mathcal{Z}_{31}$ and their divided powers with the following relations:

$$\mathcal{Z}_{32}^{(a)} \mathcal{Z}_{31}^{(b)} = \mathcal{Z}_{31}^{(b)} \mathcal{Z}_{32}^{(a)} \quad \text{and} \quad \mathcal{Z}_{21}^{(a)} \mathcal{Z}_{31}^{(b)} = \mathcal{Z}_{31}^{(b)} \mathcal{Z}_{21}^{(a)}$$

And the module $\mathcal{M}$ is the direct sum of $\mathcal{D}_p \otimes \mathcal{D}_q \otimes \mathcal{D}_r$ for suitable p, q, and r with the action of $\mathcal{Z}_{21}, \mathcal{Z}_{32}$, and $\mathcal{Z}_{31}$ and their divided powers.

The terms of the characteristic-free resolution (3.1.1); where $b, b_1, b_2, b_3, b_4, b_5, b_6, b_7, c_1, c_2 \in \mathbb{Z}^+$ are:

∘ In dimension zero $(\mathcal{X}_0)$ we have $\mathcal{D}_8 \otimes \mathcal{D}_7 \otimes \mathcal{D}_3$.

∘ In dimension one $(\mathcal{X}_1)$ we have

- $\mathcal{Z}_{21}^{(b)} x \mathcal{D}_{8+b} \otimes \mathcal{D}_{7-b} \otimes \mathcal{D}_3$; where $1 \leq b \leq 7$.
- $\mathcal{Z}_{32}^{(b)} y \mathcal{D}_8 \otimes \mathcal{D}_{7+b} \otimes \mathcal{D}_{3-b}$; where $1 \leq b \leq 3$.

∘ In dimension two $(\mathcal{X}_2)$ we have the sum of the following terms:

- $\mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{D}_{8+|b|} \otimes \mathcal{D}_{7-|b|} \otimes \mathcal{D}_3$; where $2 \leq |b| = b_1 + b_2 \leq 7$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(b)} x \mathcal{D}_{8+b} \otimes \mathcal{D}_{8-b} \otimes \mathcal{D}_2$; where $2 \leq b \leq 8$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(b)} x \mathcal{D}_{8+b} \otimes \mathcal{D}_{9-b} \otimes \mathcal{D}_1$; where $3 \leq b \leq 9$.
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(b)} x \mathcal{D}_{8+b} \otimes \mathcal{D}_{10-b} \otimes \mathcal{D}_0$; where $4 \leq b \leq 10$.
- $\mathcal{Z}_{32}^{(b_1)} y \mathcal{Z}_{32}^{(b_2)} y \mathcal{D}_8 \otimes \mathcal{D}_{7+|b|} \otimes \mathcal{D}_{3-|b|}$; where $2 \leq |b| = b_1 + b_2 \leq 3$.
- $\mathcal{Z}_{32}^{(b)} y \mathcal{Z}_{31} z \mathcal{D}_9 \otimes \mathcal{D}_{7+b} \otimes \mathcal{D}_{2-b}$; where $1 \leq b \leq 2$.

∘ In dimension three $(\mathcal{X}_3)$ we have the sum of the following terms:

- $\mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{D}_{8+|b|} \otimes \mathcal{D}_{7-|b|} \otimes \mathcal{D}_3$; where $3 \leq |b| = \sum_{i=1}^{3} b_i \leq 7$ and $b_1 \geq 1$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{D}_{8+|b|} \otimes \mathcal{D}_{8-|b|} \otimes \mathcal{D}_2$; where $3 \leq |b| = b_1 + b_2 \leq 8$ and $b_1 \geq 2$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{D}_{8+|b|} \otimes \mathcal{D}_{9-|b|} \otimes \mathcal{D}_1$; where $4 \leq |b| = b_1 + b_2 \leq 9$ and $b_1 \geq 3$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(b)} x \mathcal{D}_{8+b} \otimes \mathcal{D}_{9-b} \otimes \mathcal{D}_1$; where $3 \leq b \leq 9$.
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{D}_{8+|b|} \otimes \mathcal{D}_{10-|b|} \otimes \mathcal{D}_0$; where $5 \leq |b| = b_1 + b_2 \leq 10$ and $b_1 \geq 4$.

- $\mathcal{Z}_{32}^{(c_1)}\mathcal{y}\mathcal{Z}_{32}^{(c_2)}\mathcal{y}\mathcal{Z}_{21}^{(b)}\mathcal{x}\mathcal{D}_{8+b}\otimes\mathcal{D}_{10-b}\otimes\mathcal{D}_0$; where $c_1+c_2=3$ and $4\le b\le 10$.
- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{D}_8\otimes\mathcal{D}_{10}\otimes\mathcal{D}_0$.
- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\mathcal{Z}_{21}^{(b)}\mathcal{x}\mathcal{D}_{9+b}\otimes\mathcal{D}_{8-b}\otimes\mathcal{D}_1$; where $1\le b\le 8$.
- $\mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\mathcal{Z}_{21}^{(b)}\mathcal{x}\mathcal{D}_{9+b}\otimes\mathcal{D}_{9-b}\otimes\mathcal{D}_0$; where $2\le b\le 9$.
- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\mathcal{D}_9\otimes\mathcal{D}_9\otimes\mathcal{D}_0$.

◦ In dimension four $(\mathcal{X}_4)$ we have the sum of the following terms:

- $\mathcal{Z}_{21}^{(b_1)}\mathcal{x}\mathcal{Z}_{21}^{(b_2)}\mathcal{x}\mathcal{Z}_{21}^{(b_3)}\mathcal{x}\mathcal{Z}_{21}^{(b_4)}\mathcal{x}\mathcal{D}_{8+|b|}\otimes\mathcal{D}_{7-|b|}\otimes\mathcal{D}_3$;where $4\le|b|=\sum_{i=1}^{4}b_i\le 7$ and $b_1\ge 1$.
- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(b_1)}\mathcal{x}\mathcal{Z}_{21}^{(b_2)}\mathcal{x}\mathcal{Z}_{21}^{(b_3)}\mathcal{x}\mathcal{D}_{8+|b|}\otimes\mathcal{D}_{8-|b|}\otimes\mathcal{D}_2$; where $4\le |b|=\sum_{i=1}^{3}b_i\le 8$ and $b_1\ge 2$.
- $\mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{21}^{(b_1)}\mathcal{x}\mathcal{Z}_{21}^{(b_2)}\mathcal{x}\mathcal{Z}_{21}^{(b_3)}\mathcal{x}\mathcal{D}_{8+|b|}\otimes\mathcal{D}_{9-|b|}\otimes\mathcal{D}_1$; where $5\le|b|=\sum_{i=1}^{3}b_i\le 9$ and $b_1\ge 3$.
- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(b_1)}\mathcal{x}\mathcal{Z}_{21}^{(b_2)}\mathcal{x}\mathcal{D}_{8+|b|}\otimes\mathcal{D}_{9-|b|}\otimes\mathcal{D}_1$; where $4\le|b|=b_1+b_2\le 9$; and $b_1\ge 3$.
- $\mathcal{Z}_{32}^{(3)}\mathcal{y}\mathcal{Z}_{21}^{(b_1)}\mathcal{x}\mathcal{Z}_{21}^{(b_2)}\mathcal{x}\mathcal{Z}_{21}^{(b_3)}\mathcal{x}\mathcal{D}_{8+|b|}\otimes\mathcal{D}_{10-|b|}\otimes\mathcal{D}_0$; where $6\le|b|=\sum_{i=1}^{3}b_i\le 10$ and $b_1\ge 4$.
- $\mathcal{Z}_{32}^{(c_1)}\mathcal{y}\mathcal{Z}_{32}^{(c_2)}\mathcal{y}\mathcal{Z}_{21}^{(b_1)}\mathcal{x}\mathcal{Z}_{21}^{(b_2)}\mathcal{x}\mathcal{D}_{8+|b|}\otimes\mathcal{D}_{10-|b|}\otimes\mathcal{D}_0$; where $c_1+c_2=3$, $5\le|b|=b_1+b_2\le 10$ and $b_1\ge 4$.
- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(b)}\mathcal{x}\mathcal{D}_{8+b}\otimes\mathcal{D}_{10-b}\otimes\mathcal{D}_0$; where $4\le b\le 10$.
- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\mathcal{Z}_{21}^{(b_1)}\mathcal{x}\mathcal{Z}_{21}^{(b_2)}\mathcal{x}\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{8-|b|}\otimes\mathcal{D}_1$; where $2\le|b|=b_1+b_2\le 8$ and $b_1\ge 1$.
- $\mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\mathcal{Z}_{21}^{(b_1)}\mathcal{x}\mathcal{Z}_{21}^{(b_2)}\mathcal{x}\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{9-|b|}\otimes\mathcal{D}_0$; where $3\le|b|=b_1+b_2\le 9$ and $b_1\ge 2$.
- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\mathcal{Z}_{21}^{(b)}\mathcal{x}\mathcal{D}_{9+b}\otimes\mathcal{D}_{9-b}\otimes\mathcal{D}_0$; where $2\le b\le 9$.

∘ In dimension five $(\mathcal{X}_5)$ we have the sum of the following terms:

- $\mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{Z}_{21}^{(b_5)} x \mathcal{D}_{8+|b|} \otimes \mathcal{D}_{7-|b|} \otimes \mathcal{D}_3$; where
  $5 \leq |b| = \sum_{i=1}^{5} b_i \leq 7$ and $b_1 \geq 1$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{D}_{8+|b|} \otimes \mathcal{D}_{8-|b|} \otimes \mathcal{D}_2$; where
  $5 \leq |b| = \sum_{i=1}^{4} b_i \leq 8$ and $b_1 \geq 2$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{D}_{8+|b|} \otimes \mathcal{D}_{9-|b|} \otimes \mathcal{D}_1$; where
  $6 \leq |b| = \sum_{i=1}^{4} b_i \leq 9$ and $b_1 \geq 3$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{D}_{8+|b|} \otimes \mathcal{D}_{9-|b|} \otimes \mathcal{D}_1$; where
  $5 \leq |b| = \sum_{i=1}^{3} b_i \leq 9$ and $b_1 \geq 3$.
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{D}_{8+|b|} \otimes \mathcal{D}_{10-|b|} \otimes \mathcal{D}_0$; where
  $7 \leq |b| = \sum_{i=1}^{4} b_i \leq 10$ and $b_1 \geq 4$.
- $\mathcal{Z}_{32}^{(c_1)} y \mathcal{Z}_{32}^{(c_2)} y \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{D}_{8+|b|} \otimes \mathcal{D}_{10-|b|} \otimes \mathcal{D}_0$; where $c_1 + c_2 = 3$,
  $6 \leq |b| = \sum_{i=1}^{3} b_i \leq 10$ and $b_1 \geq 4$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{32} y \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{D}_{8+|b|} \otimes \mathcal{D}_{10-|b|} \otimes \mathcal{D}_0$; where
  $5 \leq |b| = b_1 + b_2 \leq 10$ and $b_1 \geq 4$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{8-|b|} \otimes \mathcal{D}_1$; where
  $3 \leq |b| = \sum_{i=1}^{3} b_i \leq 8$ and $b_1 \geq 1$.
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{31} z \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{9-|b|} \otimes \mathcal{D}_0$; where
  $4 \leq |b| = \sum_{i=1}^{3} b_i \leq 9$ and $b_1 \geq 2$.
- $\mathcal{Z}_{32} y \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{D}_{9+|b|} \otimes \mathcal{D}_{9-|b|} \otimes \mathcal{D}_0$; where
  $3 \leq |b| = b_1 + b_2 \leq 9$ and $b_1 \geq 2$.

∘ In dimension six $(\mathcal{X}_6)$ we have the sum of the following terms:

- $\mathcal{Z}_{21}^{(b_1)} x \mathcal{Z}_{21}^{(b_2)} x \mathcal{Z}_{21}^{(b_3)} x \mathcal{Z}_{21}^{(b_4)} x \mathcal{Z}_{21}^{(b_5)} x \mathcal{Z}_{21}^{(b_6)} x \mathcal{D}_{8+|b|} \otimes \mathcal{D}_{7-|b|} \otimes \mathcal{D}_3$; where
  $6 \leq |b| = \sum_{i=1}^{6} b_i \leq 7$ and $b_1 \geq 1$.

- $\mathcal{Z}_{32}\mathcal{y}\, \mathcal{Z}_{21}^{(b_1)}x\mathcal{Z}_{21}^{(b_2)}x\mathcal{Z}_{21}^{(b_3)}x\mathcal{Z}_{21}^{(b_4)}x\mathcal{Z}_{21}^{(b_5)}x\mathcal{D}_{8+|b|}\otimes\mathcal{D}_{8-|b|}\otimes\mathcal{D}_2$; where $6 \le |b| = \sum_{i=1}^{5} b_i \le 8$ and $b_1 \ge 2$.
- $\mathcal{Z}_{32}^{(2)}\mathcal{y}\, \mathcal{Z}_{21}^{(b_1)}x\mathcal{Z}_{21}^{(b_2)}x\mathcal{Z}_{21}^{(b_3)}x\mathcal{Z}_{21}^{(b_4)}x\mathcal{Z}_{21}^{(b_5)}x\mathcal{D}_{8+|b|}\otimes\mathcal{D}_{9-|b|}\otimes\mathcal{D}_1$; where $7 \le |b| = \sum_{i=1}^{5} b_i \le 9$ and $b_1 \ge 3$.
- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(b_1)}x\mathcal{Z}_{21}^{(b_2)}x\mathcal{Z}_{21}^{(b_3)}x\mathcal{Z}_{21}^{(b_4)}x\mathcal{D}_{8+|b|}\otimes\mathcal{D}_{9-|b|}\otimes\mathcal{D}_1$; where $6 \le |b| = \sum_{i=1}^{4} b_i \le 9$ and $b_1 \ge 3$.
- $\mathcal{Z}_{32}^{(3)}\mathcal{y}\, \mathcal{Z}_{21}^{(b_1)}x\mathcal{Z}_{21}^{(b_2)}x\mathcal{Z}_{21}^{(b_3)}x\mathcal{Z}_{21}^{(b_4)}x\mathcal{Z}_{21}^{(b_5)}x\mathcal{D}_{8+|b|}\otimes\mathcal{D}_{10-|b|}\otimes\mathcal{D}_0$; where $8 \le |b| = \sum_{i=1}^{5} b_i \le 10$ and $b_1 \ge 4$.
- $\mathcal{Z}_{32}^{(c_1)}\mathcal{y}\mathcal{Z}_{32}^{(c_2)}\mathcal{y}\mathcal{Z}_{21}^{(b_1)}x\mathcal{Z}_{21}^{(b_2)}x\mathcal{Z}_{21}^{(b_3)}x\mathcal{Z}_{21}^{(b_4)}x\mathcal{D}_{8+|b|}\otimes\mathcal{D}_{10-|b|}\otimes\mathcal{D}_0$; where $c_1 + c_2 = 3$, $7 \le |b| = \sum_{i=1}^{4} b_i \le 10$ and $b_1 \ge 4$.
- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(b_1)}x\mathcal{Z}_{21}^{(b_2)}x\mathcal{Z}_{21}^{(b_3)}x\mathcal{D}_{8+|b|}\otimes\mathcal{D}_{10-|b|}\otimes\mathcal{D}_0$; where $6 \le |b| = \sum_{i=1}^{3} b_i \le 10$ and $b_1 \ge 4$.
- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(b_1)}x\mathcal{Z}_{21}^{(b_2)}x\mathcal{Z}_{21}^{(b_3)}x\mathcal{Z}_{21}^{(b_4)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{8-|b|}\otimes\mathcal{D}_1$; where $4 \le |b| = \sum_{i=1}^{4} b_i \le 8$ and $b_1 \ge 1$.
- $\mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(b_1)}x\mathcal{Z}_{21}^{(b_2)}x\mathcal{Z}_{21}^{(b_3)}x\mathcal{Z}_{21}^{(b_4)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{9-|b|}\otimes\mathcal{D}_0$; where $5 \le |b| = \sum_{i=1}^{4} b_i \le 9$ and $b_1 \ge 2$.
- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(b_1)}x\mathcal{Z}_{21}^{(b_2)}x\mathcal{Z}_{21}^{(b_3)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{9-|b|}\otimes\mathcal{D}_0$; where $4 \le |b| = \sum_{i=1}^{3} b_i \le 9$ and $b_1 \ge 2$.

◦ In dimension seven $(\mathcal{X}_7)$ we have the sum of the following terms:

- $\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{D}_{15}\otimes\mathcal{D}_0\otimes\mathcal{D}_3$.
- $\mathcal{Z}_{32}\mathcal{y}\, \mathcal{Z}_{21}^{(b_1)}x\mathcal{Z}_{21}^{(b_2)}x\mathcal{Z}_{21}^{(b_3)}x\mathcal{Z}_{21}^{(b_4)}x\mathcal{Z}_{21}^{(b_5)}x\mathcal{Z}_{21}^{(b_6)}x\mathcal{D}_{8+|b|}\otimes\mathcal{D}_{8-|b|}\otimes\mathcal{D}_2$; where $7 \le |b| = \sum_{i=1}^{6} b_i \le 8$ and $b_1 \ge 2$.
- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(b_1)}x\mathcal{Z}_{21}^{(b_2)}x\mathcal{Z}_{21}^{(b_3)}x\mathcal{Z}_{21}^{(b_4)}x\mathcal{Z}_{21}^{(b_5)}x\mathcal{D}_{8+|b|}\otimes\mathcal{D}_{9-|b|}\otimes\mathcal{D}_1$; where $7 \le |b| = \sum_{i=1}^{5} b_i \le 9$ and e $b_1 \ge 3$.

- $\mathcal{Z}_{32}^{(2)}\mathcal{Y}\,\mathcal{Z}_{21}^{(b_1)}x\mathcal{Z}_{21}^{(b_2)}x\mathcal{Z}_{21}^{(b_3)}x\mathcal{Z}_{21}^{(b_4)}x\mathcal{Z}_{21}^{(b_5)}x\mathcal{Z}_{21}^{(b_6)}x\mathcal{D}_{8+|b|}\otimes\mathcal{D}_{9-|b|}\otimes\mathcal{D}_1$; where $8 \leq |b| = \sum_{i=1}^{6} b_i \leq 9$ and $b_1 \geq 3$.
- $\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(b_1)}x\mathcal{Z}_{21}^{(b_2)}x\mathcal{Z}_{21}^{(b_3)}x\mathcal{Z}_{21}^{(b_4)}x\mathcal{D}_{8+|b|}\otimes\mathcal{D}_{10-|b|}\otimes\mathcal{D}_1$; where $7 \leq |b| = \sum_{i=1}^{4} b_i \leq 10$ and $b_1 \geq 4$.
- $\mathcal{Z}_{32}^{(c_1)}\mathcal{Y}\mathcal{Z}_{32}^{(c_2)}\mathcal{Y}\mathcal{Z}_{21}^{(b_1)}x\mathcal{Z}_{21}^{(b_2)}x\mathcal{Z}_{21}^{(b_3)}x\mathcal{Z}_{21}^{(b_4)}x\mathcal{Z}_{21}^{(b_5)}x\mathcal{D}_{8+|b|}\otimes\mathcal{D}_{10-|b|}\otimes\mathcal{D}_0$; where $c_1 + c_2 = 3,\ 8 \leq |b| = \sum_{i=1}^{5} b_i \leq 10$ and $b_1 \geq 4$.
- $\mathcal{Z}_{32}^{(3)}\mathcal{Y}\,\mathcal{Z}_{21}^{(b_1)}x\mathcal{Z}_{21}^{(b_2)}x\mathcal{Z}_{21}^{(b_3)}x\mathcal{Z}_{21}^{(b_4)}x\mathcal{Z}_{21}^{(b_5)}x\mathcal{Z}_{21}^{(b_6)}x\mathcal{D}_{8+|b|}\otimes\mathcal{D}_{10-|b|}\otimes\mathcal{D}_0$; where $9 \leq |b| = \sum_{i=1}^{6} b_i \leq 10$ and $b_1 \geq 4$.
- $\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(b_1)}x\mathcal{Z}_{21}^{(b_2)}x\mathcal{Z}_{21}^{(b_3)}x\mathcal{Z}_{21}^{(b_4)}x\mathcal{Z}_{21}^{(b_5)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{8-|b|}\otimes\mathcal{D}_1$; where $5 \leq |b| = \sum_{i=1}^{5} b_i \leq 8$ and $b_1 \geq 1$.
- $\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(b_1)}x\mathcal{Z}_{21}^{(b_2)}x\mathcal{Z}_{21}^{(b_3)}x\mathcal{Z}_{21}^{(b_4)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{9-|b|}\otimes\mathcal{D}_0$; where $5 \leq |b| = \sum_{i=1}^{4} b_i \leq 9$ and $b_1 \geq 2$.
- $\mathcal{Z}_{32}^{(2)}\mathcal{Y}\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(b_1)}x\mathcal{Z}_{21}^{(b_2)}x\mathcal{Z}_{21}^{(b_3)}x\mathcal{Z}_{21}^{(b_4)}x\mathcal{Z}_{21}^{(b_5)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{9-|b|}\otimes\mathcal{D}_0$; where $6 \leq |b| = \sum_{i=1}^{5} b_i \leq 9$ and $b_1 \geq 2$.

◦ In dimension eight $(\mathcal{X}_8)$ we have the sum of the following terms:

- $\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{D}_{16}\otimes\mathcal{D}_0\otimes\mathcal{D}_2$.
- $\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{32}\mathcal{Y}\,\mathcal{Z}_{21}^{(b_1)}x\mathcal{Z}_{21}^{(b_2)}x\mathcal{Z}_{21}^{(b_3)}x\mathcal{Z}_{21}^{(b_4)}x\mathcal{Z}_{21}^{(b_5)}x\mathcal{Z}_{21}^{(b_6)}x\mathcal{D}_{8+|b|}\otimes\mathcal{D}_{9-|b|}\otimes\mathcal{D}_2$; where $8 \leq |b| = \sum_{i=1}^{6} b_i \leq 9$ and $b_1 \geq 3$.
- $\mathcal{Z}_{32}^{(2)}\mathcal{Y}\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{D}_{17}\otimes\mathcal{D}_0\otimes\mathcal{D}_1$.
- $\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(b_1)}x\mathcal{Z}_{21}^{(b_2)}x\mathcal{Z}_{21}^{(b_3)}x\mathcal{Z}_{21}^{(b_4)}x\mathcal{Z}_{21}^{(b_5)}x\mathcal{D}_{8+|b|}\otimes\mathcal{D}_{10-|b|}\otimes\mathcal{D}_0$; where $8 \leq |b| = \sum_{i=1}^{5} b_i \leq 10$ and $b_1 \geq 4$.
- $\mathcal{Z}_{32}^{(c_1)}\mathcal{Y}\mathcal{Z}_{32}^{(c_2)}\mathcal{Y}\mathcal{Z}_{21}^{(b_1)}x\mathcal{Z}_{21}^{(b_2)}x\mathcal{Z}_{21}^{(b_3)}x\mathcal{Z}_{21}^{(b_4)}x\mathcal{Z}_{21}^{(b_5)}x\mathcal{Z}_{21}^{(b_6)}x\mathcal{D}_{8+|b|}\otimes\mathcal{D}_{10-|b|}\otimes\mathcal{D}_0$; where $c_1 + c_2 = 3,\ 9 \leq |b| = \sum_{i=1}^{6} b_i \leq 10$ and $b_1 \geq 4$.
- $\mathcal{Z}_{32}^{(3)}\mathcal{Y}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{D}_{15}\otimes\mathcal{D}_2\otimes\mathcal{D}_1$.

- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(b_1)}x\mathcal{Z}_{21}^{(b_2)}x\mathcal{Z}_{21}^{(b_3)}x\mathcal{Z}_{21}^{(b_4)}x\mathcal{Z}_{21}^{(b_5)}x\mathcal{Z}_{21}^{(b_6)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{8-|b|}\otimes\mathcal{D}_1$;
  where $7 \le |b| = \sum_{i=1}^{6} b_i \le 8$ and $b_1 \ge 2$.
- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(b_1)}x\mathcal{Z}_{21}^{(b_2)}x\mathcal{Z}_{21}^{(b_3)}x\mathcal{Z}_{21}^{(b_4)}x\mathcal{Z}_{21}^{(b_5)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{9-|b|}\otimes\mathcal{D}_0$;
  where $6 \le |b| = \sum_{i=1}^{5} b_i \le 9$ and $b_1 \ge 2$.
- $\mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(b_1)}x\mathcal{Z}_{21}^{(b_2)}x\mathcal{Z}_{21}^{(b_3)}x\mathcal{Z}_{21}^{(b_4)}x\mathcal{Z}_{21}^{(b_5)}x\mathcal{Z}_{21}^{(b_6)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{9-|b|}\otimes\mathcal{D}_0$;
  where $7 \le |b| = \sum_{i=1}^{6} b_i \le 9$ and $b_1 \ge 2$.

∘ In dimension nine $(\mathcal{X}_9)$ we have the sum of the following terms:

- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{D}_{17}\otimes\mathcal{D}_0\otimes\mathcal{D}_1$.
- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(b_1)}x\mathcal{Z}_{21}^{(b_2)}x\mathcal{Z}_{21}^{(b_3)}x\mathcal{Z}_{21}^{(b_4)}x\mathcal{Z}_{21}^{(b_5)}x\mathcal{Z}_{21}^{(b_6)}x\mathcal{D}_{8+|b|}\otimes\mathcal{D}_{10-|b|}\otimes\mathcal{D}_0$;
  where $9 \le |b| = \sum_{i=1}^{6} b_i \le 10$ and $b_1 \ge 4$.
- $\mathcal{Z}_{32}^{(c_1)}\mathcal{y}\mathcal{Z}_{32}^{(c_2)}\mathcal{y}\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{D}_{18}\otimes\mathcal{D}_0\otimes\mathcal{D}_0$; where
  $c_1 + c_2 = 3$.
- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(b_1)}x\mathcal{Z}_{21}^{(b_2)}x\mathcal{Z}_{21}^{(b_3)}x\mathcal{Z}_{21}^{(b_4)}x\mathcal{Z}_{21}^{(b_5)}x\mathcal{Z}_{21}^{(b_6)}x\mathcal{Z}_{21}^{(b_7)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{8-|b|}\otimes\mathcal{D}_1$;
  where $7 \le |b| = \sum_{i=1}^{6} b_i \le 8$ and $b_1 \ge 1$.
- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(b_1)}x\mathcal{Z}_{21}^{(b_2)}x\mathcal{Z}_{21}^{(b_3)}x\mathcal{Z}_{21}^{(b_4)}x\mathcal{Z}_{21}^{(b_5)}x\mathcal{Z}_{21}^{(b_6)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{9-|b|}\otimes\mathcal{D}_0$;
  where $7 \le |b| = \sum_{i=1}^{5} b_i \le 9$ and $b_1 \ge 2$.
- $\mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(b_1)}x\mathcal{Z}_{21}^{(b_2)}x\mathcal{Z}_{21}^{(b_3)}x\mathcal{Z}_{21}^{(b_4)}x\mathcal{Z}_{21}^{(b_5)}x\mathcal{Z}_{21}^{(b_6)}x\mathcal{Z}_{21}^{(b_7)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{9-|b|}\otimes$
  $\mathcal{D}_0$; where $8 \le |b| = \sum_{i=1}^{7} b_i \le 9$ and $b_1 \ge 1$.

∘ In dimension ten $(\mathcal{X}_{10})$ we have the sum of the following terms:

- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{D}_{18}\otimes\mathcal{D}_0\otimes\mathcal{D}_0$.
- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{D}_{17}\otimes\mathcal{D}_0\otimes\mathcal{D}_1$.
- $\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(b_1)}x\mathcal{Z}_{21}^{(b_2)}x\mathcal{Z}_{21}^{(b_3)}x\mathcal{Z}_{21}^{(b_4)}x\mathcal{Z}_{21}^{(b_5)}x\mathcal{Z}_{21}^{(b_6)}x\mathcal{Z}_{21}^{(b_7)}x\mathcal{D}_{9+|b|}\otimes\mathcal{D}_{9-|b|}$
  $\otimes\mathcal{D}_0$; where $8 \le |b| = \sum_{i=1}^{5} b_i \le 9$ and $b_1 \ge 2$.
- $\mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{D}_{18}\otimes\mathcal{D}_0\otimes\mathcal{D}_0$.

Finally, in dimension eleven ($\mathcal{X}_{11}$) we have

- $\mathcal{Z}_{32}y\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{D}_{18}\otimes\mathcal{D}_{0}\otimes\mathcal{D}_{0}$.

3.2 Complex of Lascoux in the case of partition (8,7,3)

Proposition (3.2.1): The terms of Lascoux complex

The terms of the Lascoux complex are obtained by the determinantal expansion of the Jacobi-Trudi matrix of the partition, [3]. The positions of the terms of the complex are determined by the length of the permutation to which they correspond, [4].

In the case of partition (p, q, r) the matrix is

$$\begin{bmatrix} \mathcal{D}_{p}\mathcal{F} & \mathcal{D}_{q-1}\mathcal{F} & \mathcal{D}_{r-2}\mathcal{F} \\ \mathcal{D}_{p+1}\mathcal{F} & \mathcal{D}_{q}\mathcal{F} & \mathcal{D}_{r-1}\mathcal{F} \\ \mathcal{D}_{p+2}\mathcal{F} & \mathcal{D}_{q+1}\mathcal{F} & \mathcal{D}_{r}\mathcal{F} \end{bmatrix}$$

In our case i.e. (8,7,3) we have the following matrix:

$$\begin{bmatrix} \mathcal{D}_{8}\mathcal{F} & \mathcal{D}_{6}\mathcal{F} & \mathcal{D}_{1}\mathcal{F} \\ \mathcal{D}_{9}\mathcal{F} & \mathcal{D}_{7}\mathcal{F} & \mathcal{D}_{2}\mathcal{F} \\ \mathcal{D}_{10}\mathcal{F} & \mathcal{D}_{8}\mathcal{F} & \mathcal{D}_{3}\mathcal{F} \end{bmatrix}$$

Then the Lascaux complex has the correspondence between its terms as follows:

$\mathcal{D}_{8}\mathcal{F} \otimes \mathcal{D}_{7}\mathcal{F} \otimes \mathcal{D}_{3}\mathcal{F} \leftrightarrow$ identity

$\mathcal{D}_{9}\mathcal{F} \otimes \mathcal{D}_{6}\mathcal{F} \otimes \mathcal{D}_{3}\mathcal{F} \leftrightarrow (12)$

$\mathcal{D}_{8}\mathcal{F} \otimes \mathcal{D}_{8}\mathcal{F} \otimes \mathcal{D}_{2}\mathcal{F} \leftrightarrow (23)$

$\mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_{6}\mathcal{F} \otimes \mathcal{D}_{2}\mathcal{F} \leftrightarrow (132)$

$\mathcal{D}_{9}\mathcal{F} \otimes \mathcal{D}_{8}\mathcal{F} \otimes \mathcal{D}_{1}\mathcal{F} \leftrightarrow (123)$

$\mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_{7}\mathcal{F} \otimes \mathcal{D}_{1}\mathcal{F} \leftrightarrow (13)$

Thus the formulation of the Lascoux resolution in the case of the partition (8,7,3) is

$$\mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_{7}\mathcal{F} \otimes \mathcal{D}_{1}\mathcal{F} \longrightarrow \begin{matrix} \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_{6}\mathcal{F} \otimes \mathcal{D}_{2}\mathcal{F} \\ \oplus \\ \mathcal{D}_{9}\mathcal{F} \otimes \mathcal{D}_{8}\mathcal{F} \otimes \mathcal{D}_{1}\mathcal{F} \end{matrix} \longrightarrow \begin{matrix} \mathcal{D}_{9}\mathcal{F} \otimes \mathcal{D}_{6}\mathcal{F} \otimes \mathcal{D}_{3}\mathcal{F} \\ \oplus \\ \mathcal{D}_{8}\mathcal{F} \otimes \mathcal{D}_{8}\mathcal{F} \otimes \mathcal{D}_{2}\mathcal{F} \end{matrix} \longrightarrow \mathcal{D}_{8}\mathcal{F} \otimes \mathcal{D}_{7}\mathcal{F} \otimes \mathcal{D}_{3}\mathcal{F}$$

Proposition (3.2.2): Diagrams of the complex of Lascoux

Consider the following diagram:

$$\begin{array}{ccccc} \mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_7\mathcal{F}\otimes\mathcal{D}_1\mathcal{F} & \xrightarrow{d_1} & \mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_6\mathcal{F}\otimes\mathcal{D}_2\mathcal{F} & \xrightarrow{d_2} & \mathcal{D}_9\mathcal{F}\otimes\mathcal{D}_6\mathcal{F}\otimes\mathcal{D}_3\mathcal{F} \\ \hbar_1\downarrow & \mathcal{Q} & \hbar_2\downarrow & \mathcal{T} & \downarrow\hbar_3 \\ \mathcal{D}_9\mathcal{F}\otimes\mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_1\mathcal{F} & \xrightarrow{g_1} & \mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_2\mathcal{F} & \xrightarrow{g_2} & \mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_7\mathcal{F}\otimes\mathcal{D}_3\mathcal{F} \end{array}$$

If we define

$d_1: \mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_7\mathcal{F}\otimes\mathcal{D}_1\mathcal{F}\longrightarrow\mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_6\mathcal{F}\otimes\mathcal{D}_2\mathcal{F}$ as

$d_1(v)=\partial_{32}(v)$; where $v\in\mathcal{D}_{10}\otimes\mathcal{D}_7\otimes\mathcal{D}_1$

$\hbar_1: \mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_7\mathcal{F}\otimes\mathcal{D}_1\mathcal{F}\longrightarrow\mathcal{D}_9\mathcal{F}\otimes\mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_1\mathcal{F}$ as

$\hbar_1(v)=\partial_{21}(v)$; where $v\in\mathcal{D}_{10}\otimes\mathcal{D}_7\otimes\mathcal{D}_1$,

and

$\hbar_2: \mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_6\mathcal{F}\otimes\mathcal{D}_2\mathcal{F}\longrightarrow\mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_2\mathcal{F}$ as

$\hbar_2(v)=\partial_{21}^{(2)}(v)$; where $v\in\mathcal{D}_{10}\otimes\mathcal{D}_6\otimes\mathcal{D}_2$.

Now, we have to define the map

$g_1: \mathcal{D}_9\mathcal{F}\otimes\mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_1\mathcal{F}\longrightarrow\mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_2\mathcal{F}$,

which make the diagram $\mathcal{Q}$ commute, i.e.

$g_1\circ\hbar_1=\hbar_2\circ d_1$

Which implies that

$g_1\circ\partial_{21}=\partial_{21}^{(2)}\partial_{32}$

By employing Capelli identities we get:

$$\begin{aligned}\partial_{21}^{(2)}\partial_{32} &= \partial_{32}\partial_{21}^{(2)}-\partial_{31}\partial_{21}\\ &= (\tfrac{1}{2}\partial_{32}\partial_{21}-\partial_{31})\partial_{21}\end{aligned}$$

Thus, $g_1=\frac{1}{2}\partial_{32}\partial_{21}-\partial_{31}$

On the other hand, if we define the map

$g_2: \mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_2\mathcal{F}\longrightarrow\mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_7\mathcal{F}\otimes\mathcal{D}_3\mathcal{F}$ as

$g_2(v)=\partial_{32}(v)$; where $v\in\mathcal{D}_8\otimes\mathcal{D}_8\otimes\mathcal{D}_2$,

and

$\hbar_3: \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} \longrightarrow \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_3\mathcal{F}$ as

$\hbar_3(v) = \partial_{21}(v)$; where $v \in \mathcal{D}_{10} \otimes \mathcal{D}_6 \otimes \mathcal{D}_2$

We need to define d_2 to make the diagram $\mathcal{T}$ commute

$d_2: \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} \longrightarrow \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_3\mathcal{F}$ such that

$\hbar_3 \circ d_2 = g_2 \circ \hbar_2$ i.e. $\partial_{21} \circ d_2 = \partial_{32}\partial_{21}^{(2)}$

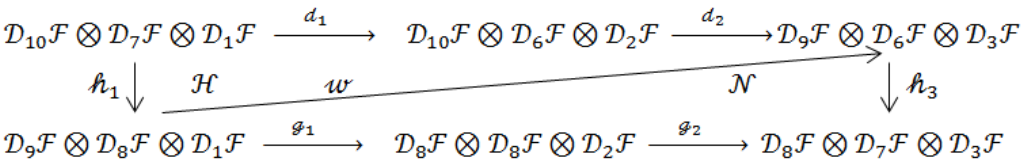
By employing Capelli identities we have

$$\partial_{32}\partial_{21}^{(2)} = \partial_{21}^{(2)}\partial_{32} + \partial_{21}\partial_{31}$$

$$= \partial_{21}\left(\frac{1}{2}\partial_{21}\partial_{32} + \partial_{31}\right)$$

Thus, $d_2 = \frac{1}{2}\partial_{21}\partial_{32} + \partial_{31}$

Now consider the following diagram:



Define $w: \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_1\mathcal{F} \longrightarrow \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_3\mathcal{F}$ by

$w(v) = \partial_{32}^{(2)}(v)$; where $v \in \mathcal{D}_9 \otimes \mathcal{D}_8 \otimes \mathcal{D}_1$

Remark (3.2.3):

The diagram $\mathcal{H}$ is commute.

Proof:

To prove the diagram $\mathcal{H}$ is commute, it is sufficient to prove that

$$d_2 \circ d_1 = w \circ \hbar_1$$

$$d_2 \circ d_1 = \left(\frac{1}{2}\partial_{21}\partial_{32} + \partial_{31}\right)\partial_{32}$$

$$= \partial_{21}\partial_{32}^{(2)} + \partial_{31}\partial_{32}$$

$$= \partial_{32}^{(2)}\partial_{21} - \partial_{32}\partial_{31} + \partial_{31}\partial_{32}$$

$$= \partial_{32}^{(2)}\partial_{21}$$

$$= w \circ \hbar_1 \quad \blacksquare$$

Remark (3.2.4):

The diagram $\mathcal{N}$ is commute.

Proof:

$$g_2 \circ g_1 = \partial_{32}(\tfrac{1}{2}\partial_{32}\partial_{21} - \partial_{31})$$

$$= \partial_{32}^{(2)}\partial_{21} - \partial_{32}\partial_{31}$$

$$= \partial_{21}\partial_{32}^{(2)} + \partial_{32}\partial_{31} - \partial_{32}\partial_{31}$$

$$= \partial_{21}\partial_{32}^{(2)}$$

$$= h_3 \circ w \quad \blacksquare$$

Eventually, we define the maps σ_1, σ_2, and σ_3 as follows:

- $\sigma_3(x) = \big(d_1(x), h_1(x)\big); \forall\, x \in \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_1\mathcal{F}$

- $\sigma_2((x_1, x_2)) = \big(d_2(x_1) - w(x_2), g_1(x_2) - h_2(x_1)\big)$;
  $\forall\, x \in \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} \oplus \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_1\mathcal{F}$

- $\sigma_1((x_1, x_2)) = \big(h_3(x_1) + g_2(x_2)\big)$;
  $\forall\, x \in \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} \oplus \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_2\mathcal{F}$;

where

$$\sigma_3:\ \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_1\mathcal{F} \longrightarrow \begin{array}{c} \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} \\ \oplus \\ \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_1\mathcal{F} \end{array}$$

$$\sigma_2:\ \begin{array}{c} \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} \\ \oplus \\ \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_1\mathcal{F} \end{array} \longrightarrow \begin{array}{c} \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} \\ \oplus \\ \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} \end{array}$$

and

$$\sigma_1:\ \begin{array}{c} \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} \\ \oplus \\ \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} \end{array} \longrightarrow \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_3\mathcal{F}$$

Lemma (3.2.5):

$$0 \longrightarrow \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_1\mathcal{F} \xrightarrow{\sigma_3} \begin{matrix} \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} \\ \oplus \\ \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_1\mathcal{F} \end{matrix} \xrightarrow{\sigma_2} \begin{matrix} \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} \\ \oplus \\ \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} \end{matrix} \xrightarrow{\sigma_1} \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_3\mathcal{F}$$

is complex.

Proof:

From the definition of ∂_{21} and ∂_{32} they are injective [10], then we get σ_3 is injective.

$$(\sigma_2 \circ \sigma_3)(x) = \sigma_2\big(d_1(x), h_1(x)\big)$$
$$= \sigma_2\big(\partial_{32}(x), \partial_{21}(x)\big)$$
$$= \Big(d_2\big(\partial_{32}(x)\big) - w\big(\partial_{21}(x)\big), g_1\big(\partial_{21}(x)\big) - h_2\big(\partial_{32}(x)\big)\Big)$$

$$d_2\big(\partial_{32}(x)\big) - w\big(\partial_{21}(x)\big) = \left(\tfrac{1}{2}\partial_{21}\partial_{32} + \partial_{31}\right)\partial_{32}(x) - \partial_{32}^{(2)}\partial_{21}(x)$$
$$= (\partial_{21}\partial_{32}^{(2)} + \partial_{31}\partial_{32} - \partial_{32}^{(2)}\partial_{21})(x)$$
$$= (\partial_{32}^{(2)}\partial_{21} - \partial_{32}\partial_{31} + \partial_{31}\partial_{32} - \partial_{32}^{(2)}\partial_{21})(x)$$
$$= 0$$

$$g_1\big(\partial_{21}(x)\big) - h_2\big(\partial_{32}(x)\big) = \left(\tfrac{1}{2}\partial_{32}\partial_{21} - \partial_{31}\right)\partial_{21}(x) - \partial_{21}^{(2)}\partial_{32}(x)$$
$$= (\partial_{32}\partial_{21}^{(2)} - \partial_{31}\partial_{21} - \partial_{21}^{(2)}\partial_{32})(x)$$
$$= (\partial_{21}^{(2)}\partial_{32} + \partial_{21}\partial_{31} - \partial_{31}\partial_{21} - \partial_{21}^{(2)}\partial_{32})(x)$$
$$= 0$$

We have $(\sigma_2 \circ \sigma_3)(x) = 0$, and

$$(\sigma_1 \circ \sigma_2)(x_1, x_2) = \sigma_1(d_2(x_1) - w(x_2), g_1(x_2) - h_2(x_1))$$
$$= \sigma_1((\tfrac{1}{2}\partial_{21}\partial_{32} + \partial_{31})(x_1) - \partial_{32}^{(2)}(x_2), (\tfrac{1}{2}\partial_{32}\partial_{21} - \partial_{31})(x_2) - \partial_{21}^{(2)}(x_1))$$
$$= \partial_{21}(\frac{1}{2}\partial_{21}\partial_{32} + \partial_{31})(x_1) - \partial_{32}^{(2)}(x_2)) + \partial_{32}((\frac{1}{2}\partial_{32}\partial_{21} - \partial_{31})(x_2) - \partial_{21}^{(2)}(x_1))$$
$$= \left(\partial_{21}^{(2)}\partial_{32} + \partial_{21}\partial_{31} - \partial_{32}\partial_{21}^{(2)}\right)(x_1) + \left(\partial_{32}^{(2)}\partial_{21} - \partial_{32}\partial_{31} - \partial_{21}\partial_{32}^{(2)}\right)(x_2)$$
$$= \left(\partial_{32}\partial_{21}^{(2)} - \partial_{21}\partial_{31} + \partial_{21}\partial_{31} - \partial_{32}\partial_{21}^{(2)}\right)(x_1) +$$
$$\left(\partial_{32}^{(2)}\partial_{21} + \partial_{32}\partial_{31} - \partial_{32}\partial_{31} - \partial_{21}\partial_{32}^{(2)}\right)(x_2) = 0 \quad \blacksquare$$

3.3 Reduction from characteristic-free resolution to Lascoux resolution in the case of partition (8,7,3)

This section exhibits the connection between the characteristic-free resolution and the Lascoux resolution of the partition (8,7,3).

The Lascoux resolution of the partition (8,7,3) has the formulation

$$\mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_7\mathcal{F}\otimes\mathcal{D}_1\mathcal{F}\longrightarrow\begin{array}{c}\mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_6\mathcal{F}\otimes\mathcal{D}_2\mathcal{F}\\ \oplus\\ \mathcal{D}_9\mathcal{F}\otimes\mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_1\mathcal{F}\end{array}\longrightarrow\begin{array}{c}\mathcal{D}_9\mathcal{F}\otimes\mathcal{D}_6\mathcal{F}\otimes\mathcal{D}_3\mathcal{F}\\ \oplus\\ \mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_2\mathcal{F}\end{array}\longrightarrow\mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_7\mathcal{F}\otimes\mathcal{D}_3\mathcal{F}$$

As in [18], we exhibit the terms of the complex (3.1.1) as:

$\mathcal{X}_0=\mathcal{L}_0=\mathcal{M}_0$

$\mathcal{X}_1=\mathcal{L}_1\oplus\mathcal{M}_1$

$\mathcal{X}_2=\mathcal{L}_2\oplus\mathcal{M}_2$

$\mathcal{X}_3=\mathcal{L}_3\oplus\mathcal{M}_3$

$\mathcal{X}_{\mathrm{j}}=\mathcal{M}_{\mathrm{j}}$; for $\mathrm{j}=4,5,\dots,11$;

where $\mathcal{L}_e$ are the sum of the Lascoux terms and $\mathcal{M}_e$ are the sum of the others.

Now, we define the map $\sigma_1:\mathcal{M}_1\longrightarrow\mathcal{L}_1$ such that

$\delta_{\mathcal{L}_1\mathcal{L}_0}\circ\sigma_1=\delta_{\mathcal{M}_1\mathcal{M}_0}$ …(3.3.1)

As follows:

- $\mathcal{Z}_{21}^{(2)}x(v)\longmapsto\frac{1}{2}\,\mathcal{Z}_{21}x\partial_{21}(v)$; where $v\in\mathcal{D}_{10}\otimes\mathcal{D}_5\otimes\mathcal{D}_3$
- $\mathcal{Z}_{21}^{(3)}x(v)\longmapsto\frac{1}{3}\,\mathcal{Z}_{21}x\partial_{21}^{(2)}(v)$; where $v\in\mathcal{D}_{11}\otimes\mathcal{D}_4\otimes\mathcal{D}_3$
- $\mathcal{Z}_{21}^{(4)}x(v)\longmapsto\frac{1}{4}\,\mathcal{Z}_{21}x\partial_{21}^{(3)}(v)$; where $v\in\mathcal{D}_{12}\otimes\mathcal{D}_3\otimes\mathcal{D}_3$
- $\mathcal{Z}_{21}^{(5)}x(v)\longmapsto\frac{1}{5}\,\mathcal{Z}_{21}x\partial_{21}^{(4)}(v)$; where $v\in\mathcal{D}_{13}\otimes\mathcal{D}_2\otimes\mathcal{D}_3$
- $\mathcal{Z}_{21}^{(6)}x(v)\longmapsto\frac{1}{6}\,\mathcal{Z}_{21}x\partial_{21}^{(5)}(v)$; where $v\in\mathcal{D}_{14}\otimes\mathcal{D}_1\otimes\mathcal{D}_3$
- $\mathcal{Z}_{21}^{(7)}x(v)\longmapsto\frac{1}{7}\,\mathcal{Z}_{21}x\partial_{21}^{(6)}(v)$; where $v\in\mathcal{D}_{15}\otimes\mathcal{D}_0\otimes\mathcal{D}_3$
- $\mathcal{Z}_{32}^{(2)}y(v)\longmapsto\frac{1}{2}\,\mathcal{Z}_{32}y\partial_{32}(v)$; where $v\in\mathcal{D}_8\otimes\mathcal{D}_9\otimes\mathcal{D}_1$
- $\mathcal{Z}_{32}^{(3)}y(v)\longmapsto\frac{1}{3}\,\mathcal{Z}_{32}y\partial_{32}^{(2)}(v)$; where $v\in\mathcal{D}_8\otimes\mathcal{D}_{10}\otimes\mathcal{D}_0$

It is clear that σ_1 satisfies (3.3.1), then we can define

$$\partial_1 : \mathcal{L}_1 \longrightarrow \mathcal{L}_0 \quad \text{as} \quad \partial_1 = \delta_{\mathcal{L}_1\mathcal{L}_0}$$

At this point, we are in a position to define

$$\partial_2 : \mathcal{L}_2 \longrightarrow \mathcal{L}_1 \quad \text{by} \quad \partial_2 = \delta_{\mathcal{L}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1}$$

Lemma (3.3.1):

The composition $\partial_1\partial_2$ equal to zero.

Proof:

$$\partial_1\partial_2(a) = \delta_{\mathcal{L}_1\mathcal{L}_0} \circ \left(\delta_{\mathcal{L}_2\mathcal{L}_1}(a) + (\sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1})(a)\right)$$

$$= \delta_{\mathcal{L}_1\mathcal{L}_0} \circ \delta_{\mathcal{L}_2\mathcal{L}_1}(a) + \delta_{\mathcal{L}_1\mathcal{L}_0} \circ (\sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1})(a)$$

But $\delta_{\mathcal{L}_1\mathcal{L}_0} \circ \sigma_1 = \delta_{\mathcal{M}_1\mathcal{M}_0}$ then we get

$$\partial_1\partial_2(a) = \delta_{\mathcal{L}_1\mathcal{L}_0} \circ \delta_{\mathcal{L}_2\mathcal{L}_1}(a) + \delta_{\mathcal{M}_1\mathcal{M}_0} \circ \delta_{\mathcal{L}_2\mathcal{M}_1}(a)$$

By properties of the boundary map δ we get

$\partial_1\partial_2 = 0$ ■

We need to define the map $\sigma_2 : \mathcal{M}_2 \longrightarrow \mathcal{L}_2$ such that

$$\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1} = \left(\delta_{\mathcal{L}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1}\right) \circ \sigma_2 \qquad \ldots(3.3.2)$$

As follows:

- $\mathcal{Z}_{21}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{10} \otimes \mathcal{D}_5 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_4 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_4 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_3 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(3)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_3 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_3 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(4)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$

- $\mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(2)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_{2} \otimes \mathcal{D}_{3}$
- $\mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(3)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_{2} \otimes \mathcal{D}_{3}$
- $\mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_{1} \otimes \mathcal{D}_{3}$
- $\mathcal{Z}_{21} x \mathcal{Z}_{21}^{(5)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_{1} \otimes \mathcal{D}_{3}$
- $\mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(3)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_{1} \otimes \mathcal{D}_{3}$
- $\mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(2)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_{1} \otimes \mathcal{D}_{3}$
- $\mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(4)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_{1} \otimes \mathcal{D}_{3}$
- $\mathcal{Z}_{21}^{(6)} x \mathcal{Z}_{21} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_{0} \otimes \mathcal{D}_{3}$
- $\mathcal{Z}_{21} x \mathcal{Z}_{21}^{(6)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_{0} \otimes \mathcal{D}_{3}$
- $\mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21}^{(2)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_{0} \otimes \mathcal{D}_{3}$
- $\mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(5)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_{0} \otimes \mathcal{D}_{3}$
- $\mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(3)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_{0} \otimes \mathcal{D}_{3}$
- $\mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(4)} x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_{0} \otimes \mathcal{D}_{3}$
- $\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(3)} x(v) \longmapsto \frac{1}{3}\, \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}(v)$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_{5} \otimes \mathcal{D}_{2}$
- $\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(4)} x(v) \longmapsto \frac{1}{6}\, \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(2)}(v)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_{4} \otimes \mathcal{D}_{2}$
- $\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(5)} x(v) \longmapsto \frac{1}{10}\, \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(3)}(v)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_{3} \otimes \mathcal{D}_{2}$
- $\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(6)} x(v) \longmapsto \frac{1}{15}\, \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(4)}(v)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_{2} \otimes \mathcal{D}_{2}$
- $\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(7)} x(v) \longmapsto \frac{1}{21}\, \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_{1} \otimes \mathcal{D}_{2}$
- $\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(8)} x(v) \longmapsto \frac{1}{28}\, \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(6)}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_{0} \otimes \mathcal{D}_{2}$
- $\mathcal{Z}_{32} y \mathcal{Z}_{32} y(v) \longmapsto 0$; where $v \in \mathcal{D}_{8} \otimes \mathcal{D}_{9} \otimes \mathcal{D}_{1}$
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(3)} x(v) \longmapsto \frac{1}{3}\, (\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{31}(v) - \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(2)}(v))$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_{6} \otimes \mathcal{D}_{1}$
- $\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(4)} x(v) \longmapsto \frac{1}{12}\, \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21} \partial_{31}(v) - \frac{1}{4} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(3)}(v)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_{5} \otimes \mathcal{D}_{1}$

- $\mathcal{Z}_{32}^{(2)}\mathcal{Y}\mathcal{Z}_{21}^{(5)}x(v) \longmapsto \frac{1}{30}\ \mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}(v) - \frac{1}{5}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}z\partial_{21}^{(4)}(v)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_4 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}^{(2)}\mathcal{Y}\mathcal{Z}_{21}^{(6)}x(v) \longmapsto \frac{1}{60}\ \mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}(v) - \frac{1}{6}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}z\partial_{21}^{(5)}(v)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_3 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}^{(2)}\mathcal{Y}\mathcal{Z}_{21}^{(7)}x(v) \longmapsto \frac{1}{105}\ \mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}(v) - \frac{1}{7}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}z\partial_{21}^{(6)}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_2 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}^{(2)}\mathcal{Y}\mathcal{Z}_{21}^{(8)}x(v) \longmapsto \frac{1}{168}\ \mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}(v) - \frac{1}{8}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}z\partial_{21}^{(7)}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_1 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}^{(2)}\mathcal{Y}\mathcal{Z}_{21}^{(9)}x(v) \longmapsto \frac{1}{252}\ \mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}(v) - \frac{1}{9}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}z\partial_{21}^{(8)}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}^{(2)}\mathcal{Y}\mathcal{Z}_{32}\mathcal{Y}(v) \longmapsto 0$; where $v \in \mathcal{D}_8 \otimes \mathcal{D}_{10} \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{32}^{(2)}\mathcal{Y}(v) \longmapsto 0$; where $v \in \mathcal{D}_8 \otimes \mathcal{D}_{10} \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(3)}\mathcal{Y}\mathcal{Z}_{21}^{(4)}x(v) \longmapsto \frac{1}{3}\ \mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{31}^{(2)}(v) - \frac{1}{6}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}^{(2)}(v) - \frac{1}{3}\ \mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}z\partial_{21}^{(3)}\partial_{32}(v)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_6 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(3)}\mathcal{Y}\mathcal{Z}_{21}^{(5)}x(v) \longmapsto \frac{1}{9}\ \mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{31}^{(2)}(v) - \frac{7}{90}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}^{(2)}(v) - \frac{2}{9}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}z\partial_{21}^{(4)}\partial_{32}(v)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_5 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(3)}\mathcal{Y}\mathcal{Z}_{21}^{(6)}x(v) \longmapsto \frac{1}{18}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}^{(2)}(v) - \frac{2}{45}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}^{(2)}(v) - \frac{1}{6}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{32}(v)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_4 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(3)}\mathcal{Y}\mathcal{Z}_{21}^{(7)}x(v) \longmapsto \frac{1}{30}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}^{(2)}(v) - \frac{1}{35}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}^{(2)}(v) - \frac{2}{15}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{32}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_3 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(3)}\mathcal{Y}\mathcal{Z}_{21}^{(8)}x(v) \longmapsto \frac{1}{45}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v) - \frac{5}{252}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v) - \frac{1}{9}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_2 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(3)}\mathcal{Y}\mathcal{Z}_{21}^{(9)}x(v) \longmapsto \frac{1}{63}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}^{(2)}(v) + \frac{1}{84}\mathcal{Z}_{32}\mathcal{Y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}\partial_{31}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$

• $\mathcal{Z}_{32}^{(3)}\mathcal{y}\mathcal{Z}_{21}^{(10)}x(v) \longmapsto \frac{1}{84}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v)$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$

• $\mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{31}z(v) \longmapsto \frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{32}(v)$; where $v \in \mathcal{D}_9 \otimes \mathcal{D}_9 \otimes \mathcal{D}_0$

Proposition (3.3.2):

The map σ_2 defined above satisfies (3.3.2).

Proof:

We can see that

• $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{21}x\mathcal{Z}_{21}x(v)\right)$; where $v \in \mathcal{D}_{10} \otimes \mathcal{D}_5 \otimes \mathcal{D}_3$

$= \sigma_1\left(2\mathcal{Z}_{21}^{(2)}x(v)\right) - \mathcal{Z}_{21}x\partial_{21}(v)$

$= \frac{2}{2}\mathcal{Z}_{21}x\partial_{21}(v) - \mathcal{Z}_{21}x\partial_{21}(v) = 0$

• $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x(v)\right)$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_4 \otimes \mathcal{D}_3$

$= \sigma_1\left(3\mathcal{Z}_{21}^{(3)}x(v) - \mathcal{Z}_{21}^{(2)}x\partial_{21}(v)\right)$

$= \frac{3}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}(v) - \frac{1}{2}\mathcal{Z}_{21}x\partial_{21}\partial_{21}(v)$

$= \mathcal{Z}_{21}x\partial_{21}^{(2)}(v) - \frac{2}{2}\mathcal{Z}_{21}x\partial_{21}^{(2)}(v) = 0$

• $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x(v)\right)$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_4 \otimes \mathcal{D}_3$

$= \sigma_1\left(3\mathcal{Z}_{21}^{(3)}x(v)\right) - \mathcal{Z}_{21}x\partial_{21}^{(2)}(v)$

$= \frac{3}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}(v) - \mathcal{Z}_{21}x\partial_{21}^{(2)}(v)$

$= 0$

• $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x(v)\right)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_3 \otimes \mathcal{D}_3$

$= \sigma_1\left(4\mathcal{Z}_{21}^{(4)}x(v) - \mathcal{Z}_{21}^{(3)}x\partial_{21}(v)\right)$

$= \frac{4}{4}\mathcal{Z}_{21}x\partial_{21}^{(3)}(v) - \frac{3}{3}\mathcal{Z}_{21}x\partial_{21}^{(3)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(3)}x(v)\right)$; where $v\in\mathcal{D}_{12}\otimes\mathcal{D}_3\otimes\mathcal{D}_3$

$=\sigma_1\left(4\mathcal{Z}_{21}^{(4)}x(v)\right)-\mathcal{Z}_{21}x\partial_{21}^{(3)}(v)$

$=\frac{4}{4}\mathcal{Z}_{21}x\partial_{21}^{(3)}(v)-\mathcal{Z}_{21}x\partial_{21}^{(3)}(v)$

$=0$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x(v)\right)$; where $v\in\mathcal{D}_{12}\otimes\mathcal{D}_3\otimes\mathcal{D}_3$

$=\sigma_1\left(6\mathcal{Z}_{21}^{(4)}x(v)-\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}(v)\right)$

$=\frac{6}{4}\mathcal{Z}_{21}x\partial_{21}^{(3)}(v)-\frac{1}{2}\mathcal{Z}_{21}x\partial_{21}\partial_{21}^{(2)}(v)$

$=\frac{3}{2}\mathcal{Z}_{21}x\partial_{21}^{(3)}(v)-\frac{3}{2}\mathcal{Z}_{21}x\partial_{21}^{(3)}(v)$

$=0$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x(v)\right)$; where $v\in\mathcal{D}_{13}\otimes\mathcal{D}_2\otimes\mathcal{D}_3$

$=\sigma_1\left(5\mathcal{Z}_{21}^{(5)}x(v)-\mathcal{Z}_{21}^{(4)}x\partial_{21}(v)\right)$

$=\frac{5}{5}\mathcal{Z}_{21}x\partial_{21}^{(4)}(v)-\frac{1}{4}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{21}(v)$

$=\mathcal{Z}_{21}x\partial_{21}^{(4)}(v)-\frac{4}{4}\mathcal{Z}_{21}x\partial_{21}^{(4)}(v)$

$=0$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(4)}x(v)\right)$; where $v\in\mathcal{D}_{13}\otimes\mathcal{D}_2\otimes\mathcal{D}_3$

$=\sigma_1\left(5\mathcal{Z}_{21}^{(5)}x(v)\right)-\mathcal{Z}_{21}x\partial_{21}^{(4)}(v)$

$=\frac{5}{5}\mathcal{Z}_{21}x\partial_{21}^{(4)}(v)-\mathcal{Z}_{21}x\partial_{21}^{(4)}(v)$

$=0$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(2)}x(v)\right)$; where $v\in\mathcal{D}_{13}\otimes\mathcal{D}_2\otimes\mathcal{D}_3$

$=\sigma_1\left(10\mathcal{Z}_{21}^{(5)}x(v)-\mathcal{Z}_{21}^{(3)}x\partial_{21}^{(2)}(v)\right)$

$=\frac{10}{5}\mathcal{Z}_{21}x\partial_{21}^{(4)}(v)-\frac{1}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{21}^{(2)}(v)$

$= 2\, \mathcal{Z}_{21} x \partial_{21}^{(4)}(v) - \frac{6}{3} \mathcal{Z}_{21} x \partial_{21}^{(4)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(3)} x(v)\right)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$

$= \sigma_1 \left(10\mathcal{Z}_{21}^{(5)} x(v) - \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(3)}(v)\right)$

$= \frac{10}{5} \mathcal{Z}_{21} x \partial_{21}^{(4)}(v) - \frac{1}{2} \mathcal{Z}_{21} x \partial_{21} \partial_{21}^{(3)}(v)$

$= 2\, \mathcal{Z}_{21} x \partial_{21}^{(4)}(v) - \frac{4}{2} \mathcal{Z}_{21} x \partial_{21}^{(4)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21} x(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$

$= \sigma_1 \left(6\mathcal{Z}_{21}^{(6)} x(v) - \mathcal{Z}_{21}^{(5)} x \partial_{21}(v)\right)$

$= \frac{6}{6} \mathcal{Z}_{21} x \partial_{21}^{(5)}(v) - \frac{1}{5} \mathcal{Z}_{21} x \partial_{21}^{(4)} \partial_{21}(v)$

$= \mathcal{Z}_{21} x \partial_{21}^{(5)}(v) - \frac{5}{5} \mathcal{Z}_{21} x \partial_{21}^{(5)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{21} x \mathcal{Z}_{21}^{(5)} x(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$

$= \sigma_1 \left(6\mathcal{Z}_{21}^{(6)} x(v)\right) - \mathcal{Z}_{21} x \partial_{21}^{(5)}(v)$

$= \frac{6}{6} \mathcal{Z}_{21} x \partial_{21}^{(5)}(v) - \mathcal{Z}_{21} x \partial_{21}^{(5)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(3)} x(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$

$= \sigma_1 \left(20\mathcal{Z}_{21}^{(6)} x(v) - \mathcal{Z}_{21}^{(3)} x \partial_{21}^{(3)}(v)\right)$

$= \frac{20}{6} \mathcal{Z}_{21} x \partial_{21}^{(5)}(v) - \frac{1}{3} \mathcal{Z}_{21} x \partial_{21}^{(2)} \partial_{21}^{(3)}(v)$

$= \frac{10}{3} \mathcal{Z}_{21} x \partial_{21}^{(5)}(v) - \frac{10}{3} \mathcal{Z}_{21} x \partial_{21}^{(5)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(2)} x(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$

$= \sigma_1\left(15\mathcal{Z}_{21}^{(6)} x(v) - \mathcal{Z}_{21}^{(4)} x \partial_{21}^{(2)}(v)\right)$

$= \frac{15}{6}\mathcal{Z}_{21} x \partial_{21}^{(5)}(v) - \frac{1}{4}\mathcal{Z}_{21} x \partial_{21}^{(3)}\partial_{21}^{(2)}(v)$

$= \frac{5}{2}\mathcal{Z}_{21} x \partial_{21}^{(5)}(v) - \frac{10}{4}\mathcal{Z}_{21} x \partial_{21}^{(5)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(4)} x(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$

$= \sigma_1\left(15\mathcal{Z}_{21}^{(6)} x(v) - \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(4)}(v)\right)$

$= \frac{15}{6}\mathcal{Z}_{21} x \partial_{21}^{(5)}(v) - \frac{1}{2}\mathcal{Z}_{21} x \partial_{21}\partial_{21}^{(4)}(v)$

$= \frac{5}{2}\mathcal{Z}_{21} x \partial_{21}^{(5)}(v) - \frac{5}{2}\mathcal{Z}_{21} x \partial_{21}^{(5)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{21}^{(6)} x \mathcal{Z}_{21} x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$

$= \sigma_1\left(7\mathcal{Z}_{21}^{(7)} x(v) - \mathcal{Z}_{21}^{(6)} x \partial_{21}(v)\right)$

$= \frac{7}{7}\mathcal{Z}_{21} x \partial_{21}^{(6)}(v) - \frac{1}{6}\mathcal{Z}_{21} x \partial_{21}^{(5)}\partial_{21}(v)$

$= \mathcal{Z}_{21} x \partial_{21}^{(6)}(v) - \frac{6}{6}\mathcal{Z}_{21} x \partial_{21}^{(6)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{21} x \mathcal{Z}_{21}^{(6)} x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$

$= \sigma_1\left(7\mathcal{Z}_{21}^{(7)} x(v)\right) - \mathcal{Z}_{21} x \partial_{21}^{(6)}(v)$

$= \frac{7}{7}\mathcal{Z}_{21} x \partial_{21}^{(6)}(v) - \mathcal{Z}_{21} x \partial_{21}^{(6)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21}^{(2)} x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$

$= \sigma_1\left(21\mathcal{Z}_{21}^{(7)} x(v) - \mathcal{Z}_{21}^{(5)} x \partial_{21}^{(2)}(v)\right)$

$= \frac{21}{7} \mathcal{Z}_{21} x \partial_{21}^{(6)}(v) - \frac{1}{5} \mathcal{Z}_{21} x \partial_{21}^{(4)} \partial_{21}^{(2)}(v)$

$= 3\, \mathcal{Z}_{21} x \partial_{21}^{(6)}(v) - \frac{15}{5} \mathcal{Z}_{21} x \partial_{21}^{(6)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(5)} x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$

$= \sigma_1 \left(21 \mathcal{Z}_{21}^{(7)} x(v) - \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)}(v)\right)$

$= \frac{21}{7} \mathcal{Z}_{21} x \partial_{21}^{(6)}(v) - \frac{1}{2} \mathcal{Z}_{21} x \partial_{21} \partial_{21}^{(5)}(v)$

$= 3\, \mathcal{Z}_{21} x \partial_{21}^{(6)}(v) - \frac{6}{2} \mathcal{Z}_{21} x \partial_{21}^{(6)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(3)} x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$

$= \sigma_1 \left(35 \mathcal{Z}_{21}^{(7)} x(v) - \mathcal{Z}_{21}^{(4)} x \partial_{21}^{(3)}(v)\right)$

$= \frac{35}{7} \mathcal{Z}_{21} x \partial_{21}^{(6)}(v) - \frac{1}{4} \mathcal{Z}_{21} x \partial_{21}^{(3)} \partial_{21}^{(3)}(v)$

$= 5\, \mathcal{Z}_{21} x \partial_{21}^{(6)}(v) - \frac{20}{4} \mathcal{Z}_{21} x \partial_{21}^{(6)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(4)} x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$

$= \sigma_1 \left(35 \mathcal{Z}_{21}^{(7)} x(v) - \mathcal{Z}_{21}^{(3)} x \partial_{21}^{(4)}(v)\right)$

$= \frac{35}{7} \mathcal{Z}_{21} x \partial_{21}^{(6)}(v) - \frac{1}{3} \mathcal{Z}_{21} x \partial_{21}^{(2)} \partial_{21}^{(4)}(v)$

$= 5\, \mathcal{Z}_{21} x \partial_{21}^{(6)}(v) - \frac{15}{3} \mathcal{Z}_{21} x \partial_{21}^{(6)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(3)} x(v)\right)$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_5 \otimes \mathcal{D}_2$

$= \sigma_1 \left(\mathcal{Z}_{21}^{(3)} x \partial_{32}(v) + \mathcal{Z}_{21}^{(2)} x \partial_{31}(v)\right) - \mathcal{Z}_{32} y \partial_{21}^{(3)}(v)$

$= \frac{1}{3} \mathcal{Z}_{21} x \partial_{21}^{(2)} \partial_{32}(v) + \frac{1}{2} \mathcal{Z}_{21} x \partial_{21} \partial_{31}(v) - \mathcal{Z}_{32} y \partial_{21}^{(3)}(v)$

And

$$\left(\delta_{\mathcal{L}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{L}_2\mathcal{M}_1}\right)\left(\frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}(v)\right)$$

$$=\sigma_1\left(\frac{1}{3}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}(v)\right)+\frac{1}{3}\mathcal{Z}_{21}x\partial_{31}\partial_{21}(v)-\frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(2)}\partial_{21}(v)$$

$$=\frac{1}{6}\mathcal{Z}_{21}x\partial_{21}\partial_{21}\partial_{32}(v)+\frac{1}{6}\mathcal{Z}_{21}x\partial_{21}\partial_{31}(v)+\frac{1}{3}\mathcal{Z}_{21}x\partial_{31}\partial_{21}(v)-\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(3)}(v)$$

$$=\frac{1}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{32}(v)+\frac{1}{2}\mathcal{Z}_{21}x\partial_{21}\partial_{31}(v)-\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(3)}(v)$$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(4)}x(v)\right)$; where $v\in\mathcal{D}_{12}\otimes\mathcal{D}_4\otimes\mathcal{D}_2$

$$=\sigma_1\left(\mathcal{Z}_{21}^{(4)}x\partial_{32}(v)+\mathcal{Z}_{21}^{(3)}x\partial_{31}(v)\right)-\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(4)}(v)$$

$$=\frac{1}{4}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}(v)+\frac{1}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{31}(v)-\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(4)}(v)$$

And

$$\left(\delta_{\mathcal{L}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{L}_2\mathcal{M}_1}\right)\left(\frac{1}{6}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}(v)\right)$$

$$=\sigma_1\left(\frac{1}{6}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(2)}(v)\right)+\frac{1}{6}\mathcal{Z}_{21}x\partial_{31}\partial_{21}^{(2)}(v)-\frac{1}{6}\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(2)}\partial_{21}^{(2)}(v)$$

$$=\frac{1}{12}\mathcal{Z}_{21}x\partial_{21}\partial_{21}^{(2)}\partial_{32}(v)+\frac{1}{12}\mathcal{Z}_{21}x\partial_{21}\partial_{21}\partial_{31}(v)+\frac{1}{6}\mathcal{Z}_{21}x\partial_{31}\partial_{21}^{(2)}(v)-$$
$$\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(4)}(v)$$

$$=\frac{1}{4}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}(v)+\frac{1}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{31}(v)-\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(4)}(v)$$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(5)}x(v)\right)$; where $v\in\mathcal{D}_{13}\otimes\mathcal{D}_3\otimes\mathcal{D}_2$

$$=\sigma_1\left(\mathcal{Z}_{21}^{(5)}x\partial_{32}(v)+\mathcal{Z}_{21}^{(4)}x\partial_{31}(v)\right)-\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(5)}(v)$$

$$=\frac{1}{5}\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}(v)+\frac{1}{4}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{31}(v)-\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(5)}(v)$$

And

$$\left(\delta_{\mathcal{L}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{L}_2\mathcal{M}_1}\right)\left(\frac{1}{10}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}(v)\right)$$

$$=\sigma_1\left(\frac{1}{10}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(3)}(v)\right)+\frac{1}{10}\mathcal{Z}_{21}x\partial_{31}\partial_{21}^{(3)}(v)-\frac{1}{10}\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(2)}\partial_{21}^{(3)}(v)$$

$= \frac{1}{20}\mathcal{Z}_{21}x\partial_{21}\partial_{21}^{(3)}\partial_{32}(v) + \frac{1}{20}\mathcal{Z}_{21}x\partial_{21}\partial_{21}^{(2)}\partial_{31}(v) + \frac{1}{10}\mathcal{Z}_{21}x\partial_{31}\partial_{21}^{(3)}(v) -$

$\mathcal{Z}_{32}y\partial_{21}^{(5)}(v)$

$= \frac{1}{5}\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}(v) + \frac{1}{4}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{31}(v) - \mathcal{Z}_{32}y\partial_{21}^{(5)}(v)$

- $(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1})\left(\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(6)}x(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_2 \otimes \mathcal{D}_2$

$= \sigma_1\left(\mathcal{Z}_{21}^{(6)}x\partial_{32}(v) + \mathcal{Z}_{21}^{(5)}x\partial_{31}(v)\right) - \mathcal{Z}_{32}y\partial_{21}^{(6)}(v)$

$= \frac{1}{6}\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{32}(v) + \frac{1}{5}\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{31}(v) - \mathcal{Z}_{32}y\partial_{21}^{(6)}(v)$

And

$(\delta_{\mathcal{L}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1})\left(\frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}(v)\right)$

$= \sigma_1\left(\frac{1}{15}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(4)}(v)\right) + \frac{1}{15}\mathcal{Z}_{21}x\partial_{31}\partial_{21}^{(4)}(v) - \frac{1}{15}\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(4)}(v)$

$= \frac{1}{30}\mathcal{Z}_{21}x\partial_{21}\partial_{21}^{(4)}\partial_{32}(v) + \frac{1}{30}\mathcal{Z}_{21}x\partial_{21}\partial_{21}^{(3)}\partial_{31}(v) + \frac{1}{15}\mathcal{Z}_{21}x\partial_{31}\partial_{21}^{(4)}(v) -$

$\mathcal{Z}_{32}y\partial_{21}^{(6)}(v)$

$= \frac{1}{6}\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{32}(v) + \frac{1}{5}\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{31}(v) - \mathcal{Z}_{32}y\partial_{21}^{(6)}(v)$

- $(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1})\left(\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(7)}x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_1 \otimes \mathcal{D}_2$

$= \sigma_1\left(\mathcal{Z}_{21}^{(7)}x\partial_{32}(v) + \mathcal{Z}_{21}^{(6)}x\partial_{31}(v)\right) - \mathcal{Z}_{32}y\partial_{21}^{(7)}(v)$

$= \frac{1}{7}\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{32}(v) + \frac{1}{6}\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{31}(v) - \mathcal{Z}_{32}y\partial_{21}^{(7)}(v)$

And

$(\delta_{\mathcal{L}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1})\left(\frac{1}{21}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}(v)\right)$

$= \sigma_1\left(\frac{1}{21}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(5)}(v)\right) + \frac{1}{21}\mathcal{Z}_{21}x\partial_{31}\partial_{21}^{(5)}(v) - \frac{1}{21}\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(5)}(v)$

$= \frac{1}{42}\mathcal{Z}_{21}x\partial_{21}\partial_{21}^{(5)}\partial_{32}(v) + \frac{1}{42}\mathcal{Z}_{21}x\partial_{21}\partial_{21}^{(4)}\partial_{31}(v) + \frac{1}{21}\mathcal{Z}_{21}x\partial_{31}\partial_{21}^{(5)}(v) -$

$\mathcal{Z}_{32}y\partial_{21}^{(7)}(v)$

$= \frac{1}{7}\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{32}(v) + \frac{1}{6}\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{31}(v) - \mathcal{Z}_{32}y\partial_{21}^{(7)}(v)$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(8)}x(v)\right)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_0 \otimes \mathcal{D}_2$

$= \mathcal{Z}_{21}^{(8)}x\partial_{32}(v) + \sigma_1\left(\mathcal{Z}_{21}^{(7)}x\partial_{31}(v)\right) - \mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(8)}(v)$

$= \frac{1}{7}\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{31}(v) - \mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(8)}(v)$

And

$\left(\delta_{\mathcal{L}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1}\right)\left(\frac{1}{28}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}(v)\right)$

$= \sigma_1\left(\frac{1}{28}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(5)}(v)\right) + \frac{1}{28}\mathcal{Z}_{21}x\partial_{31}\partial_{21}^{(6)}(v) - \frac{1}{28}\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(2)}\partial_{21}^{(6)}(v)$

$= \frac{1}{56}\mathcal{Z}_{21}x\partial_{21}\partial_{21}^{(5)}\partial_{31}(v) + \frac{1}{28}\mathcal{Z}_{21}x\partial_{31}\partial_{21}^{(6)}(v) - \mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(8)}(v)$

$= \frac{1}{7}\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{31}(v) - \mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(8)}(v)$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}(v)\right)$; where $v \in \mathcal{D}_8 \otimes \mathcal{D}_9 \otimes \mathcal{D}_1$

$= \sigma_1\left(2\,\mathcal{Z}_{32}^{(2)}\mathcal{y}(v)\right) - \mathcal{Z}_{32}\mathcal{y}\partial_{32}(v)$

$= \frac{2}{2}\mathcal{Z}_{32}\mathcal{y}\partial_{32}(v) - \mathcal{Z}_{32}\mathcal{y}\partial_{32}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{21}^{(3)}x(v)\right)$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_6 \otimes \mathcal{D}_1$

$= \sigma_1\left(\mathcal{Z}_{21}^{(3)}x\partial_{32}^{(2)}(v) + \mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{31}(v)\right) + \mathcal{Z}_{21}x\partial_{31}^{(2)}(v) - \sigma_1\left(\mathcal{Z}_{32}^{(2)}\mathcal{y}\partial_{21}^{(3)}(v)\right)$

$= \frac{1}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{32}^{(2)}(v) + \frac{1}{2}\mathcal{Z}_{21}x\partial_{21}\partial_{32}\partial_{31}(v) + \mathcal{Z}_{21}x\partial_{31}^{(2)}(v) - \frac{1}{2}\mathcal{Z}_{32}\mathcal{y}\partial_{32}\partial_{21}^{(3)}(v)$

And

$\left(\delta_{\mathcal{L}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1}\right)\left(\frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{31}(v) - \frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(2)}(v)\right)$

$= \sigma_1\left(\frac{1}{3}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{31}(v)\right) + \frac{1}{3}\mathcal{Z}_{21}x\partial_{31}\partial_{31}(v) - \frac{1}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{31}(v) -$

$\sigma_1\left(\frac{1}{3}\mathcal{Z}_{32}^{(2)}\mathcal{y}\partial_{21}\partial_{21}^{(2)}(v)\right) + \frac{1}{3}\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(2)}(v) + \frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\partial_{31}\partial_{21}^{(2)}(v)$

$= \frac{1}{6}\mathcal{Z}_{21}x\partial_{21}\partial_{32}\partial_{31}(v) + \frac{2}{3}\mathcal{Z}_{21}x\partial_{31}^{(2)}(v) - \frac{1}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{31}(v) -$

$\frac{1}{2}\mathcal{Z}_{32}\mathcal{y}\partial_{32}\partial_{21}^{(3)}(v) + \frac{1}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{32}^{(2)}(v) + \frac{1}{3}\mathcal{Z}_{21}x\partial_{21}\partial_{32}\partial_{31}(v) +$

$\frac{1}{3}\mathcal{Z}_{21}x\partial_{31}^{(2)}(v)+\frac{1}{3}\mathcal{Z}_{32}y\partial_{31}\partial_{21}^{(2)}(v)$

$=\frac{1}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{32}^{(2)}(v)+\frac{1}{2}\mathcal{Z}_{21}x\partial_{21}\partial_{32}\partial_{31}(v)+\mathcal{Z}_{21}x\partial_{31}^{(2)}(v)-\frac{1}{2}\mathcal{Z}_{32}y\partial_{32}\partial_{21}^{(3)}(v)$

- $(\delta_{\mathcal{M}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{M}_2\mathcal{M}_1})\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(4)}x(v)\right)$; where $v\in\mathcal{D}_{12}\otimes\mathcal{D}_5\otimes\mathcal{D}_1$

$=\sigma_1\left(\mathcal{Z}_{21}^{(4)}x\partial_{32}^{(2)}(v)+\mathcal{Z}_{21}^{(3)}x\partial_{32}\partial_{31}(v)+\mathcal{Z}_{21}^{(2)}x\partial_{31}^{(2)}(v)-\mathcal{Z}_{32}^{(2)}y\partial_{21}^{(4)}(v)\right)$

$=\frac{1}{4}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}^{(2)}(v)+\frac{1}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{32}\partial_{31}(v)+\frac{1}{2}\mathcal{Z}_{21}x\partial_{21}\partial_{31}^{(2)}(v)-$
$\frac{1}{2}\mathcal{Z}_{32}y\partial_{32}\partial_{21}^{(4)}(v)$

And

$(\delta_{\mathcal{L}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{L}_2\mathcal{M}_1})\left(\frac{1}{12}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{31}(v)-\frac{1}{4}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(3)}(v)\right)$

$=\sigma_1\left(\frac{1}{12}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}\partial_{31}(v)\right)+\frac{1}{12}\mathcal{Z}_{21}x\partial_{31}\partial_{21}\partial_{31}(v)-$
$\frac{1}{12}\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}\partial_{31}(v)-\sigma_1\left(\frac{1}{4}\mathcal{Z}_{32}^{(2)}y\partial_{21}\partial_{21}^{(3)}(v)\right)+\frac{1}{4}\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(3)}(v)+$
$\frac{1}{4}\mathcal{Z}_{32}y\partial_{31}\partial_{21}^{(3)}(v)$

$=\frac{1}{12}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{32}\partial_{31}(v)+\frac{3}{12}\mathcal{Z}_{21}x\partial_{21}\partial_{31}^{(2)}(v)-\frac{3}{12}\mathcal{Z}_{32}y\partial_{21}^{(3)}\partial_{31}(v)-$
$\frac{1}{2}\mathcal{Z}_{32}y\partial_{32}\partial_{21}^{(4)}(v)+\frac{1}{4}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}^{(2)}(v)+\frac{1}{4}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{32}\partial_{31}(v)+$
$\frac{1}{4}\mathcal{Z}_{21}x\partial_{21}\partial_{31}^{(2)}(v)+\frac{1}{4}\mathcal{Z}_{32}y\partial_{31}\partial_{21}^{(3)}(v)$

$=\frac{1}{4}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}^{(2)}(v)+\frac{1}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{32}\partial_{31}(v)+\frac{1}{2}\mathcal{Z}_{21}x\partial_{21}\partial_{31}^{(2)}(v)-$
$\frac{1}{2}\mathcal{Z}_{32}y\partial_{32}\partial_{21}^{(4)}(v)$

- $(\delta_{\mathcal{M}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{M}_2\mathcal{M}_1})\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(5)}x(v)\right)$; where $v\in\mathcal{D}_{13}\otimes\mathcal{D}_4\otimes\mathcal{D}_1$

$=\sigma_1\left(\mathcal{Z}_{21}^{(5)}x\partial_{32}^{(2)}(v)+\mathcal{Z}_{21}^{(4)}x\partial_{32}\partial_{31}(v)+\mathcal{Z}_{21}^{(3)}x\partial_{31}^{(2)}(v)-\mathcal{Z}_{32}^{(2)}y\partial_{21}^{(5)}(v)\right)$

$=\frac{1}{5}\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}^{(2)}(v)+\frac{1}{4}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}\partial_{31}(v)+\frac{1}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{31}^{(2)}(v)-$
$\frac{1}{2}\mathcal{Z}_{32}y\partial_{32}\partial_{21}^{(5)}(v)$

And

$(\delta_{\mathcal{L}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1})\left(\frac{1}{30}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}(v) - \frac{1}{5}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(4)}(v)\right)$

$= \sigma_1\left(\frac{1}{30}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(2)}\partial_{31}(v)\right) + \frac{1}{30}\mathcal{Z}_{21}x\partial_{31}\partial_{21}^{(2)}\partial_{31}(v) -$

$\frac{1}{30}\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(2)}\partial_{21}^{(2)}\partial_{31}(v) - \sigma_1\left(\frac{1}{5}\mathcal{Z}_{32}^{(2)}\mathcal{y}\partial_{21}\partial_{21}^{(4)}(v)\right) + \frac{1}{5}\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(4)}(v) +$

$\frac{1}{5}\mathcal{Z}_{32}\mathcal{y}\partial_{31}\partial_{21}^{(4)}(v)$

$= \frac{3}{60}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}\partial_{31}(v) + \frac{4}{30}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{31}^{(2)}(v) - \frac{6}{30}\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(4)}\partial_{31}(v) -$

$\frac{5}{10}\mathcal{Z}_{32}\mathcal{y}\partial_{32}\partial_{21}^{(5)}(v) + \frac{1}{5}\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}^{(2)}(v) + \frac{1}{5}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}\partial_{31}(v) +$

$\frac{1}{5}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{31}^{(2)}(v) + \frac{1}{5}\mathcal{Z}_{32}\mathcal{y}\partial_{31}\partial_{21}^{(4)}(v)$

$= \frac{1}{5}\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}^{(2)}(v) + \frac{1}{4}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}\partial_{31}(v) + \frac{1}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{31}^{(2)}(v) -$

$\frac{1}{2}\mathcal{Z}_{32}\mathcal{y}\partial_{32}\partial_{21}^{(5)}(v)$

- $(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1})\left(\mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{21}^{(6)}x(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_3 \otimes \mathcal{D}_1$

$= \sigma_1\left(\mathcal{Z}_{21}^{(6)}x\partial_{32}^{(2)}(v) + \mathcal{Z}_{21}^{(5)}x\partial_{32}\partial_{31}(v) + \mathcal{Z}_{21}^{(4)}x\partial_{31}^{(2)}(v) - \mathcal{Z}_{32}^{(2)}\mathcal{y}\partial_{21}^{(6)}(v)\right)$

$= \frac{1}{6}\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{32}^{(2)}(v) + \frac{1}{5}\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}\partial_{31}(v) + \frac{1}{4}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{31}^{(2)}(v) -$

$\frac{1}{2}\mathcal{Z}_{32}\mathcal{y}\partial_{32}\partial_{21}^{(6)}(v)$

And

$(\delta_{\mathcal{L}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1})\left(\frac{1}{60}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}(v) - \frac{1}{6}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(5)}(v)\right)$

$= \sigma_1\left(\frac{1}{60}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(3)}\partial_{31}(v)\right) + \frac{1}{60}\mathcal{Z}_{21}x\partial_{31}\partial_{21}^{(3)}\partial_{31}(v) -$

$\frac{1}{60}\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(\ )}\partial_{21}^{(\ )}\partial_{31}(v) - \sigma_1\left(\frac{1}{6}\mathcal{Z}_{32}^{(2)}\mathcal{y}\partial_{21}\partial_{21}^{(5)}(v)\right) + \frac{1}{6}\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(5)}(v) +$

$\frac{1}{6}\mathcal{Z}_{32}\mathcal{y}\partial_{31}\partial_{21}^{(5)}(v)$

$= \frac{2}{60}\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}\partial_{31}(v) + \frac{5}{60}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{31}^{(2)}(v) - \frac{10}{60}\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(5)}\partial_{31}(v) -$

$\frac{3}{6}\mathcal{Z}_{32}\mathcal{y}\partial_{32}\partial_{21}^{(6)}(v) + \frac{1}{6}\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{32}^{(2)}(v) + \frac{1}{6}\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}\partial_{31}(v) +$

$\frac{1}{6}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{31}^{(2)}(v)+\frac{1}{6}\mathcal{Z}_{32}y\partial_{31}\partial_{21}^{(5)}(v)$

$=\frac{1}{6}\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{32}^{(2)}(v)+\frac{1}{5}\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}\partial_{31}(v)+\frac{1}{4}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{31}^{(2)}(v)-\frac{1}{2}\mathcal{Z}_{32}y\partial_{32}\partial_{21}^{(6)}(v)$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(7)}x(v)\right)$; where $v\in\mathcal{D}_{15}\otimes\mathcal{D}_2\otimes\mathcal{D}_1$

$=\sigma_1(\mathcal{Z}_{21}^{(7)}x\partial_{32}^{(2)}(v)+\mathcal{Z}_{21}^{(6)}x\partial_{32}\partial_{31}(v)+\mathcal{Z}_{21}^{(5)}x\partial_{31}^{(2)}(v)-\mathcal{Z}_{32}^{(2)}y\partial_{21}^{(7)}(v))$

$=\frac{1}{7}\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v)+\frac{1}{6}\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v)+\frac{1}{5}\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v)-\frac{1}{2}\mathcal{Z}_{32}y\partial_{32}\partial_{21}^{(7)}(v)$

And

$\left(\delta_{\mathcal{L}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{L}_2\mathcal{M}_1}\right)\left(\frac{1}{105}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}(v)-\frac{1}{7}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}(v)\right)$

$=\sigma_1\left(\frac{1}{105}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(4)}\partial_{31}(v)\right)+\frac{1}{105}\mathcal{Z}_{21}x\partial_{31}\partial_{21}^{(4)}\partial_{31}(v)-\frac{1}{105}\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(4)}\partial_{31}(v)-\sigma_1\left(\frac{1}{7}\mathcal{Z}_{32}^{(2)}y\partial_{21}\partial_{21}^{(6)}(v)\right)+\frac{1}{7}\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(6)}(v)+\frac{1}{7}\mathcal{Z}_{32}y\partial_{31}\partial_{21}^{(6)}(v)$

$=\frac{5}{210}\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v)+\frac{6}{105}\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v)-\frac{15}{105}\mathcal{Z}_{32}y\partial_{21}^{(6)}\partial_{31}(v)-\frac{7}{14}\mathcal{Z}_{32}y\partial_{32}\partial_{21}^{(7)}(v)+\frac{1}{7}\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v)+\frac{1}{7}\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v)+\frac{1}{7}\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v)+\frac{1}{7}\mathcal{Z}_{32}y\partial_{31}\partial_{21}^{(6)}(v)$

$=\frac{1}{6}\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v)+\frac{1}{5}\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v)-\frac{1}{2}\mathcal{Z}_{32}y\partial_{32}\partial_{21}^{(7)}(v)+\frac{1}{7}\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v)$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(8)}x(v)\right)$; where $v\in\mathcal{D}_{16}\otimes\mathcal{D}_1\otimes\mathcal{D}_1$

$=\mathcal{Z}_{21}^{(8)}x\partial_{32}^{(2)}(v)+\sigma_1\left(\mathcal{Z}_{21}^{(7)}x\partial_{32}\partial_{31}(v)+\mathcal{Z}_{21}^{(6)}x\partial_{31}^{(2)}(v)-\mathcal{Z}_{32}^{(2)}y\partial_{21}^{(8)}(v)\right)$

$=\frac{1}{7}\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{32}\partial_{31}(v)+\frac{1}{6}\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{31}^{(2)}(v)-\frac{1}{2}\mathcal{Z}_{32}y\partial_{32}\partial_{21}^{(8)}(v)$

And

$$\left(\delta_{\mathcal{L}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{L}_2\mathcal{M}_1}\right)\left(\frac{1}{168}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}(v)-\frac{1}{8}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}(v)\right)$$

$$=\sigma_1\left(\frac{1}{168}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(5)}\partial_{31}(v)\right)+\frac{1}{168}\mathcal{Z}_{21}x\partial_{31}\partial_{21}^{(5)}\partial_{31}(v)-$$

$$\frac{1}{168}\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(5)}\partial_{31}(v)-\sigma_1\left(\frac{1}{8}\mathcal{Z}_{32}^{(2)}y\partial_{21}\partial_{21}^{(7)}(v)\right)+\frac{1}{8}\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(7)}(v)+$$

$$\frac{1}{8}\mathcal{Z}_{32}y\partial_{31}\partial_{21}^{(7)}(v)$$

$$=\frac{3}{168}\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{32}\partial_{31}(v)+\frac{7}{168}\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{31}^{(2)}(v)-\frac{21}{168}\mathcal{Z}_{32}y\partial_{21}^{(7)}\partial_{31}(v)-$$

$$\frac{4}{8}\mathcal{Z}_{32}y\partial_{32}\partial_{21}^{(8)}(v)+\frac{1}{8}\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{32}\partial_{31}(v)+\frac{1}{8}\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{31}^{(2)}(v)+$$

$$\frac{1}{8}\mathcal{Z}_{32}y\partial_{31}\partial_{21}^{(7)}(v)$$

$$=\frac{1}{7}\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{32}\partial_{31}(v)+\frac{1}{6}\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{31}^{(2)}(v)-\frac{1}{2}\mathcal{Z}_{32}y\partial_{32}\partial_{21}^{(8)}(v)$$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(9)}x(v)\right)$; where $v\in\mathcal{D}_{17}\otimes\mathcal{D}_0\otimes\mathcal{D}_1$

$$=\mathcal{Z}_{21}^{(9)}x\partial_{31}^{(2)}(v)+\mathcal{Z}_{21}^{(8)}x\partial_{32}\partial_{31}(v)+\sigma_1\left(\mathcal{Z}_{21}^{(7)}x\partial_{31}^{(2)}(v)-\mathcal{Z}_{32}^{(2)}y\partial_{21}^{(9)}(v)\right)$$

$$=\frac{1}{7}\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v)-\frac{1}{2}\mathcal{Z}_{32}y\partial_{32}\partial_{21}^{(9)}(v)$$

And

$$\left(\delta_{\mathcal{L}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{L}_2\mathcal{M}_1}\right)\left(\frac{1}{252}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}(v)-\frac{1}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}(v)\right)$$

$$=\sigma_1\left(\frac{1}{252}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(6)}\partial_{31}(v)\right)+\frac{1}{252}\mathcal{Z}_{21}x\partial_{31}\partial_{21}^{(6)}\partial_{31}(v)-$$

$$\frac{1}{252}\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(6)}\partial_{31}(v)-\sigma_1\left(\frac{1}{9}\mathcal{Z}_{32}^{(2)}y\partial_{21}\partial_{21}^{(8)}(v)\right)+\frac{1}{9}\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(8)}(v)+$$

$$\frac{1}{9}\mathcal{Z}_{32}y\partial_{31}\partial_{21}^{(8)}(v)$$

$$=\frac{8}{252}\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v)-\frac{28}{252}\mathcal{Z}_{32}y\partial_{21}^{(8)}\partial_{31}(v)-\frac{9}{18}\mathcal{Z}_{32}y\partial_{32}\partial_{21}^{(9)}(v)+$$

$$\frac{1}{9}\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v)+\frac{1}{9}\mathcal{Z}_{32}y\partial_{31}\partial_{21}^{(8)}(v)$$

$$=\frac{1}{7}\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v)-\frac{1}{2}\mathcal{Z}_{32}y\partial_{32}\partial_{21}^{(9)}(v)$$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{32}y(v)\right)$; where $v\in\mathcal{D}_8\otimes\mathcal{D}_{10}\otimes\mathcal{D}_0$

$=\sigma_1\left(3\,\mathcal{Z}_{32}^{(3)}y(v)-\mathcal{Z}_{32}^{(2)}y\partial_{32}(v)\right)$

$=\frac{3}{3}\mathcal{Z}_{32}y\partial_{32}^{(2)}(v)-\frac{2}{2}\mathcal{Z}_{32}y\partial_{32}^{(2)}(v)$

$=0$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{32}y\mathcal{Z}_{32}^{(2)}y(v)\right)$; where $v\in\mathcal{D}_8\otimes\mathcal{D}_{10}\otimes\mathcal{D}_0$

$=\sigma_1\left(3\,\mathcal{Z}_{32}^{(3)}y(v)\right)-\mathcal{Z}_{32}y\partial_{32}^{(2)}(v)$

$=\frac{3}{3}\mathcal{Z}_{32}y\partial_{32}^{(2)}(v)-\mathcal{Z}_{32}y\partial_{32}^{(2)}(v)$

$=0$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(4)}x(v)\right)$; where $v\in\mathcal{D}_{12}\otimes\mathcal{D}_6\otimes\mathcal{D}_0$

$=\sigma_1\left(\mathcal{Z}_{21}^{(4)}x\partial_{32}^{(3)}(v)+\mathcal{Z}_{21}^{(3)}x\partial_{32}^{(2)}\partial_{31}(v)+\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{31}^{(2)}(v)\right)+\mathcal{Z}_{21}x\partial_{31}^{(3)}(v)-$
$\sigma_1\left(\mathcal{Z}_{32}^{(3)}y\partial_{21}^{(4)}(v)\right)$

$=\frac{1}{4}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}^{(3)}(v)+\frac{1}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{32}^{(2)}\partial_{31}(v)+\frac{1}{2}\mathcal{Z}_{21}x\partial_{21}\partial_{32}\partial_{31}^{(2)}(v)+$
$\mathcal{Z}_{21}x\partial_{31}^{(3)}(v)-\frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(4)}\partial_{32}^{(2)}(v)-\frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(3)}\partial_{32}\partial_{31}-\frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{31}^{(2)}(v)$

And

$\left(\delta_{\mathcal{L}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{L}_2\mathcal{M}_1}\right)\left(\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{31}^{(2)}(v)-\frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}^{(2)}(v)-\right.$
$\left.\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(3)}\partial_{32}(v)\right)$

$=\sigma_1\left(\frac{1}{3}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{31}^{(2)}(v)\right)+\frac{1}{3}\mathcal{Z}_{21}x\partial_{31}\partial_{31}^{(2)}(v)-\frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{31}^{(2)}(v)-$
$\sigma_1\left(\frac{1}{6}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(2)}\partial_{32}^{(2)}(v)\right)-\frac{1}{6}\mathcal{Z}_{21}x\partial_{31}\partial_{21}^{(2)}\partial_{32}^{(2)}(v)+$
$\mathcal{Z}_{32}y\partial_{21}^{(4)}\partial_{32}^{(2)}(v)-\sigma_1\left(\frac{1}{3}\mathcal{Z}_{32}^{(2)}y\partial_{21}\partial_{21}^{(3)}\partial_{32}(v)\right)+\frac{1}{3}\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(3)}\partial_{32}(v)+$
$\frac{1}{3}\mathcal{Z}_{32}y\partial_{31}\partial_{21}^{(3)}\partial_{32}(v)$

$=\frac{1}{6}\mathcal{Z}_{21}x\partial_{21}\partial_{32}\partial_{31}^{(2)}(v)+\frac{3}{3}\mathcal{Z}_{21}x\partial_{31}^{(3)}(v)-\frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{31}^{(2)}(v)-$

$\frac{9}{12}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}^{(3)}(v)-\frac{2}{6}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{32}^{(2)}\partial_{31}(v)+\frac{6}{6}\mathcal{Z}_{32}y\partial_{21}^{(4)}\partial_{32}^{(2)}(v)-$
$\frac{4}{3}\mathcal{Z}_{32}y\partial_{21}^{(4)}\partial_{32}^{(2)}(v)-\frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(3)}\partial_{32}\partial_{31}(v)+\frac{3}{3}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}^{(3)}(v)+$
$\frac{2}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{32}^{(2)}\partial_{31}(v)+\frac{1}{3}\mathcal{Z}_{21}x\partial_{21}\partial_{32}\partial_{31}^{(2)}(v)$
$=\frac{1}{4}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}^{(3)}(v)+\frac{1}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{32}^{(2)}\partial_{31}(v)+\frac{1}{2}\mathcal{Z}_{21}x\partial_{21}\partial_{32}\partial_{31}^{(2)}(v)+$
$\mathcal{Z}_{21}x\partial_{31}^{(3)}(v)-\frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(4)}\partial_{32}^{(2)}(v)-\frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(3)}\partial_{32}\partial_{31}-\frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{31}^{(2)}(v)$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(5)}x(v)\right)$; where $v\in\mathcal{D}_{13}\otimes\mathcal{D}_5\otimes\mathcal{D}_0$

$=\sigma_1\left(\mathcal{Z}_{21}^{(5)}x\partial_{32}^{(3)}(v)+\mathcal{Z}_{21}^{(4)}x\partial_{32}^{(2)}\partial_{31}(v)+\mathcal{Z}_{21}^{(3)}x\partial_{32}\partial_{31}^{(2)}(v)+\mathcal{Z}_{21}^{(2)}x\partial_{31}^{(3)}(v)-\right.$
$\left.\mathcal{Z}_{32}^{(3)}y\partial_{21}^{(5)}(v)\right)$
$=\frac{1}{5}\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}^{(3)}(v)+\frac{1}{4}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}^{(2)}\partial_{31}(v)+\frac{1}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{32}\partial_{31}^{(2)}(v)+$
$\frac{1}{2}\mathcal{Z}_{21}x\partial_{21}\partial_{31}^{(3)}(v)-\frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(5)}\partial_{32}^{(2)}(v)-\frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(4)}\partial_{32}\partial_{31}-$
$\frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(3)}\partial_{31}^{(2)}(v)$

And

$\left(\delta_{\mathcal{L}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{L}_2\mathcal{M}_1}\right)\left(\frac{1}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{31}^{(2)}(v)-\frac{7}{90}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}^{(2)}(v)-\right.$
$\left.\frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(4)}\partial_{32}(v)\right)$

$=\sigma_1\left(\frac{1}{9}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}\partial_{31}^{(2)}(v)\right)+\frac{1}{9}\mathcal{Z}_{21}x\partial_{31}\partial_{21}\partial_{31}^{(2)}(v)-$
$\frac{1}{9}\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}\partial_{31}^{(2)}(v)-\sigma_1\left(\frac{7}{90}\mathcal{Z}_{21}^{(2)}x\partial_{23}\partial_{21}^{(3)}\partial_{32}^{(2)}(v)\right)-$
$\frac{7}{90}\mathcal{Z}_{21}x\partial_{31}\partial_{21}^{(3)}\partial_{32}^{(2)}(v)+\frac{7}{90}\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(3)}\partial_{32}^{(2)}(v)-$
$\sigma_1\left(\frac{2}{9}\mathcal{Z}_{32}^{(2)}y\partial_{21}\partial_{21}^{(4)}\partial_{32}(v)\right)+\frac{2}{9}\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(4)}\partial_{32}(v)+\frac{2}{9}\mathcal{Z}_{32}y\partial_{31}\partial_{21}^{(4)}\partial_{32}(v)$
$=\frac{1}{9}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{32}\partial_{31}^{(2)}(v)+\frac{9}{18}\mathcal{Z}_{21}x\partial_{21}\partial_{31}^{(3)}(v)-\frac{3}{9}\mathcal{Z}_{32}y\partial_{21}^{(3)}\partial_{31}^{(2)}(v)-$
$\frac{42}{90}\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}^{(3)}(v)-\frac{35}{180}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}^{(2)}\partial_{31}(v)+\frac{70}{90}\mathcal{Z}_{32}y\partial_{21}^{(5)}\partial_{32}^{(2)}(v)-$
$\frac{10}{9}\mathcal{Z}_{32}y\partial_{21}^{(5)}\partial_{32}^{(2)}(v)-\frac{6}{18}\mathcal{Z}_{32}y\partial_{21}^{(4)}\partial_{32}\partial_{31}(v)+\frac{6}{9}\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}^{(3)}(v)+$

$\frac{4}{9}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}^{(2)}\partial_{31}(v)+\frac{2}{9}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{32}\partial_{31}^{(2)}(v)$

$=\frac{1}{5}\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}^{(3)}(v)+\frac{1}{4}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}^{(2)}\partial_{31}(v)+\frac{1}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{32}\partial_{31}^{(2)}(v)+$
$\frac{1}{2}\mathcal{Z}_{21}x\partial_{21}\partial_{31}^{(3)}(v)-\frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(5)}\partial_{32}^{(2)}(v)-\frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(4)}\partial_{32}\partial_{31}-$
$\frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(3)}\partial_{31}^{(2)}(v)$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(6)}x(v)\right)$; where $v\in\mathcal{D}_{14}\otimes\mathcal{D}_4\otimes\mathcal{D}_0$

$=\sigma_1\left(\mathcal{Z}_{21}^{(6)}x\partial_{32}^{(3)}(v)+\mathcal{Z}_{21}^{(5)}x\partial_{32}^{(2)}\partial_{31}(v)+\mathcal{Z}_{21}^{(4)}x\partial_{32}\partial_{31}^{(2)}(v)+\mathcal{Z}_{21}^{(3)}x\partial_{31}^{(3)}(v)-\right.$
$\left.\mathcal{Z}_{32}^{(3)}y\partial_{21}^{(6)}(v)\right)$

$=\frac{1}{6}\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{32}^{(3)}(v)+\frac{1}{5}\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}^{(2)}\partial_{31}(v)+\frac{1}{4}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}\partial_{31}^{(2)}(v)+$
$\frac{1}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{31}^{(3)}(v)-\frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(6)}\partial_{32}^{(2)}(v)-\frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(5)}\partial_{32}\partial_{31}-$
$\frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(4)}\partial_{31}^{(2)}(v)$

And

$\left(\delta_{\mathcal{L}_2\mathcal{L}_1}+\sigma_1\circ\delta_{\mathcal{L}_2\mathcal{M}_1}\right)\left(\frac{1}{18}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}^{(2)}(v)-\frac{2}{45}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}^{(2)}(v)-\right.$
$\left.\frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{32}(v)\right)$

$=\sigma_1\left(\frac{1}{18}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(2)}\partial_{31}^{(2)}(v)\right)+\frac{1}{18}\mathcal{Z}_{21}x\partial_{31}\partial_{21}^{(2)}\partial_{31}^{(2)}(v)-$
$\frac{1}{18}\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(2)}\partial_{31}^{(2)}(v)-\sigma_1\left(\frac{2}{45}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(4)}\partial_{32}^{(2)}(v)\right)-$
$\frac{2}{45}\mathcal{Z}_{21}x\partial_{31}\partial_{21}^{(4)}\partial_{32}^{(2)}(v)+\frac{2}{45}\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(4)}\partial_{32}^{(2)}(v)-$
$\sigma_1\left(\frac{1}{6}\mathcal{Z}_{32}^{(2)}y\partial_{21}\partial_{21}^{(5)}\partial_{32}(v)\right)+\frac{1}{6}\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(5)}\partial_{32}(v)+\frac{1}{6}\mathcal{Z}_{32}y\partial_{31}\partial_{21}^{(5)}\partial_{32}(v)$

$=\frac{3}{36}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}\partial_{31}^{(2)}(v)+\frac{6}{18}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{31}^{(3)}(v)-\frac{6}{18}\mathcal{Z}_{32}y\partial_{21}^{(4)}\partial_{31}^{(2)}(v)-$
$\frac{30}{90}\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{32}^{(3)}(v)-\frac{6}{45}\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}^{(2)}\partial_{31}(v)+\frac{30}{45}\mathcal{Z}_{32}y\partial_{21}^{(6)}\partial_{32}^{(2)}(v)-$
$\frac{6}{6}\mathcal{Z}_{32}y\partial_{21}^{(6)}\partial_{32}^{(2)}(v)-\frac{2}{6}\mathcal{Z}_{32}y\partial_{21}^{(5)}\partial_{32}\partial_{31}(v)+\frac{3}{6}\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{32}^{(3)}(v)+$
$\frac{2}{6}\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}^{(2)}\partial_{31}(v)+\frac{1}{6}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}\partial_{31}^{(2)}(v)$

$= \frac{1}{6}\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{32}^{(3)}(v) + \frac{1}{5}\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}^{(2)}\partial_{31}(v) + \frac{1}{4}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}\partial_{31}^{(2)}(v) +$
$\frac{1}{3}\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{31}^{(3)}(v) - \frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(6)}\partial_{32}^{(2)}(v) - \frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(5)}\partial_{32}\partial_{31} - \frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(4)}\partial_{31}^{(2)}(v)$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(7)}x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_3 \otimes \mathcal{D}_0$

$= \sigma_1\left(\mathcal{Z}_{21}^{(7)}x\partial_{32}^{(3)}(v) + \mathcal{Z}_{21}^{(6)}x\partial_{32}^{(2)}\partial_{31}(v) + \mathcal{Z}_{21}^{(5)}x\partial_{32}\partial_{31}^{(2)}(v) + \mathcal{Z}_{21}^{(4)}x\partial_{31}^{(3)}(v) - \right.$
$\left.\mathcal{Z}_{32}^{(3)}y\partial_{21}^{(7)}(v)\right)$

$= \frac{1}{7}\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{32}^{(3)}(v) + \frac{1}{6}\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{32}^{(2)}\partial_{31}(v) + \frac{1}{5}\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}\partial_{31}^{(2)}(v) +$
$\frac{1}{4}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{31}^{(3)}(v) - \frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(7)}\partial_{32}^{(2)}(v) - \frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(6)}\partial_{32}\partial_{31} -$
$\frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(5)}\partial_{31}^{(2)}(v)$

And

$\left(\delta_{\mathcal{L}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1}\right)\left(\frac{1}{30}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}^{(2)}(v) - \frac{1}{35}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}^{(2)}(v) - \right.$
$\left.\frac{2}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{32}(v)\right)$

$= \sigma_1\left(\frac{1}{30}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(3)}\partial_{31}^{(2)}(v)\right) + \frac{1}{30}\mathcal{Z}_{21}x\partial_{31}\partial_{21}^{(3)}\partial_{31}^{(2)}(v) -$
$\frac{1}{30}\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(3)}\partial_{31}^{(2)}(v) - \sigma_1\left(\frac{1}{35}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(5)}\partial_{32}^{(2)}(v)\right) -$
$\frac{1}{35}\mathcal{Z}_{21}x\partial_{31}\partial_{21}^{(5)}\partial_{32}^{(2)}(v) + \frac{1}{35}\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(5)}\partial_{32}^{(2)}(v) -$
$\sigma_1\left(\frac{2}{15}\mathcal{Z}_{32}^{(2)}y\partial_{21}\partial_{21}^{(6)}\partial_{32}(v)\right) + \frac{2}{15}\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(6)}\partial_{32}(v) +$
$\frac{2}{15}\mathcal{Z}_{32}y\partial_{31}\partial_{21}^{(6)}\partial_{32}(v)$

$= \frac{2}{30}\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}\partial_{31}^{(2)}(v) + \frac{15}{60}\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{31}^{(3)}(v) - \frac{10}{30}\mathcal{Z}_{32}y\partial_{21}^{(5)}\partial_{31}^{(2)}(v) -$
$\frac{9}{35}\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{32}^{(3)}(v) - \frac{7}{70}\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{32}^{(2)}\partial_{31}(v) + \frac{21}{35}\mathcal{Z}_{32}y\partial_{21}^{(7)}\partial_{32}^{(2)}(v) -$
$\frac{14}{15}\mathcal{Z}_{32}y\partial_{21}^{(7)}\partial_{32}^{(2)}(v) - \frac{10}{30}\mathcal{Z}_{32}y\partial_{21}^{(6)}\partial_{32}\partial_{31}(v) + \frac{6}{15}\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{32}^{(3)}(v) +$
$\frac{4}{15}\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{32}^{(2)}\partial_{31}(v) + \frac{2}{15}\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}\partial_{31}^{(2)}(v)$

$= \frac{1}{7} \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{32}^{(3)}(v) + \frac{1}{6} \mathcal{Z}_{21} x \partial_{21}^{(5)} \partial_{32}^{(2)} \partial_{31}(v) + \frac{1}{5} \mathcal{Z}_{21} x \partial_{21}^{(4)} \partial_{32} \partial_{31}^{(2)}(v) +$
$\frac{1}{4} \mathcal{Z}_{21} x \partial_{21}^{(3)} \partial_{31}^{(3)}(v) - \frac{1}{3} \mathcal{Z}_{32} y \partial_{21}^{(7)} \partial_{32}^{(2)}(v) - \frac{1}{3} \mathcal{Z}_{32} y \partial_{21}^{(6)} \partial_{32} \partial_{31} -$
$\frac{1}{3} \mathcal{Z}_{32} y \partial_{21}^{(5)} \partial_{31}^{(2)}(v)$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(8)} x(v)\right)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_2 \otimes \mathcal{D}_0$

$= \mathcal{Z}_{21}^{(8)} x \partial_{32}^{(3)}(v) + \sigma_1 \left(\mathcal{Z}_{21}^{(7)} x \partial_{32}^{(2)} \partial_{31}(v) + \mathcal{Z}_{21}^{(6)} x \partial_{32} \partial_{31}^{(2)}(v) + \mathcal{Z}_{21}^{(5)} x \partial_{31}^{(3)}(v) -\right.$
$\left.\mathcal{Z}_{32}^{(3)} y \partial_{21}^{(8)}(v)\right)$

$= \frac{1}{7} \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{32}^{(2)} \partial_{31}(v) + \frac{1}{6} \mathcal{Z}_{21} x \partial_{21}^{(5)} \partial_{32} \partial_{31}^{(2)}(v) + \frac{1}{5} \mathcal{Z}_{21} x \partial_{21}^{(4)} \partial_{31}^{(3)}(v) -$
$\frac{1}{3} \mathcal{Z}_{32} y \partial_{21}^{(8)} \partial_{32}^{(2)}(v) - \frac{1}{3} \mathcal{Z}_{32} y \partial_{21}^{(7)} \partial_{32} \partial_{31} - \frac{1}{3} \mathcal{Z}_{32} y \partial_{21}^{(6)} \partial_{31}^{(2)}(v)$

And

$\left(\delta_{\mathcal{L}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1}\right)\left(\frac{1}{45} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(4)} \partial_{31}^{(2)}(v) - \frac{5}{252} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(6)} \partial_{32}^{(2)}(v) -\right.$
$\left.\frac{1}{9} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(7)} \partial_{32}(v)\right)$

$= \sigma_1 \left(\frac{1}{45} \mathcal{Z}_{21}^{(2)} x \partial_{32} \partial_{21}^{(4)} \partial_{31}^{(2)}(v)\right) + \frac{1}{45} \mathcal{Z}_{21} x \partial_{31} \partial_{21}^{(4)} \partial_{31}^{(2)}(v) -$
$\frac{1}{45} \mathcal{Z}_{32} y \partial_{21}^{(2)} \partial_{21}^{(4)} \partial_{31}^{(2)}(v) - \sigma_1 \left(\frac{5}{252} \mathcal{Z}_{21}^{(2)} x \partial_{32} \partial_{21}^{(6)} \partial_{32}^{(2)}(v)\right) -$
$\frac{5}{252} \mathcal{Z}_{21} x \partial_{31} \partial_{21}^{(6)} \partial_{32}^{(2)}(v) + \frac{5}{252} \mathcal{Z}_{32} y \partial_{21}^{(2)} \partial_{21}^{(6)} \partial_{32}^{(2)}(v) -$
$\sigma_1 \left(\frac{1}{9} \mathcal{Z}_{32}^{(2)} y \partial_{21} \partial_{21}^{(7)} \partial_{32}(v)\right) + \frac{1}{9} \mathcal{Z}_{21} x \partial_{32}^{(2)} \partial_{21}^{(7)} \partial_{32}(v) + \frac{1}{9} \mathcal{Z}_{32} y \partial_{31} \partial_{21}^{(7)} \partial_{32}(v)$

$= \frac{5}{90} \mathcal{Z}_{21} x \partial_{21}^{(5)} \partial_{32} \partial_{31}^{(2)}(v) + \frac{9}{45} \mathcal{Z}_{21} x \partial_{21}^{(4)} \partial_{31}^{(3)}(v) - \frac{15}{45} \mathcal{Z}_{32} y \partial_{21}^{(6)} \partial_{31}^{(2)}(v) -$
$\frac{20}{252} \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{32}^{(2)} \partial_{31}(v) + \frac{140}{252} \mathcal{Z}_{21} x \partial_{21}^{(8)} \partial_{32}^{(2)}(v) - \frac{8}{9} \mathcal{Z}_{32} y \partial_{21}^{(8)} \partial_{32}^{(2)}(v) -$
$\frac{3}{9} \mathcal{Z}_{32} y \partial_{21}^{(7)} \partial_{32} \partial_{31}(v) + \frac{2}{9} \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{32}^{(2)} \partial_{31}(v) + \frac{1}{9} \mathcal{Z}_{21} x \partial_{21}^{(5)} \partial_{32} \partial_{31}^{(2)}(v)$

$= \frac{1}{7} \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{32}^{(2)} \partial_{31}(v) + \frac{1}{6} \mathcal{Z}_{21} x \partial_{21}^{(5)} \partial_{32} \partial_{31}^{(2)}(v) + \frac{1}{5} \mathcal{Z}_{21} x \partial_{21}^{(4)} \partial_{31}^{(3)}(v) -$
$\frac{1}{3} \mathcal{Z}_{32} y \partial_{21}^{(8)} \partial_{32}^{(2)}(v) - \frac{1}{3} \mathcal{Z}_{32} y \partial_{21}^{(7)} \partial_{32} \partial_{31} - \frac{1}{3} \mathcal{Z}_{32} y \partial_{21}^{(6)} \partial_{31}^{(2)}(v)$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(9)} x(v)\right)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$

$= \mathcal{Z}_{21}^{(9)} x \partial_{32}^{(3)}(v) + \mathcal{Z}_{21}^{(8)} x \partial_{32}^{(2)} \partial_{31}(v) + \sigma_1\left(\mathcal{Z}_{21}^{(7)} x \partial_{32} \partial_{31}^{(2)}(v) + \mathcal{Z}_{21}^{(6)} x \partial_{31}^{(3)}(v) - \mathcal{Z}_{32}^{(3)} y \partial_{21}^{(9)}(v)\right)$

$= \frac{1}{7} \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{32} \partial_{31}^{(2)}(v) + \frac{1}{6} \mathcal{Z}_{21} x \partial_{21}^{(5)} \partial_{31}^{(3)}(v) - \frac{1}{3} \mathcal{Z}_{32} y \partial_{21}^{(8)} \partial_{32} \partial_{31} - \frac{1}{3} \mathcal{Z}_{32} y \partial_{21}^{(7)} \partial_{31}^{(2)}(v)$

And

$\left(\delta_{\mathcal{L}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1}\right)\left(\frac{1}{63} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}^{(2)}(v) + \frac{1}{84} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(6)} \partial_{32} \partial_{31}(v)\right)$

$= \sigma_1\left(\frac{1}{63} \mathcal{Z}_{21}^{(2)} x \partial_{32} \partial_{21}^{(5)} \partial_{31}^{(2)}(v)\right) + \frac{1}{63} \mathcal{Z}_{21} x \partial_{31} \partial_{21}^{(5)} \partial_{31}^{(2)}(v) - \frac{1}{63} \mathcal{Z}_{32} y \partial_{21}^{(2)} \partial_{21}^{(5)} \partial_{31}^{(2)}(v) + \sigma_1\left(\frac{1}{84} \mathcal{Z}_{21}^{(2)} x \partial_{32} \partial_{21}^{(6)} \partial_{32} \partial_{31}(v)\right) + \frac{1}{84} \mathcal{Z}_{21} x \partial_{31} \partial_{21}^{(6)} \partial_{32} \partial_{31}(v) - \frac{1}{84} \mathcal{Z}_{32} y \partial_{21}^{(2)} \partial_{21}^{(6)} \partial_{32} \partial_{31}(v)$

$= \frac{3}{63} \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{32} \partial_{31}^{(2)}(v) + \frac{21}{126} \mathcal{Z}_{21} x \partial_{21}^{(5)} \partial_{31}^{(3)}(v) - \frac{21}{63} \mathcal{Z}_{32} y \partial_{21}^{(7)} \partial_{31}^{(2)}(v) + \frac{8}{84} \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{32} \partial_{31}^{(2)}(v) - \frac{28}{84} \mathcal{Z}_{32} y \partial_{21}^{(8)} \partial_{32} \partial_{31}(v)$

$= \frac{1}{7} \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{32} \partial_{31}^{(2)}(v) + \frac{1}{6} \mathcal{Z}_{21} x \partial_{21}^{(5)} \partial_{31}^{(3)}(v) - \frac{1}{3} \mathcal{Z}_{32} y \partial_{21}^{(9)} \partial_{32}^{(2)}(v) - \frac{1}{3} \mathcal{Z}_{32} y \partial_{21}^{(8)} \partial_{32} \partial_{31} - \frac{1}{3} \mathcal{Z}_{32} y \partial_{21}^{(7)} \partial_{31}^{(2)}(v)$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(10)} x(v)\right)$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$

$= \mathcal{Z}_{21}^{(10)} x \partial_{32}^{(3)}(v) + \mathcal{Z}_{21}^{(9)} x \partial_{32}^{(2)} \partial_{31}(v) + \mathcal{Z}_{21}^{(8)} x \partial_{32} \partial_{31}^{(2)}(v) + \sigma_1\left(\mathcal{Z}_{21}^{(7)} x \partial_{31}^{(3)}(v) - \mathcal{Z}_{32}^{(3)} y \partial_{21}^{(10)}(v)\right)$

$= \frac{1}{7} \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{31}^{(3)}(v) - \frac{1}{3} \mathcal{Z}_{32} y \partial_{21}^{(8)} \partial_{31}^{(2)}(v)$

And

$\left(\delta_{\mathcal{L}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1}\right)\left(\frac{1}{84} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(6)} \partial_{31}^{(2)}(v)\right)$

$= \sigma_1\left(\frac{1}{84} \mathcal{Z}_{21}^{(2)} x \partial_{32} \partial_{21}^{(6)} \partial_{31}^{(2)}(v)\right) + \frac{1}{84} \mathcal{Z}_{21} x \partial_{31} \partial_{21}^{(6)} \partial_{31}^{(2)}(v) -$

$\frac{1}{84}\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(6)}\partial_{31}^{(2)}(v)$

$= \frac{1}{56}\mathcal{Z}_{21}x\partial_{21}\partial_{21}^{(5)}\partial_{31}^{(3)}(v) + \frac{1}{28}\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{31}^{(3)}(v) - \frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(8)}\partial_{31}^{(2)}(v) +$

$= \frac{1}{7}\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{31}^{(3)}(v) - \frac{1}{3}\mathcal{Z}_{32}y\partial_{21}^{(8)}\partial_{31}^{(2)}(v)$

- $\left(\delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}\right)\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{31}z(v)\right)$; where $v \in \mathcal{D}_9 \otimes \mathcal{D}_9 \otimes \mathcal{D}_0$

$= \sigma_1\left(\mathcal{Z}_{32}^{(3)}y\partial_{21}(v)\right) - \mathcal{Z}_{21}x\partial_{32}^{(3)}(v) - \sigma_1\left(\mathcal{Z}_{32}^{(2)}y\partial_{31}(v)\right)$

$= \frac{1}{3}\mathcal{Z}_{32}y\partial_{32}^{(2)}\partial_{21}(v) - \mathcal{Z}_{21}x\partial_{32}^{(3)}(v) - \frac{1}{2}\mathcal{Z}_{32}y\partial_{32}\partial_{31}(v)$

And

$\left(\delta_{\mathcal{L}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1}\right)\left(\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{32}(v)\right)$

$= \sigma_1\left(\frac{1}{3}\mathcal{Z}_{32}^{(2)}y\partial_{21}\partial_{32}(v)\right) - \frac{1}{3}\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{32}(v) - \frac{1}{3}\mathcal{Z}_{32}y\partial_{31}\partial_{32}(v)$

$= \frac{1}{6}\mathcal{Z}_{32}y\partial_{32}\partial_{21}\partial_{32}(v) - \mathcal{Z}_{21}x\partial_{32}^{(3)}(v) - \frac{1}{3}\mathcal{Z}_{32}y\partial_{32}\partial_{31}(v)$

$= \frac{1}{3}\mathcal{Z}_{32}y\partial_{32}^{(2)}\partial_{21}(v) - \mathcal{Z}_{21}x\partial_{32}^{(3)}(v) - \frac{1}{2}\mathcal{Z}_{32}y\partial_{32}\partial_{31}(v)$ ■

Now by employ σ_2 we can also define

$$\partial_3 : \mathcal{L}_3 \longrightarrow \mathcal{L}_2 \quad \text{as} \quad \partial_3 = \delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}$$

Lemma (3.3.3):

The composition $\partial_2\partial_3$ equal to zero.

Proof:

$\partial_2\partial_3(a) = \left(\delta_{\mathcal{L}_2\mathcal{L}_1}(a) + (\sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1})(a)\right) \circ \left(\delta_{\mathcal{L}_3\mathcal{L}_2}(a) + (\sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2})(a)\right)$

$= (\delta_{\mathcal{L}_2\mathcal{L}_1} \circ \delta_{\mathcal{L}_3\mathcal{L}_2})(a) + (\delta_{\mathcal{L}_2\mathcal{L}_1} \circ \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2})(a) +$

$(\sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1} \circ \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2})(a)$

But $\delta_{\mathcal{L}_2\mathcal{L}_1} \circ \sigma_2 + \sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{M}_1} \circ \sigma_2 = \delta_{\mathcal{M}_2\mathcal{L}_1} + \sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1}$ so we get

$$\partial_2\partial_3(a) = (\delta_{\mathcal{L}_2\mathcal{L}_1} \circ \delta_{\mathcal{L}_3\mathcal{L}_2})(a) + (\delta_{\mathcal{M}_2\mathcal{L}_1} \circ \delta_{\mathcal{L}_3\mathcal{M}_2})(a) + (\sigma_1 \circ \delta_{\mathcal{L}_2\mathcal{L}_1} \circ \delta_{\mathcal{L}_3\mathcal{L}_2})(a)$$
$$(\sigma_1 \circ \delta_{\mathcal{M}_2\mathcal{M}_1} \circ \delta_{\mathcal{L}_2\mathcal{M}_2})(a)$$

By properties of the boundary map δ we get

$\partial_2\partial_3 = 0$ ■

We need the definition of a map $\sigma_3\colon \mathcal{M}_3 \longrightarrow \mathcal{L}_3$ such that

$$\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2} = \left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right) \circ \sigma_3 \qquad \ldots(3.3.3)$$

As follows:

- $\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_4 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_3 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_3 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_3 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(3)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(4)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(3)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$

- $\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(3)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(5)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(3)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(3)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(3)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(4)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(4)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$
- $\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_5 \otimes \mathcal{D}_2$
- $\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_4 \otimes \mathcal{D}_2$
- $\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_4 \otimes \mathcal{D}_2$
- $\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_3 \otimes \mathcal{D}_2$
- $\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(3)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_3 \otimes \mathcal{D}_2$
- $\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_3 \otimes \mathcal{D}_2$
- $\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_2 \otimes \mathcal{D}_2$

- $Z_{32}yZ_{21}^{(4)}xZ_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_2 \otimes \mathcal{D}_2$
- $Z_{32}yZ_{21}^{(2)}xZ_{21}^{(4)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_2 \otimes \mathcal{D}_2$
- $Z_{32}yZ_{21}^{(3)}xZ_{21}^{(3)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_2 \otimes \mathcal{D}_2$
- $Z_{32}yZ_{21}^{(6)}xZ_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_1 \otimes \mathcal{D}_2$
- $Z_{32}yZ_{21}^{(5)}xZ_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_1 \otimes \mathcal{D}_2$
- $Z_{32}yZ_{21}^{(2)}xZ_{21}^{(5)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_1 \otimes \mathcal{D}_2$
- $Z_{32}yZ_{21}^{(4)}xZ_{21}^{(3)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_1 \otimes \mathcal{D}_2$
- $Z_{32}yZ_{21}^{(3)}xZ_{21}^{(4)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_1 \otimes \mathcal{D}_2$
- $Z_{32}yZ_{21}^{(7)}xZ_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_0 \otimes \mathcal{D}_2$
- $Z_{32}yZ_{21}^{(6)}xZ_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_0 \otimes \mathcal{D}_2$
- $Z_{32}yZ_{21}^{(2)}xZ_{21}^{(6)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_0 \otimes \mathcal{D}_2$
- $Z_{32}yZ_{21}^{(5)}xZ_{21}^{(3)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_0 \otimes \mathcal{D}_2$
- $Z_{32}yZ_{21}^{(3)}xZ_{21}^{(5)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_0 \otimes \mathcal{D}_2$
- $Z_{32}yZ_{21}^{(4)}xZ_{21}^{(4)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_0 \otimes \mathcal{D}_2$
- $Z_{32}yZ_{32}yZ_{21}^{(3)}x(v) \longmapsto -\frac{1}{3}Z_{32}yZ_{31}zZ_{21}x\partial_{21}(v)$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_6 \otimes \mathcal{D}_1$
- $Z_{32}yZ_{32}yZ_{21}^{(4)}x(v) \longmapsto -\frac{1}{6}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(2)}(v)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_5 \otimes \mathcal{D}_1$
- $Z_{32}yZ_{32}yZ_{21}^{(5)}x(v) \longmapsto -\frac{1}{10}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(3)}(v)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_4 \otimes \mathcal{D}_1$
- $Z_{32}yZ_{32}yZ_{21}^{(6)}x(v) \longmapsto -\frac{1}{15}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(4)}(v)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_3 \otimes \mathcal{D}_1$
- $Z_{32}yZ_{32}yZ_{21}^{(7)}x(v) \longmapsto -\frac{1}{21}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(5)}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_2 \otimes \mathcal{D}_1$

- $\mathcal{Z}_{32}y\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(8)}x(v) \longmapsto -\frac{1}{28}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(6)}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_1 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}y\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(9)}x(v) \longmapsto -\frac{1}{36}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(7)}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_5 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_4 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_4 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_3 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_3 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(3)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_3 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(6)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_2 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_2 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(3)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_2 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(4)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_2 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(7)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_1 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(6)}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_1 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}^{(3)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_1 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(4)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_1 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(5)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_1 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(8)}x\mathcal{Z}_{21}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(7)}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(6)}x\mathcal{Z}_{21}^{(3)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}^{(4)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(5)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(6)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$

- $\mathcal{Z}_{32}y\mathcal{Z}_{32}y\mathcal{Z}_{32}y(v) \longmapsto 0$; where $v \in \mathcal{D}_8 \otimes \mathcal{D}_{10} \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(4)}x(v) \longmapsto \frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}\partial_{31}(v) - \frac{1}{4}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{32}(v)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_6 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(5)}x(v) \longmapsto \frac{1}{12}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{31}(v) - \frac{7}{60}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}(v)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_5 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(6)}x(v) \longmapsto \frac{1}{20}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{31}(v) - \frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}(v)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_4 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(7)}x(v) \longmapsto \frac{1}{30}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{31}(v) - \frac{3}{70}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{32}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_3 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(8)}x(v) \longmapsto \frac{1}{42}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{31}(v) - \frac{5}{168}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{32}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_2 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(9)}x(v) \longmapsto \frac{1}{56}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{31}(v) + \frac{1}{72}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(7)}\partial_{32}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(10)}x(v) \longmapsto \frac{1}{72}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(7)}\partial_{31}(v)$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}y\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(4)}x(v) \longmapsto -\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{32}(v)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_6 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}y\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(5)}x(v) \longmapsto -\frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}(v)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_5 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}y\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(6)}x(v) \longmapsto -\frac{1}{10}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}(v)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_4 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}y\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(7)}x(v) \longmapsto -\frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{32}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_3 \otimes \mathcal{D}_0$

- $\mathcal{Z}_{32}y\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(8)}x(v) \longmapsto -\frac{1}{21}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{32}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_2 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}y\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(9)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}y\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(10)}x(v) \longmapsto 0$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x(v) \longmapsto -\frac{1}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{31}(v) - \frac{1}{18}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}(v)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_5 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}x(v) \longmapsto -\frac{1}{18}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{31}(v) - \frac{1}{45}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}(v)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_4 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto -\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{31}(v) - \frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}(v)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_4 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(6)}x\mathcal{Z}_{21}x(v) \longmapsto -\frac{1}{30}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{31}(v) - \frac{1}{90}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{32}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_3 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto -\frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{31}(v) - \frac{4}{45}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{32}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_3 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(3)}x(v) \longmapsto -\frac{2}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{31}(v) - \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{32}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_3 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(7)}x\mathcal{Z}_{21}x(v) \longmapsto -\frac{1}{45}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{31}(v) - \frac{2}{315}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{32}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_2 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(6)}x\mathcal{Z}_{21}^{(2)}x(v) \longmapsto -\frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{31}(v) - \frac{1}{18}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{32}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_2 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}^{(3)}x(v) \longmapsto -\frac{5}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{31}(v) - \frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{32}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_2 \otimes \mathcal{D}_0$

- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(4)} x(v) \mapsto -\frac{10}{9} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(5)} \partial_{31}(v) -$
$\frac{5}{9} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{32}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_2 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(8)} x \mathcal{Z}_{21} x(v) \mapsto -\frac{1}{63} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{31}(v) -$
$\frac{1}{9} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(7)} \partial_{32}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(7)} x \mathcal{Z}_{21}^{(2)} x(v) \mapsto -\frac{2}{15} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{31}(v) -$
$\frac{7}{15} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(7)} \partial_{32}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(6)} x \mathcal{Z}_{21}^{(3)} x(v) \mapsto -\frac{1}{2} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{31}(v) -$
$\frac{7}{6} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(7)} \partial_{32}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21}^{(4)} x(v) \mapsto -\frac{10}{9} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{31}(v) -$
$\frac{35}{18} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(7)} \partial_{32}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(5)} x(v) \mapsto -\frac{5}{3} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{31}(v) -$
$\frac{7}{3} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(7)} \partial_{32}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(9)} x \mathcal{Z}_{21} x(v) \mapsto 0$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(8)} x \mathcal{Z}_{21}^{(2)} x(v) \mapsto -\frac{1}{9} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(7)} \partial_{31}(v)$; where
$v \in \mathcal{D}_{18} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(7)} x \mathcal{Z}_{21}^{(3)} x(v) \mapsto -\frac{7}{15} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(7)} \partial_{31}(v)$; where
$v \in \mathcal{D}_{18} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(6)} x \mathcal{Z}_{21}^{(4)} x(v) \mapsto -\frac{7}{6} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(7)} \partial_{31}(v)$; where
$v \in \mathcal{D}_{18} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21}^{(5)} x(v) \mapsto -\frac{35}{18} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(7)} \partial_{31}(v)$; where
$v \in \mathcal{D}_{18} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(6)} x(v) \mapsto -\frac{7}{3} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(7)} \partial_{31}(v)$; where
$v \in \mathcal{D}_{18} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$

- $\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(2)}x(v) \longmapsto \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}(v)$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_6 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(3)}x(v) \longmapsto \frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(2)}(v)$;where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_5 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(4)}x(v) \longmapsto \frac{1}{10}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(3)}(v)$;where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_4 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(5)}x(v) \longmapsto \frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(4)}(v)$;where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_3 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(6)}x(v) \longmapsto \frac{1}{21}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(5)}(v)$;where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_2 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(7)}x(v) \longmapsto \frac{1}{28}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(6)}(v)$;where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_1 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(8)}x(v) \longmapsto \frac{1}{36}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(7)}(v)$;where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$
- $\mathcal{Z}_{32}y\mathcal{Z}_{32}y\mathcal{Z}_{31}z(v) \longmapsto 0$;where $v \in \mathcal{D}_9 \otimes \mathcal{D}_9 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(2)}x(v) \longmapsto \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{31}(v)$;where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_7 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(3)}x(v) \longmapsto \frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}\partial_{31}(v) -$
  $\frac{1}{12}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{32}(v)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_6 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(4)}x(v) \longmapsto \frac{1}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{31}(v) -$
  $\frac{7}{90}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}(v)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_5 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(5)}x(v) \longmapsto \frac{1}{12}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{31}(v) -$
  $\frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}(v)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_4 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(6)}x(v) \longmapsto \frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{31}(v) -$
  $\frac{2}{35}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{32}(v)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_3 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(7)}x(v) \longmapsto \frac{1}{18}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{31}(v) -$
  $\frac{25}{504}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{32}(v)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_2 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(8)}x(v) \longmapsto \frac{1}{21}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{31}(v) +$
  $\frac{1}{36}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(7)}\partial_{32}(v)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$
- $\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(9)}x(v) \longmapsto \frac{1}{24}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(7)}\partial_{31}(v)$; where
  $v \in \mathcal{D}_{18} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$

Proposition (3.3.4):

The map σ_3 defined above satisfies (3.3.3).

Proof:

We can see that

- $(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2})(\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x(v))$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_4 \otimes \mathcal{D}_3$

$= \sigma_2 \left(2\, \mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x(v) - 2\, \mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x(v) + \mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{21}(v)\right)$

$= 0$

- $(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2})\left(\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x(v)\right)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_3 \otimes \mathcal{D}_3$

$= \sigma_2 \left(3\, \mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x(v) - 2\, \mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x(v) + \mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x\partial_{21}(v)\right)$

$= 0$

- $(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2})\left(\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x(v)\right)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_3 \otimes \mathcal{D}_3$

$= \sigma_2 \left(3\, \mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x(v) - 3\, \mathcal{Z}_{21}x\mathcal{Z}_{21}^{(3)}x(v) + \mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x\partial_{21}(v)\right) = 0$

- $(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2})\left(\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x(v)\right)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_3 \otimes \mathcal{D}_3$

$= \sigma_2 \left(2\, \mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x(v) - 3\, \mathcal{Z}_{21}x\mathcal{Z}_{21}^{(3)}x(v) + \mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{21}^{(2)}(v)\right)$

$= 0$

- $(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2})\left(\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x(v)\right)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$

$= \sigma_2 \left(4\, \mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x(v) - 2\, \mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(2)}x(v) + \mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x\partial_{21}(v)\right)$

$= 0$

- $(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2})\left(\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x(v)\right)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$

$= \sigma_2 \left(4\, \mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x(v) - 4\, \mathcal{Z}_{21}x\mathcal{Z}_{21}^{(4)}x(v) + \mathcal{Z}_{21}x\mathcal{Z}_{21}^{(3)}x\partial_{21}(v)\right)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(3)}x(v)\right)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$

$= \sigma_2\left(2\, \mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(3)}x(v) - 4\, \mathcal{Z}_{21}x\mathcal{Z}_{21}^{(4)}x(v) + \mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{21}^{(3)}(v)\right)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x(v)\right)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$

$= \sigma_2\left(6\, \mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x(v) - 3\, \mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(3)}x(v) + \mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x\partial_{21}(v)\right)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x(v)\right)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$

$= \sigma_2\left(3\, \mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(2)}x(v) - 3\, \mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(3)}x(v) + \mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x\partial_{21}^{(2)}(v)\right)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x(v)\right)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_2 \otimes \mathcal{D}_3$

$= \sigma_2\left(3\, \mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x(v) - 6\, \mathcal{Z}_{21}x\mathcal{Z}_{21}^{(4)}x(v) + \mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}(v)\right)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x\mathcal{Z}_{21}x(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$

$= \sigma_2\left(5\, \mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}x(v) - 2\, \mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(2)}x(v) + \mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x\partial_{21}(v)\right)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$

$= \sigma_2\left(5\, \mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}x(v) - 5\, \mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}x(v) + \mathcal{Z}_{21}x\mathcal{Z}_{21}^{(4)}x\partial_{21}(v)\right)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{21}x\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(4)}x(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$

$= \sigma_2\left(2\, \mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(4)}x(v) - 5\, \mathcal{Z}_{21}x\mathcal{Z}_{21}^{(5)}x(v) + \mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{21}^{(4)}(v)\right)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21} x(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$

$= \sigma_2\left(10\, \mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21} x(v) - 3\, \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(3)} x(v) + \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(2)} x \partial_{21}(v)\right)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21} x \mathcal{Z}_{21}^{(2)} x(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$

$= \sigma_2\left(4\, \mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(2)} x(v) - 3\, \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(3)} x(v) + \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21} x \partial_{21}^{(2)}(v)\right)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21} x(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$

$= \sigma_2\left(10\, \mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21} x(v) - 4\, \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(4)} x(v) + \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(3)} x \partial_{21}(v)\right)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21} x \mathcal{Z}_{21}^{(3)} x(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$

$= \sigma_2\left(3\, \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(3)} x(v) - 4\, \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(4)} x(v) + \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21} x \partial_{21}^{(3)}(v)\right)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{21} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(3)} x(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$

$= \sigma_2\left(3\, \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(3)} x(v) - 10\, \mathcal{Z}_{21} x \mathcal{Z}_{21}^{(5)} x(v) + \mathcal{Z}_{21} x \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(3)}(v)\right)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{21} x \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(2)} x(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$

$= \sigma_2\left(4\, \mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(2)} x(v) - 10\, \mathcal{Z}_{21} x \mathcal{Z}_{21}^{(5)} x(v) + \mathcal{Z}_{21} x \mathcal{Z}_{21}^{(3)} x \partial_{21}^{(2)}(v)\right)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(2)} x(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_1 \otimes \mathcal{D}_3$

$= \sigma_2\left(6\, \mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(2)} x(v) - 6\, \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(4)} x(v) + \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(2)}(v)\right)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21} x \mathcal{Z}_{21} x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$

$= \sigma_2\left(6\, \mathcal{Z}_{21}^{(6)} x \mathcal{Z}_{21} x(v) - 2\, \mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21}^{(2)} x(v) + \mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21} x \partial_{21}(v)\right)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{21} x \mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21} x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$

$= \sigma_2\left(6\, \mathcal{Z}_{21}^{(6)} x \mathcal{Z}_{21} x(v) - 6\, \mathcal{Z}_{21} x \mathcal{Z}_{21}^{(6)} x(v) + \mathcal{Z}_{21} x \mathcal{Z}_{21}^{(5)} x \partial_{21}(v)\right)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{21} x \mathcal{Z}_{21} x \mathcal{Z}_{21}^{(5)} x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$

$= \sigma_2\left(2\, \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(5)} x(v) - 6\, \mathcal{Z}_{21} x \mathcal{Z}_{21}^{(6)} x(v) + \mathcal{Z}_{21} x \mathcal{Z}_{21} x \partial_{21}^{(5)}(v)\right)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(3)} x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$

$= \sigma_2\left(6\, \mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(3)} x(v) - 10\, \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(5)} x(v) + \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(3)}(v)\right)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(2)} x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$

$= \sigma_2\left(10\, \mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21}^{(2)} x(v) - 10\, \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(5)} x(v) + \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(3)} x \partial_{21}^{(2)}(v)\right)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(2)} x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$

$= \sigma_2\left(10\, \mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21}^{(2)} x(v) - 6\, \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(4)} x(v) + \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(2)}(v)\right)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21} x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$

$= \sigma_2\left(20\, \mathcal{Z}_{21}^{(6)} x \mathcal{Z}_{21} x(v) - 4\, \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(4)} x(v) + \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(3)} x \partial_{21}(v)\right)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(Z_{21}^{(3)} x Z_{21} x Z_{21}^{(3)} x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$

$= \sigma_2\left(4\, Z_{21}^{(4)} x Z_{21}^{(3)} x(v) - 4\, Z_{21}^{(3)} x Z_{21}^{(4)} x(v) + Z_{21}^{(3)} x Z_{21} x \partial_{21}^{(3)}(v)\right)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(Z_{21} x Z_{21}^{(3)} x Z_{21}^{(3)} x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$

$= \sigma_2\left(4\, Z_{21}^{(4)} x Z_{21}^{(3)} x(v) - 20\, Z_{21} x Z_{21}^{(6)} x(v) + Z_{21} x Z_{21}^{(3)} x \partial_{21}^{(3)}(v)\right)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(Z_{21}^{(2)} x Z_{21}^{(4)} x Z_{21} x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$

$= \sigma_2\left(15\, Z_{21}^{(6)} x Z_{21} x(v) - 5\, Z_{21}^{(2)} x Z_{21}^{(5)} x(v) + Z_{21}^{(2)} x Z_{21}^{(4)} x \partial_{21}(v)\right)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(Z_{21}^{(2)} x Z_{21} x Z_{21}^{(4)} x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$

$= \sigma_2\left(3\, Z_{21}^{(3)} x Z_{21}^{(4)} x(v) - 5\, Z_{21}^{(2)} x Z_{21}^{(5)} x(v) + Z_{21}^{(2)} x Z_{21} x \partial_{21}^{(4)}(v)\right)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(Z_{21}^{(4)} x Z_{21}^{(2)} x Z_{21} x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$

$= \sigma_2\left(15\, Z_{21}^{(6)} x Z_{21} x(v) - 3\, Z_{21}^{(4)} x Z_{21}^{(3)} x(v) + Z_{21}^{(4)} x Z_{21}^{(2)} x \partial_{21}(v)\right)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(Z_{21}^{(4)} x Z_{21} x Z_{21}^{(2)} x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$

$= \sigma_2\left(5\, Z_{21}^{(5)} x Z_{21}^{(2)} x(v) - 3\, Z_{21}^{(4)} x Z_{21}^{(3)} x(v) + Z_{21}^{(4)} x Z_{21} x \partial_{21}^{(2)}(v)\right)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(Z_{21} x Z_{21}^{(4)} x Z_{21}^{(2)} x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$

$= \sigma_2\left(5\, Z_{21}^{(5)} x Z_{21}^{(2)} x(v) - 15\, Z_{21} x Z_{21}^{(6)} x(v) + Z_{21} x Z_{21}^{(4)} x \partial_{21}^{(2)}(v)\right)$

$= 0$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(4)}x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_0 \otimes \mathcal{D}_3$

$= \sigma_2\left(3\, \mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(4)}x(v) - 15\, \mathcal{Z}_{21}x\mathcal{Z}_{21}^{(6)}x(v) + \mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}(v)\right)$

$= 0$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x(v)\right)$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_5 \otimes \mathcal{D}_2$

$= \sigma_2\left(\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x\partial_{32}(v) + \mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{31}(v) - 3\, \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(3)}x(v)\right) +$

$\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}(v)$

$= -\frac{3}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}(v) + \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}(v)$

$= 0$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x(v)\right)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_4 \otimes \mathcal{D}_2$

$= \sigma_2\left(\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x\partial_{32}(v) + \mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x\partial_{31}(v) - 4\, \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(4)}x(v) +\right.$

$\left.\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(3)}x\partial_{21}(v)\right)$

$= -\frac{4}{6}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}(v) + \frac{2}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}(v)$

$= 0$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x(v)\right)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_4 \otimes \mathcal{D}_2$

$= \sigma_2\left(\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x\partial_{32}(v) + \mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x\partial_{31}(v) - 3\, \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(3)}x(v)\right) +$

$\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}(v)$

$= -\frac{3}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}(v) + \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}(v)$

$= 0$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x(v)\right)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_3 \otimes \mathcal{D}_2$

$= \sigma_2\left(\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x\partial_{32}(v) + \mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x\partial_{31}(v) - 5\, \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(5)}x(v) +\right.$

$\left.\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(4)}x\partial_{21}(v)\right)$

$= -\frac{5}{10} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(3)}(v) + \frac{3}{6} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(3)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(3)} x(v)\right)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_3 \otimes \mathcal{D}_2$

$= \sigma_2 \left(\mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(3)} x \partial_{32}(v) + \mathcal{Z}_{21} x \mathcal{Z}_{21}^{(3)} x \partial_{31}(v) - 10\, \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(5)} x(v)\right) +$

$\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(3)}(v)$

$= -\frac{10}{10} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(3)}(v) + \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(3)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(2)} x(v)\right)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_3 \otimes \mathcal{D}_2$

$= \sigma_2 \left(\mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(2)} x \partial_{32}(v) + \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(2)} x \partial_{31}(v) - 10\, \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(5)} x(v) +\right.$

$\left.\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(3)} x \partial_{21}^{(2)}(v)\right)$

$= -\frac{10}{10} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(3)}(v) + \frac{3}{3} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(3)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21} x(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_2 \otimes \mathcal{D}_2$

$= \sigma_2 \left(\mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21} x \partial_{32}(v) + \mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21} x \partial_{31}(v) - 6\, \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(6)} x(v) +\right.$

$\left.\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(5)} x \partial_{21}(v)\right)$

$= -\frac{6}{15} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(4)}(v) + \frac{4}{10} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(4)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(2)} x(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_2 \otimes \mathcal{D}_2$

$= \sigma_2 \left(\mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(2)} x \partial_{32}(v) + \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(2)} x \partial_{31}(v) - 15\, \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(6)} x(v) +\right.$

$\left.\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(4)} x \partial_{21}^{(2)}(v)\right)$

$= -\frac{15}{15} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(4)}(v) + \frac{6}{6} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(4)}(v) = 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(4)}x(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_2 \otimes \mathcal{D}_2$

$= \sigma_2\left(\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(4)}x\partial_{32}(v) + \mathcal{Z}_{21}x\mathcal{Z}_{21}^{(4)}x\partial_{31}(v) - 15\,\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(6)}x(v)\right) +$
$\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}(v)$

$= -\frac{15}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}(v) + \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(3)}x(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_2 \otimes \mathcal{D}_2$

$= \sigma_2\left(\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(3)}x\partial_{32}(v) + \mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(3)}x\partial_{31}(v) - 20\,\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(6)}x(v) +$
$\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(3)}x\partial_{21}^{(3)}(v)\right)$

$= -\frac{20}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}(v) + \frac{4}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(6)}x\mathcal{Z}_{21}x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_1 \otimes \mathcal{D}_2$

$= \sigma_2\left(\mathcal{Z}_{21}^{(6)}x\mathcal{Z}_{21}x\partial_{32}(v) + \mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}x\partial_{31}(v) - 7\,\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(7)}x(v) +$
$\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(6)}x\partial_{21}(v)\right)$

$= -\frac{7}{21}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}(v) + \frac{5}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}^{(2)}x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_1 \otimes \mathcal{D}_2$

$= \sigma_2\left(\mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}^{(2)}x\partial_{32}(v) + \mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(2)}x\partial_{31}(v) - 21\,\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(7)}x(v) +$
$\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(5)}x\partial_{21}^{(2)}(v)\right)$

$= -\frac{21}{21}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}(v) + \frac{10}{10}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(5)}x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_1 \otimes \mathcal{D}_2$

$= \sigma_2\left(\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(5)}x\partial_{32}(v) + \mathcal{Z}_{21}x\mathcal{Z}_{21}^{(5)}x\partial_{31}(v) - 21\,\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(7)}x(v)\right) +$

$\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}(v)$

$= -\frac{21}{21}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}(v) + \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(3)}x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_1 \otimes \mathcal{D}_2$

$= \sigma_2\left(\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(3)}x\partial_{32}(v) + \mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(3)}x\partial_{31}(v) - 35\,\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(7)}x(v) +\right.$

$\left.\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(4)}x\partial_{21}^{(3)}(v)\right)$

$= -\frac{35}{21}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}(v) + \frac{10}{6}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(4)}x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_1 \otimes \mathcal{D}_2$

$= \sigma_2\left(\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(4)}x\partial_{32}(v) + \mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(4)}x\partial_{31}(v) - 35\,\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(7)}x(v) +\right.$

$\left.\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(3)}x\partial_{21}^{(4)}(v)\right)$

$= -\frac{35}{21}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}(v) + \frac{5}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(7)}x\mathcal{Z}_{21}x(v)\right)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_0 \otimes \mathcal{D}_2$

$= \mathcal{Z}_{21}^{(7)}x\mathcal{Z}_{21}x\partial_{32}(v) + \sigma_2\left(\mathcal{Z}_{21}^{(6)}x\mathcal{Z}_{21}x\partial_{31}(v) - 8\,\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(8)}x(v) +\right.$

$\left.\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(7)}x\partial_{21}(v)\right)$

$= -\frac{8}{28}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}(v) + \frac{6}{21}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(6)}x\mathcal{Z}_{21}^{(2)}x(v)\right)$; where $v\in\mathcal{D}_{16}\otimes\mathcal{D}_0\otimes\mathcal{D}_2$

$= \mathcal{Z}_{21}^{(6)}x\mathcal{Z}_{21}^{(2)}x\partial_{32}(v)+\sigma_2\left(\mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}^{(2)}x\partial_{31}(v)-28\,\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(8)}x(v)+\right.$

$\left.\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(6)}x\partial_{21}^{(2)}(v)\right)$

$=-\frac{28}{28}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}(v)+\frac{15}{15}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}(v)$

$=0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(6)}x(v)\right)$; where $v\in\mathcal{D}_{16}\otimes\mathcal{D}_0\otimes\mathcal{D}_2$

$= \mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(6)}x\partial_{32}(v)+\sigma_2\left(\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(6)}x\partial_{31}(v)-28\,\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(8)}x(v)\right)+$

$\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}(v)$

$=-\frac{28}{28}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}(v)+\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}(v)$

$=0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}^{(3)}x(v)\right)$; where $v\in\mathcal{D}_{16}\otimes\mathcal{D}_0\otimes\mathcal{D}_2$

$= \mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}^{(3)}x\partial_{32}(v)+\sigma_2\left(\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(3)}x\partial_{31}(v)-56\,\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(8)}x(v)+\right.$

$\left.\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(5)}x\partial_{21}^{(3)}(v)\right)$

$=-\frac{56}{28}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}(v)+\frac{20}{10}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}(v)$

$=0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(5)}x(v)\right)$; where $v\in\mathcal{D}_{16}\otimes\mathcal{D}_0\otimes\mathcal{D}_2$

$= \mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(5)}x\partial_{32}(v)+\sigma_2\left(\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(5)}x\partial_{31}(v)-56\,\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(8)}x(v)+\right.$

$\left.\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(3)}x\partial_{21}^{(5)}(v)\right)$

$=-\frac{56}{28}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}(v)+\frac{6}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}(v)$

$=0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(4)}x(v)\right)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_0 \otimes \mathcal{D}_2$

$$= \mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(4)}x\partial_{32}(v) + \sigma_2\left(\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(4)}x\partial_{31}(v) - 70\,\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(8)}x(v) + \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(4)}x\partial_{21}^{(4)}(v)\right)$$

$$= -\frac{70}{28}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}(v) + \frac{15}{6}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}(v)$$

$$= 0$$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}y\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(3)}x(v)\right)$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_6 \otimes \mathcal{D}_1$

$$= \sigma_2\left(2\,\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(3)}x(v) - \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(3)}x\partial_{32}(v)\right) - \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{31}(v) + \sigma_2\left(\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(3)}(v)\right)$$

$$= \frac{2}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{31}(v) - \frac{2}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(2)}(v) - \frac{1}{3}\,\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{32}(v) - \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{31}(v)$$

$$= -\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{31}(v) - \frac{2}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(2)}(v) - \frac{1}{3}\,\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{32}(v)$$

And

$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(-\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}(v)\right)$$

$$= \sigma_2\left(\frac{1}{3}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}(v)\right) - \frac{1}{3}\,\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}(v) + \sigma_2\left(\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}(v)\right) - \frac{2}{3}\,\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(2)}(v)$$

$$= -\frac{1}{3}\,\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}(v) - \frac{2}{3}\,\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(2)}(v)$$

$$= -\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{31}(v) - \frac{2}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(2)}(v) - \frac{1}{3}\,\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{32}(v)$$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}y\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(4)}x(v)\right)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_5 \otimes \mathcal{D}_1$

$$= \sigma_2\left(2\,\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(4)}x(v) - \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(4)}x\partial_{32}(v) - \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(3)}x\partial_{31}(v) + \mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(4)}(v)\right)$$

$= \frac{2}{12} Z_{32} y Z_{21}^{(2)} x \partial_{21} \partial_{31}(v) - \frac{2}{4} Z_{32} y Z_{31} z \partial_{21}^{(3)}(v) - \frac{1}{6} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(2)} \partial_{32}(v) -$
$\frac{1}{3} Z_{32} y Z_{21}^{(2)} x \partial_{21} \partial_{31}(v)$

$= -\frac{1}{6} Z_{32} y Z_{21}^{(2)} x \partial_{21} \partial_{31}(v) - \frac{1}{2} Z_{32} y Z_{31} z \partial_{21}^{(3)}(v) - \frac{1}{6} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(2)} \partial_{32}(v)$

And

$(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2})\left(-\frac{1}{6} Z_{32} y Z_{31} z Z_{21} x \partial_{21}^{(2)}(v)\right)$

$= \sigma_2\left(\frac{1}{6} Z_{21} x Z_{21} x \partial_{32}^{(2)} \partial_{21}^{(2)}(v)\right) - \frac{1}{6} Z_{32} y Z_{21}^{(2)} x \partial_{32} \partial_{21}^{(2)}(v) +$
$\sigma_2\left(\frac{1}{6} Z_{32} y Z_{32} y \partial_{21}^{(2)} \partial_{21}^{(2)}(v)\right) - \frac{3}{6} Z_{32} y Z_{31} z \partial_{21}^{(3)}(v)$

$= -\frac{1}{6} Z_{32} y Z_{21}^{(2)} x \partial_{32} \partial_{21}^{(2)}(v) - \frac{1}{2} Z_{32} y Z_{31} z \partial_{21}^{(3)}(v)$

$= -\frac{1}{6} Z_{32} y Z_{21}^{(2)} x \partial_{21} \partial_{31}(v) - \frac{1}{2} Z_{32} y Z_{31} z \partial_{21}^{(3)}(v) - \frac{1}{6} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(2)} \partial_{32}(v)$

- $(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2})\left(Z_{32} y Z_{32} y Z_{21}^{(5)} x(v)\right)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_4 \otimes \mathcal{D}_1$

$= \sigma_2\left(2 Z_{32}^{(2)} y Z_{21}^{(5)} x(v) - Z_{32} y Z_{21}^{(5)} x \partial_{32}(v) - Z_{32} y Z_{21}^{(4)} x \partial_{31}(v) +\right.$
$\left. Z_{32} y Z_{32} y \partial_{21}^{(5)}(v)\right)$

$= \frac{2}{30} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(2)} \partial_{31}(v) - \frac{2}{5} Z_{32} y Z_{31} z \partial_{21}^{(4)}(v) - \frac{1}{10} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(3)} \partial_{32}(v) -$
$\frac{1}{6} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(2)} \partial_{31}(v)$

$= -\frac{1}{10} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(2)} \partial_{31}(v) - \frac{2}{5} Z_{32} y Z_{31} z \partial_{21}^{(4)}(v) - \frac{1}{10} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(3)} \partial_{32}(v)$

And

$(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2})\left(-\frac{1}{10} Z_{32} y Z_{31} z Z_{21} x \partial_{21}^{(3)}(v)\right)$

$= \sigma_2\left(\frac{1}{10} Z_{21} x Z_{21} x \partial_{32}^{(2)} \partial_{21}^{(3)}(v)\right) - \frac{1}{10} Z_{32} y Z_{21}^{(2)} x \partial_{32} \partial_{21}^{(3)}(v) +$
$\sigma_2\left(\frac{1}{10} Z_{32} y Z_{32} y \partial_{21}^{(2)} \partial_{21}^{(3)}(v)\right) - \frac{4}{10} Z_{32} y Z_{31} z \partial_{21}^{(4)}(v)$

$= -\frac{1}{10} Z_{32} y Z_{21}^{(2)} x \partial_{32} \partial_{21}^{(3)}(v) - \frac{2}{5} Z_{32} y Z_{31} z \partial_{21}^{(4)}(v)$

$= -\frac{1}{10} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(2)} \partial_{31}(v) - \frac{2}{5} Z_{32} y Z_{31} z \partial_{21}^{(4)}(v) - \frac{1}{10} Z_{32} y Z_{21}^{(2)} x \partial_{21}^{(3)} \partial_{32}(v)$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}y\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(6)}x(v)\right)$; where $v\in\mathcal{D}_{14}\otimes\mathcal{D}_3\otimes\mathcal{D}_1$

$=\sigma_2\Big(2\,\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(6)}x(v)-\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(6)}x\partial_{32}(v)-\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(5)}x\partial_{31}(v)+$

$\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(6)}(v)\Big)$

$=\frac{2}{60}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}(v)-\frac{2}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}(v)-$

$\frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}(v)-\frac{1}{10}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}(v)$

$=-\frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}(v)-\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}(v)-\frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}(v)$

And

$\left(\delta_{\mathcal{L}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(-\frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(4)}(v)\right)$

$=\sigma_2\left(\frac{1}{15}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(4)}(v)\right)-\frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(4)}(v)+$

$\sigma_2\left(\frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(4)}(v)\right)-\frac{5}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}(v)$

$=-\frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(4)}(v)-\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}(v)$

$=-\frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}(v)-\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}(v)-\frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}(v)$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}y\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(7)}x(v)\right)$; where $v\in\mathcal{D}_{15}\otimes\mathcal{D}_2\otimes\mathcal{D}_1$

$=\sigma_2\Big(2\,\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(7)}x(v)-\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(7)}x\partial_{32}(v)-\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(6)}x\partial_{31}(v)+$

$\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(7)}(v)\Big)$

$=\frac{2}{105}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}(v)-\frac{2}{7}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}(v)-\frac{1}{21}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}(v)$

$-\frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}(v)$

$=-\frac{1}{21}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}(v)-\frac{2}{7}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}(v)-\frac{1}{21}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}(v)$

And

$\left(\delta_{\mathcal{L}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(-\frac{1}{21}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(5)}(v)\right)$

$=\sigma_2\left(\frac{1}{21}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(5)}(v)\right)-\frac{1}{21}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(5)}(v)+$

$$\sigma_2\left(\frac{1}{21}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(5)}(v)\right)-\frac{6}{21}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}(v)$$

$$=-\frac{1}{21}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(5)}(v)-\frac{2}{7}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}(v)$$

$$=-\frac{1}{21}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}(v)-\frac{2}{7}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}(v)-\frac{1}{21}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}(v)$$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}y\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(8)}x(v)\right)$; where $v\in\mathcal{D}_{16}\otimes\mathcal{D}_1\otimes\mathcal{D}_1$

$$=\sigma_2\left(2\,\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(8)}x(v)-\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(8)}x\partial_{32}(v)-\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(7)}x\partial_{31}(v)+\right.$$
$$\left.\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(8)}(v)\right)$$

$$=\frac{2}{168}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}(v)-\frac{2}{8}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}(v)-$$
$$\frac{1}{28}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}(v)-\frac{1}{21}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}(v)$$

$$=-\frac{1}{28}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}(v)-\frac{1}{4}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}(v)-\frac{1}{28}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}(v)$$

And

$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(-\frac{1}{28}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(6)}(v)\right)$$

$$=\sigma_2\left(\frac{1}{28}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(6)}(v)\right)-\frac{1}{28}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(6)}(v)+$$
$$\sigma_2\left(\frac{1}{28}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(6)}(v)\right)-\frac{7}{28}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}(v)$$

$$=-\frac{1}{28}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(6)}(v)-\frac{1}{4}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}(v)$$

$$=-\frac{1}{28}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}(v)-\frac{1}{4}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}(v)-\frac{1}{28}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}(v)$$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}y\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(9)}x(v)\right)$; where $v\in\mathcal{D}_{17}\otimes\mathcal{D}_0\otimes\mathcal{D}_1$

$$=\sigma_2\left(2\,\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(9)}x(v)\right)-\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(9)}x\partial_{32}(v)-\sigma_2\left(\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(8)}x\partial_{31}(v)+\right.$$
$$\left.\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(9)}(v)\right)$$

$$=\frac{2}{252}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}(v)-\frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}(v)-\frac{1}{28}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}(v)$$

$$=-\frac{1}{36}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}(v)-\frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}(v)$$

And

$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(-\frac{1}{36}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(7)}(v)\right)$$

$$=\sigma_2\left(\frac{1}{36}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(7)}(v)\right)-\frac{1}{36}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(7)}(v)+$$

$$\sigma_2\left(\frac{1}{36}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(7)}(v)\right)-\frac{8}{36}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}(v)$$

$$=-\frac{1}{36}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}(v)-\frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}(v)$$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x(v)\right)$; where $v\in\mathcal{D}_{12}\otimes\mathcal{D}_5\otimes\mathcal{D}_1$

$$=\sigma_2\left(\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x\partial_{32}^{(2)}(v)+\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x\partial_{32}\partial_{31}(v)+\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{31}^{(2)}(v)-\right.$$

$$\left.4\,\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(4)}x(v)+\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(3)}x\partial_{21}(v)\right)$$

$$=-\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{31}(v)+\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(3)}(v)+\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{31}\partial_{21}(v)-$$

$$\frac{3}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(3)}(v)$$

$$=0$$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x(v)\right)$; where $v\in\mathcal{D}_{13}\otimes\mathcal{D}_4\otimes\mathcal{D}_1$

$$=\sigma_2\left(\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x\partial_{32}^{(2)}(v)+\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x\partial_{32}\partial_{31}(v)+\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x\partial_{31}^{(2)}(v)-\right.$$

$$\left.5\,\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(5)}x(v)+\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(4)}x\partial_{21}(v)\right)$$

$$=-\frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}(v)+\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(4)}(v)+$$

$$\frac{2}{12}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}(v)-\frac{4}{4}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(4)}(v)$$

$$=0$$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(2)}x(v)\right)$; where $v\in\mathcal{D}_{13}\otimes\mathcal{D}_4\otimes\mathcal{D}_1$

$$=\sigma_2\left(\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(2)}x\partial_{32}^{(2)}(v)+\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{31}(v)+\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x\partial_{31}^{(2)}(v)-\right.$$

$$\left.10\,\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(5)}x(v)+\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(3)}x\partial_{21}^{(2)}(v)\right)$$

$= -\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}(v) + 2\,\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(4)}(v) +$

$\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}(v) - \frac{6}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(4)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}x(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_3 \otimes \mathcal{D}_1$

$= \sigma_2\left(\mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}x\partial_{32}^{(2)}(v) + \mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x\partial_{32}\partial_{31}(v) + \mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x\partial_{31}^{(2)}(v) -\right.$

$\left.6\,\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(6)}x(v) + \mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(5)}x\partial_{21}(v)\right)$

$= -\frac{1}{10}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}(v) + \mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}(v) + \frac{3}{30}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}(v)$

$- \frac{5}{5}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(2)}x(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_3 \otimes \mathcal{D}_1$

$= \sigma_2\left(\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(2)}x\partial_{32}^{(2)}(v) + \mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{31}(v) + \mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x\partial_{31}^{(2)}(v) -\right.$

$\left.15\,\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(6)}x(v) + \mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(4)}x\partial_{21}^{(2)}(v)\right)$

$= -\frac{1}{4}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}(v) + \frac{5}{2}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}(v) + \frac{3}{12}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}(v)$

$-\frac{10}{4}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(3)}x(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_3 \otimes \mathcal{D}_1$

$= \sigma_2\left(\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(3)}x\partial_{32}^{(2)}(v) + \mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(3)}x\partial_{32}\partial_{31}(v) + \mathcal{Z}_{21}x\mathcal{Z}_{21}^{(3)}x\partial_{31}^{(2)}(v) -\right.$

$\left.20\,\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(6)}x(v) + \mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(3)}x\partial_{21}^{(3)}(v)\right)$

$= -\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}(v) + \frac{10}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}(v) + \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}(v)$

$- \frac{10}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(6)} x \mathcal{Z}_{21} x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_2 \otimes \mathcal{D}_1$

$= \sigma_2 \left(\mathcal{Z}_{21}^{(6)} x \mathcal{Z}_{21} x \partial_{32}^{(2)}(v) + \mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21} x \partial_{32}\partial_{31}(v) + \mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21} x \partial_{31}^{(2)}(v) - 7\, \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(7)} x(v) + \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(6)} x \partial_{21}(v)\right)$

$= -\frac{1}{15}\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(4)}\partial_{31}(v) + \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(6)}(v) + \frac{4}{60}\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(4)}\partial_{31}(v) - \frac{6}{6}\mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(6)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21}^{(2)} x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_2 \otimes \mathcal{D}_1$

$= \sigma_2 \left(\mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21}^{(2)} x \partial_{32}^{(2)}(v) + \mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(2)} x \partial_{32}\partial_{31}(v) + \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(2)} x \partial_{31}^{(2)}(v) - 21\, \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(7)} x(v) + \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(5)} x \partial_{21}^{(2)}(v)\right)$

$= -\frac{1}{5}\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(4)}\partial_{31}(v) + 3\, \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(6)}(v) + \frac{6}{30}\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(4)}\partial_{31}(v) - \frac{15}{5}\mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(6)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(3)} x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_2 \otimes \mathcal{D}_1$

$= \sigma_2 \left(\mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(3)} x \partial_{32}^{(2)}(v) + \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(3)} x \partial_{32}\partial_{31}(v) + \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(3)} x \partial_{31}^{(2)}(v) - 35\, \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(7)} x(v) + \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(4)} x \partial_{21}^{(3)}(v)\right)$

$= -\frac{1}{3}\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(4)}\partial_{31}(v) + 5\, \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(6)}(v) + \frac{4}{12}\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(4)}\partial_{31}(v) - \frac{20}{4}\mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(6)}(v) = 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(4)} x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_2 \otimes \mathcal{D}_1$

$= \sigma_2 \left(\mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(4)} x \partial_{32}^{(2)}(v) + \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(4)} x \partial_{32}\partial_{31}(v) + \mathcal{Z}_{21} x \mathcal{Z}_{21}^{(4)} x \partial_{31}^{(2)}(v) - 35\, \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(7)} x(v) + \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(3)} x \partial_{21}^{(4)}(v)\right)$

$= -\frac{1}{3} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(4)} \partial_{31}(v) + 5\, \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(6)}(v) + \frac{1}{3} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(4)} \partial_{31}(v)$
$\quad - \frac{15}{3} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(6)}(v)$
$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(7)} x \mathcal{Z}_{21} x(v)\right)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_1 \otimes \mathcal{D}_1$

$= \mathcal{Z}_{21}^{(7)} x \mathcal{Z}_{21} x \partial_{32}^{(2)}(v) + \sigma_2 \Big(\mathcal{Z}_{21}^{(6)} x \mathcal{Z}_{21} x \partial_{32} \partial_{31}(v) + \mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21} x \partial_{31}^{(2)}(v) -$
$\quad 8\, \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(8)} x(v) + \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(7)} x \partial_{21}(v)\Big)$
$= -\frac{1}{21} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}(v) + \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(7)}(v) + \frac{5}{105} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}(v)$
$\quad - \frac{7}{7} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(7)}(v)$
$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(6)} x \mathcal{Z}_{21}^{(2)} x(v)\right)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_1 \otimes \mathcal{D}_1$

$= \mathcal{Z}_{21}^{(6)} x \mathcal{Z}_{21}^{(2)} x \partial_{32}^{(2)}(v) + \sigma_2 \Big(\mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21}^{(2)} x \partial_{32} \partial_{31}(v) + \mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(2)} x \partial_{31}^{(2)}(v) -$
$\quad 28\, \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(8)} x(v) + \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(6)} x \partial_{21}^{(2)}(v)\Big)$
$= -\frac{1}{6} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}(v) + \frac{7}{2}\, \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(7)}(v) + \frac{10}{60} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}(v)$
$\quad - \frac{21}{6} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(7)}(v)$
$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21}^{(3)} x(v)\right)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_1 \otimes \mathcal{D}_1$

$= \mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21}^{(3)} x \partial_{32}^{(2)}(v) + \sigma_2 \Big(\mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(3)} x \partial_{32} \partial_{31}(v) + \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(3)} x \partial_{31}^{(2)}(v) -$
$\quad 56\, \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(8)} x(v) + \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(5)} x \partial_{21}^{(3)}(v)\Big)$
$= -\frac{1}{3} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}(v) + 7\, \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(7)}(v) + \frac{10}{30} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}(v)$
$\quad - \frac{35}{5} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(7)}(v)$
$= 0$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(4)} x(v)\right)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_1 \otimes \mathcal{D}_1$

$= \mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(4)} x \partial_{32}^{(2)}(v) + \sigma_2 \left(\mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(4)} x \partial_{32}\partial_{31}(v) + \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(4)} x \partial_{31}^{(2)}(v) -\right.$

$\left. 70\, \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(8)} x(v) + \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(4)} x \partial_{21}^{(4)}(v)\right)$

$= -\frac{5}{12} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}(v) + \frac{35}{4} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(7)}(v) +$

$\frac{5}{12} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}(v) - \frac{35}{4} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(7)}(v)$

$= 0$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(5)} x(v)\right)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_1 \otimes \mathcal{D}_1$

$= \mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(5)} x \partial_{32}^{(2)}(v) + \sigma_2 \left(\mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(5)} x \partial_{32}\partial_{31}(v) + \mathcal{Z}_{21} x \mathcal{Z}_{21}^{(5)} x \partial_{31}^{(2)}(v) -\right.$

$\left. 56\, \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(8)} x(v) + \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(3)} x \partial_{21}^{(5)}(v)\right)$

$= -\frac{1}{3} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}(v) + 7\, \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(7)}(v) +$

$\frac{1}{3} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}(v) - \frac{21}{3} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(7)}(v)$

$= 0$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(8)} x \mathcal{Z}_{21} x(v)\right)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$

$= \mathcal{Z}_{21}^{(8)} x \mathcal{Z}_{21} x \partial_{32}^{(2)}(v) + \mathcal{Z}_{21}^{(7)} x \mathcal{Z}_{21} x \partial_{32}\partial_{31}(v) + \sigma_2 \left(\mathcal{Z}_{21}^{(6)} x \mathcal{Z}_{21} x \partial_{31}^{(2)}(v) -\right.$

$\left. 9\, \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(9)} x(v) + \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(8)} x \partial_{21}(v)\right)$

$= -\frac{1}{28} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(6)} \partial_{31}(v) + \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(8)}(v) +$

$\frac{6}{168} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(6)} \partial_{31}(v) - \frac{8}{8} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(8)}(v)$

$= 0$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(7)} x \mathcal{Z}_{21}^{(2)} x(v)\right)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$

$= \mathcal{Z}_{21}^{(7)} x \mathcal{Z}_{21}^{(2)} x \partial_{32}^{(2)}(v) + \mathcal{Z}_{21}^{(6)} x \mathcal{Z}_{21}^{(2)} x \partial_{32}\partial_{31}(v) + \sigma_2 \left(\mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21}^{(2)} x \partial_{31}^{(2)}(v) -\right.$

$\left. 36\, \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(9)} x(v) + \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(7)} x \partial_{21}^{(2)}(v)\right)$

$= -\frac{1}{7}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}(v) + 4\,\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}(v) +$

$\frac{15}{105}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}(v) - \frac{28}{7}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(6)}x\mathcal{Z}_{21}^{(3)}x(v)\right)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$

$= \mathcal{Z}_{21}^{(6)}x\mathcal{Z}_{21}^{(3)}x\partial_{32}^{(2)}(v) + \mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}^{(3)}x\partial_{32}\partial_{31}(v) + \sigma_2\left(\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(3)}x\partial_{31}^{(2)}(v) -\right.$

$\left. 84\,\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(9)}x(v) + \mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(6)}x\partial_{21}^{(3)}(v)\right)$

$= -\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}(v) + \frac{28}{3}\,\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}(v) +$

$\frac{20}{60}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}(v) - \frac{56}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}^{(4)}x(v)\right)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$

$= \mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}^{(4)}x\partial_{32}^{(2)}(v) + \mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(4)}x\partial_{32}\partial_{31}(v) + \sigma_2\left(\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(4)}x\partial_{31}^{(2)}(v) -\right.$

$\left. 126\,\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(9)}x(v) + \mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(5)}x\partial_{21}^{(4)}(v)\right)$

$= -\frac{1}{2}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}(v) + 14\,\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}(v) + \frac{15}{30}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}(v)$

$- \frac{70}{5}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}(v)$

$= 0$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(5)}x(v)\right)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_0 \otimes \mathcal{D}_1$

$= \mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(5)}x\partial_{32}^{(2)}(v) + \mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(5)}x\partial_{32}\partial_{31}(v) + \sigma_2\left(\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(5)}x\partial_{31}^{(2)}(v) -\right.$

$\left. 126\,\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(9)}x(v) + \mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(4)}x\partial_{21}^{(5)}(v)\right)$

$= -\frac{1}{2}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}(v) + 14\,\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}(v) + \frac{6}{12}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}(v)$

$- \frac{56}{4}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}(v)$

$= 0$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(6)}x(v)\right)$; where $v\in\mathcal{D}_{17}\otimes\mathcal{D}_0\otimes\mathcal{D}_1$

$=\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(6)}x\partial_{32}^{(2)}(v)+\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(6)}x\partial_{32}\partial_{31}(v)+\sigma_2\left(\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(6)}x\partial_{31}^{(2)}(v)-\right.$

$\left.84\,\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(9)}x(v)+\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(3)}x\partial_{21}^{(6)}(v)\right)$

$=-\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}(v)+\frac{28}{3}\,\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}(v)+\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}(v)$

$-\frac{28}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}(v)$

$=0$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}y\mathcal{Z}_{32}y\mathcal{Z}_{32}y(v)\right)$; where $v\in\mathcal{D}_8\otimes\mathcal{D}_{10}\otimes\mathcal{D}_0$

$=\sigma_2\left(2\,\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{32}y(v)-2\,\mathcal{Z}_{32}y\mathcal{Z}_{32}^{(2)}y(v)+\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{32}(v)\right)$

$=0$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(4)}x(v)\right)$; where $v\in\mathcal{D}_{12}\otimes\mathcal{D}_6\otimes\mathcal{D}_0$

$=\sigma_2\left(3\,\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(4)}x(v)-\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(4)}x\partial_{32}(v)-\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(3)}x\partial_{31}(v)+\right.$

$\left.\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{32}y\partial_{21}^{(4)}(v)\right)$

$=\frac{3}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{31}^{(2)}(v)-\frac{3}{6}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}^{(2)}(v)-\frac{3}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(3)}\partial_{32}(v)-$

$\frac{1}{12}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{31}\partial_{32}(v)+\frac{1}{4}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(3)}\partial_{32}(v)-$

$\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{31}\partial_{31}(v)+\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(2)}\partial_{31}(v)$

$=\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{31}^{(2)}(v)-\frac{1}{2}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}^{(2)}(v)-\frac{3}{4}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(3)}\partial_{32}(v)-$

$\frac{1}{12}\,\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{32}\partial_{31}(v)+\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(2)}\partial_{31}(v)$

And

$\left(\delta_{\mathcal{L}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(\frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}\partial_{31}(v)-\right.$

$\left.\frac{1}{4}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{32}(v)\right)$

$=\sigma_2\left(-\frac{1}{6}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(2)}\partial_{32}(v)\right)+\frac{1}{6}\,\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(2)}\partial_{32}(v)-$

$\sigma_2\left(\frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(2)}\partial_{32}(v)\right)+\frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(2)}\partial_{32}(v)+$

$\sigma_2\left(\frac{1}{4}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}\partial_{31}(v)\right)-\frac{1}{4}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}\partial_{31}(v)+$

$\sigma_2\left(\frac{1}{4}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}\partial_{31}(v)\right)-\frac{1}{4}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}\partial_{31}(v)$

$=-\frac{2}{4}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}^{(2)}(v)-\frac{1}{4}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{32}\partial_{31}(v)-$

$\frac{3}{4}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(3)}\partial_{32}(v)+\frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{32}\partial_{31}(v)+\frac{2}{6}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}\partial_{31}^{(2)}(v)+$

$\frac{2}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(2)}\partial_{31}(v)$

$=\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{31}^{(2)}(v)-\frac{1}{2}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}^{(2)}(v)-\frac{3}{4}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(3)}\partial_{32}(v)-$

$\frac{1}{12}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{32}\partial_{31}(v)+\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(2)}\partial_{31}(v)$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(5)}x(v)\right)$; where $v\in\mathcal{D}_{13}\otimes\mathcal{D}_5\otimes\mathcal{D}_0$

$=\sigma_2\left(3\,\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(5)}x(v)-\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(5)}x\partial_{32}(v)-\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(4)}x\partial_{31}(v)+\right.$

$\left.\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{32}y\partial_{21}^{(5)}(v)\right)$

$=\frac{3}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{31}^{(2)}(v)-\frac{21}{90}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}^{(2)}(v)-\frac{6}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(4)}\partial_{32}(v)$

$-\frac{1}{30}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}\partial_{32}(v)+\frac{1}{5}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(4)}\partial_{32}(v)-$

$\frac{1}{12}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{31}\partial_{31}(v)+\frac{1}{4}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(3)}\partial_{31}(v)$

$=\frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{31}^{(2)}(v)-\frac{7}{30}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}^{(2)}(v)-$

$\frac{7}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(4)}\partial_{32}(v)-\frac{1}{30}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}\partial_{31}(v)+$

$\frac{1}{4}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(3)}\partial_{31}(v)$

And

$\left(\delta_{\mathcal{L}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(\frac{1}{12}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{31}(v)-\right.$

$\left.\frac{7}{60}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}(v)\right)$

$=\sigma_2\left(-\frac{1}{12}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(3)}\partial_{32}(v)\right)+\frac{1}{12}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(3)}\partial_{32}(v)-$

$$\sigma_2\left(\frac{1}{12}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(3)}\partial_{32}(v)\right)+\frac{1}{12}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(3)}\partial_{32}(v)+$$

$$\sigma_2\left(\frac{7}{60}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(2)}\partial_{31}(v)\right)-\frac{7}{60}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(2)}\partial_{31}(v)+$$

$$\sigma_2\left(\frac{7}{60}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(2)}\partial_{31}(v)\right)-\frac{7}{60}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(2)}\partial_{31}(v)$$

$$=\frac{1}{12}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}\partial_{31}(v)+\frac{2}{12}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{31}^{(2)}(v)+$$

$$\frac{3}{12}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(3)}\partial_{31}(v)-\frac{14}{60}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}^{(2)}(v)-$$

$$\frac{7}{60}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}\partial_{21}^{(2)}\partial_{32}\partial_{31}(v)-\frac{28}{60}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(4)}\partial_{32}(v)$$

$$=\frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{31}^{(2)}(v)-\frac{7}{30}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}^{(2)}(v)-$$

$$\frac{7}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(4)}\partial_{32}(v)-\frac{1}{30}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}\partial_{31}(v)+$$

$$\frac{1}{4}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(3)}\partial_{31}(v)$$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(6)}x(v)\right)$; where $v\in\mathcal{D}_{14}\otimes\mathcal{D}_4\otimes\mathcal{D}_0$

$$=\sigma_2\left(3\,\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(6)}x(v)-\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(6)}x\partial_{32}(v)-\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(5)}x\partial_{31}(v)+\right.$$

$$\left.\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{32}y\partial_{21}^{(6)}(v)\right)$$

$$=\frac{3}{18}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}^{(2)}(v)-\frac{6}{45}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}^{(2)}(v)-$$

$$\frac{3}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{32}(v)-\frac{1}{60}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}\partial_{32}(v)+$$

$$\frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{32}(v)-\frac{1}{30}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}\partial_{31}(v)+\frac{1}{5}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(4)}\partial_{31}(v)$$

$$=\frac{1}{10}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}^{(2)}(v)-\frac{2}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}^{(2)}(v)-$$

$$\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{32}(v)-\frac{1}{60}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}\partial_{31}(v)+$$

$$\frac{1}{5}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(4)}\partial_{31}(v)$$

And

$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(\frac{1}{20}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{31}(v) - \frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}(v)\right)$$

$$= \sigma_2\left(-\frac{1}{20}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(4)}\partial_{32}(v)\right) + \frac{1}{20}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(4)}\partial_{32}(v) - \sigma_2\left(\frac{1}{20}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(4)}\partial_{32}(v)\right) + \frac{1}{20}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(4)}\partial_{32}(v) + \sigma_2\left(\frac{1}{15}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(3)}\partial_{31}(v)\right) - \frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(3)}\partial_{31}(v) + \sigma_2\left(\frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(3)}\partial_{31}(v)\right) - \frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(3)}\partial_{31}(v)$$

$$= \frac{1}{20}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}\partial_{31}(v) + \frac{2}{20}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}^{(2)}(v) + \frac{4}{20}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(4)}\partial_{31}(v) - \frac{2}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}^{(2)}(v) - \frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}\partial_{21}^{(3)}\partial_{31}\partial_{32}(v) - \frac{5}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{32}(v)$$

$$= \frac{1}{10}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}^{(2)}(v) - \frac{2}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}^{(2)}(v) - \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{32}(v) - \frac{1}{60}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}\partial_{31}(v) + \frac{1}{5}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(4)}\partial_{31}(v)$$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(7)}x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_3 \otimes \mathcal{D}_0$

$$= \sigma_2\left(3\,\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(7)}x(v) - \mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(7)}x\partial_{32}(v) - \mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(6)}x\partial_{31}(v) + \mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{32}y\partial_{21}^{(7)}(v)\right)$$

$$= \frac{3}{30}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}^{(2)}(v) - \frac{3}{35}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}^{(2)}(v) - \frac{6}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{32}(v) - \frac{1}{105}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}\partial_{32}(v) + \frac{1}{7}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{32}(v) - \frac{1}{60}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}\partial_{31}(v) + \frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{31}(v)$$

$$= \frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}^{(2)}(v) - \frac{3}{35}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}^{(2)}(v) - \frac{9}{35}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{32}(v) - \frac{1}{105}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}\partial_{32}(v) + \frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{31}(v)$$

And

$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(\frac{1}{30}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(4)}\partial_{31}(v)-\frac{3}{70}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(5)}\partial_{32}(v)\right)$$

$$=\sigma_2\left(-\frac{1}{30}Z_{21}xZ_{21}x\partial_{32}^{(2)}\partial_{21}^{(5)}\partial_{32}(v)\right)+\frac{1}{30}Z_{32}yZ_{21}^{(2)}x\partial_{32}\partial_{21}^{(5)}\partial_{32}(v)-\sigma_2\left(\frac{1}{30}Z_{32}yZ_{32}y\partial_{21}^{(2)}\partial_{21}^{(5)}\partial_{32}(v)\right)+\frac{1}{30}Z_{32}yZ_{31}z\partial_{21}\partial_{21}^{(5)}\partial_{32}(v)+\sigma_2\left(\frac{3}{70}Z_{21}xZ_{21}x\partial_{32}^{(2)}\partial_{21}^{(4)}\partial_{31}(v)\right)-\frac{3}{70}Z_{32}yZ_{21}^{(2)}x\partial_{32}\partial_{21}^{(4)}\partial_{31}(v)+\sigma_2\left(\frac{3}{70}Z_{32}yZ_{32}y\partial_{21}^{(2)}\partial_{21}^{(4)}\partial_{31}(v)\right)-\frac{3}{70}Z_{32}yZ_{31}z\partial_{21}\partial_{21}^{(4)}\partial_{31}(v)$$

$$=\frac{1}{30}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}\partial_{31}(v)+\frac{2}{30}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}^{(2)}(v)+\frac{5}{30}Z_{32}yZ_{31}z\partial_{21}^{(4)}\partial_{31}(v)-\frac{6}{70}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}^{(2)}(v)-\frac{3}{70}Z_{32}yZ_{21}^{(2)}\partial_{21}^{(4)}\partial_{32}\partial_{31}(v)-\frac{18}{70}Z_{32}yZ_{31}z\partial_{21}^{(6)}\partial_{32}(v)$$

$$=\frac{1}{15}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}^{(2)}(v)-\frac{3}{35}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}^{(2)}(v)-\frac{9}{35}Z_{32}yZ_{31}z\partial_{21}^{(6)}\partial_{32}(v)-\frac{1}{105}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}\partial_{31}(v)+\frac{1}{6}Z_{32}yZ_{31}z\partial_{21}^{(5)}\partial_{31}(v)$$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(Z_{32}^{(2)}yZ_{32}yZ_{21}^{(8)}x(v)\right)$; where $v\in\mathcal{D}_{16}\otimes\mathcal{D}_2\otimes\mathcal{D}_0$

$$=\sigma_2\left(3\,Z_{32}^{(3)}yZ_{21}^{(8)}x(v)-Z_{32}^{(2)}yZ_{21}^{(8)}x\partial_{32}(v)-Z_{32}^{(2)}yZ_{21}^{(7)}x\partial_{31}(v)+Z_{32}^{(2)}yZ_{32}y\partial_{21}^{(8)}(v)\right)$$

$$=\frac{3}{45}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v)-\frac{15}{252}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v)-\frac{3}{9}Z_{32}yZ_{31}z\partial_{21}^{(7)}\partial_{32}(v)-\frac{1}{168}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}\partial_{32}(v)+\frac{1}{8}Z_{32}yZ_{31}z\partial_{21}^{(7)}\partial_{32}(v)-\frac{1}{105}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}\partial_{31}(v)+\frac{1}{7}Z_{32}yZ_{31}z\partial_{21}^{(6)}\partial_{31}(v)$$

$= \frac{1}{21} \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v) - \frac{5}{84}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v) -$
$\frac{5}{24}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v) - \frac{1}{168}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v) +$
$\frac{1}{7}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{31}(v)$

And

$(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2})\left(\frac{1}{42}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{31}(v) - \frac{5}{168}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{32}(v)\right)$

$= \sigma_2\left(-\frac{1}{42}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(6)}\partial_{32}(v)\right) + \frac{1}{42}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(6)}\partial_{32}(v) -$
$\sigma_2\left(\frac{1}{30}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(2)}\partial_{21}^{(6)}\partial_{32}(v)\right) + \frac{1}{42}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(6)}\partial_{32}(v) +$
$\sigma_2\left(\frac{5}{168}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(5)}\partial_{31}(v)\right) - \frac{5}{168}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(5)}\partial_{31}(v) +$
$\sigma_2\left(\frac{5}{168}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(2)}\partial_{21}^{(5)}\partial_{31}(v)\right) - \frac{5}{168}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(5)}\partial_{31}(v)$

$= \frac{1}{42}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v) + \frac{2}{42}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v) +$
$\frac{6}{42}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{31}(v) - \frac{10}{168}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v) -$
$\frac{5}{168}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}\partial_{21}^{(5)}\partial_{31}\partial_{32}(v) - \frac{35}{168}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v)$

$= \frac{1}{21}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v) - \frac{5}{84}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v) -$
$\frac{5}{24}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v) - \frac{1}{168}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v) +$
$\frac{1}{7}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{31}(v)$

- $(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2})\left(\mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(9)}x(v)\right)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$

$= \sigma_2\left(3\,\mathcal{Z}_{32}^{(3)}\mathcal{y}\mathcal{Z}_{21}^{(9)}x(v) - \mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{21}^{(9)}x\partial_{32}(v) - \mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{21}^{(8)}x\partial_{31}(v) +\right.$
$\left.\mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(9)}(v)\right)$

$= \frac{3}{63}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}^{(2)}(v) - \frac{3}{84}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}(v) -$
$\frac{1}{252}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}\partial_{32}(v) + \frac{1}{9}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(8)}\partial_{32}(v) -$

$\frac{1}{168}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}\partial_{31}(v)+\frac{1}{8}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{31}(v)$

$=\frac{1}{28}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}^{(2)}(v)+\frac{2}{63}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}\partial_{31}(v)+$
$\frac{1}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}\partial_{32}(v)+\frac{1}{8}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{31}(v)$

And

$\left(\delta_{\mathcal{L}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(\frac{1}{56}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{31}(v)+\frac{1}{72}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(7)}\partial_{32}(v)\right)$

$=\sigma_2\left(-\frac{1}{56}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(7)}\partial_{32}(v)\right)+\frac{1}{56}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(7)}\partial_{32}(v)-$
$\sigma_2\left(\frac{1}{56}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(7)}\partial_{32}(v)\right)+\frac{1}{56}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(7)}\partial_{32}(v)-$
$\sigma_2\left(\frac{1}{72}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(6)}\partial_{31}(v)\right)+\frac{1}{72}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(6)}\partial_{31}(v)-$
$\sigma_2\left(\frac{1}{72}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(6)}\partial_{31}(v)\right)+\frac{1}{72}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(6)}\partial_{31}(v)$

$=\frac{1}{56}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}\partial_{31}(v)+\frac{2}{56}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}(v)+$
$\frac{7}{56}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{31}(v)+\frac{1}{72}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}\partial_{21}^{(6)}\partial_{32}\partial_{31}(v)+$
$\frac{8}{72}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}\partial_{32}(v)$

$=\frac{1}{28}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}^{(2)}(v)+\frac{2}{63}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}\partial_{31}(v)+$
$\frac{1}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}\partial_{32}(v)+\frac{1}{8}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{31}(v)$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(10)}x(v)\right)$; where $v\in\mathcal{D}_{18}\otimes\mathcal{D}_0\otimes\mathcal{D}_0$

$=\sigma_2\left(3\,\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(10)}x(v)\right)-\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(10)}x\partial_{32}(v)-\sigma_2\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(9)}x\partial_{31}(v)+\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{32}y\partial_{21}^{(10)}(v)\right)$

$=\frac{3}{84}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v)-\frac{1}{252}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}\partial_{31}(v)+$
$\frac{1}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}\partial_{31}(v)$

$=\frac{1}{36}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v)+\frac{1}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}\partial_{31}(v)$

And

$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(\frac{1}{72}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(7)}\partial_{31}(v)\right)$$

$$= \sigma_2\left(-\frac{1}{72}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(7)}\partial_{31}(v)\right) + \frac{1}{72}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(7)}\partial_{31}(v) -$$

$$\sigma_2\left(\frac{1}{72}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(2)}\partial_{21}^{(7)}\partial_{31}(v)\right) + \frac{1}{72}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(7)}\partial_{31}(v)$$

$$= \frac{2}{72}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v) + \frac{8}{72}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(8)}\partial_{31}(v)$$

$$= \frac{1}{36}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v) + \frac{1}{9}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(8)}\partial_{31}(v)$$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{21}^{(4)}x(v)\right)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_6 \otimes \mathcal{D}_0$

$$= \sigma_2\left(3\,\mathcal{Z}_{32}^{(3)}\mathcal{y}\mathcal{Z}_{21}^{(4)}x(v) - \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(4)}x\partial_{32}^{(2)}(v) - \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(3)}x\partial_{32}\partial_{31}(v) -\right.$$

$$\left.\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{31}^{(2)}(v) + \mathcal{Z}_{32}\mathcal{y}\,\mathcal{Z}_{32}^{(2)}\mathcal{y}\partial_{21}^{(4)}(v)\right)$$

$$= \frac{3}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{31}^{(2)}(v) - \frac{3}{6}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}^{(2)}(v) -$$

$$\frac{3}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(3)}\partial_{32}(v) - \frac{1}{6}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}^{(2)}\mathcal{y}\partial_{21}^{(2)}\partial_{32}^{(2)}(v) -$$

$$\frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{32}\partial_{31}(v) - \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{31}^{(2)}(v)$$

$$= -\frac{2}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}^{(2)}(v) - \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(3)}\partial_{32}(v) -$$

$$\frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{32}\partial_{31}(v)$$

And

$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(-\frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{32}(v)\right)$$

$$= \sigma_2\left(\frac{1}{3}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(2)}\partial_{32}(v)\right) - \frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(2)}\partial_{32}(v) +$$

$$\sigma_2\left(\frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(2)}\partial_{21}^{(2)}\partial_{32}(v)\right) - \frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(2)}\partial_{32}(v)$$

$$= -\frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(2)}\partial_{32}(v) - \frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(2)}\partial_{32}(v)$$

$$= -\frac{2}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}^{(2)}(v) - \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(3)}\partial_{32}(v) -$$

$$\frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{32}\partial_{31}(v)$$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(Z_{32}yZ_{32}^{(2)}yZ_{21}^{(5)}x(v)\right)$; where $v\in\mathcal{D}_{13}\otimes\mathcal{D}_5\otimes\mathcal{D}_0$

$=\sigma_2\Big(3\,Z_{32}^{(3)}yZ_{21}^{(5)}x(v)-Z_{32}yZ_{21}^{(5)}x\partial_{32}^{(2)}(v)-Z_{32}yZ_{21}^{(4)}x\partial_{32}\partial_{31}(v)-$
$Z_{32}yZ_{21}^{(3)}x\partial_{31}^{(2)}(v)+Z_{32}y\,Z_{32}^{(2)}y\partial_{21}^{(5)}(v)\Big)$

$=\frac{3}{9}Z_{32}yZ_{21}^{(2)}x\partial_{21}\partial_{31}^{(2)}(v)-\frac{21}{90}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}^{(2)}(v)-$
$\frac{6}{9}Z_{32}yZ_{31}z\partial_{21}^{(4)}\partial_{32}(v)-\frac{1}{10}Z_{32}yZ_{32}^{(2)}y\partial_{21}^{(3)}\partial_{32}^{(2)}(v)-$
$\frac{1}{6}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}\partial_{31}(v)-\frac{1}{3}Z_{32}yZ_{21}^{(2)}x\partial_{21}\partial_{31}^{(2)}(v)$

$=-\frac{1}{3}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}^{(2)}(v)-\frac{2}{3}Z_{32}yZ_{31}z\partial_{21}^{(4)}\partial_{32}(v)-$
$\frac{1}{6}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}\partial_{31}(v)$

And

$\left(\delta_{\mathcal{L}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(-\frac{1}{6}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(3)}\partial_{32}(v)\right)$

$=\sigma_2\left(\frac{1}{6}Z_{21}xZ_{21}x\partial_{32}^{(2)}\partial_{21}^{(3)}\partial_{32}(v)\right)-\frac{1}{6}Z_{32}yZ_{21}^{(2)}x\partial_{32}\partial_{21}^{(3)}\partial_{32}(v)+$
$\sigma_2\left(\frac{1}{6}Z_{32}yZ_{32}y\partial_{21}^{(2)}\partial_{21}^{(3)}\partial_{32}(v)\right)-\frac{1}{6}Z_{32}yZ_{31}z\partial_{21}\partial_{21}^{(3)}\partial_{32}(v)$

$=-\frac{1}{6}Z_{32}yZ_{21}^{(2)}x\partial_{32}\partial_{21}^{(3)}\partial_{32}(v)-\frac{1}{6}Z_{32}yZ_{31}z\partial_{21}\partial_{21}^{(3)}\partial_{32}(v)$

$=-\frac{1}{3}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}^{(2)}(v)-\frac{2}{3}Z_{32}yZ_{31}z\partial_{21}^{(4)}\partial_{32}(v)-$
$\frac{1}{6}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}\partial_{31}(v)$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(Z_{32}yZ_{32}^{(2)}yZ_{21}^{(6)}x(v)\right)$; where $v\in\mathcal{D}_{14}\otimes\mathcal{D}_4\otimes\mathcal{D}_0$

$=\sigma_2\Big(3\,Z_{32}^{(3)}yZ_{21}^{(6)}x(v)-Z_{32}yZ_{21}^{(6)}x\partial_{32}^{(2)}(v)-Z_{32}yZ_{21}^{(5)}x\partial_{32}\partial_{31}(v)-$
$Z_{32}yZ_{21}^{(4)}x\partial_{31}^{(2)}(v)+Z_{32}y\,Z_{32}^{(2)}y\partial_{21}^{(6)}(v)\Big)$

$=\frac{3}{18}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}^{(2)}(v)-\frac{6}{45}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}^{(2)}(v)-$
$\frac{3}{6}Z_{32}yZ_{31}z\partial_{21}^{(5)}\partial_{32}(v)-\frac{1}{15}Z_{32}yZ_{32}^{(2)}y\partial_{21}^{(4)}\partial_{32}^{(2)}(v)-$
$\frac{1}{10}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}\partial_{31}(v)-\frac{1}{6}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}^{(2)}(v)$

$= -\frac{1}{5}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}^{(2)}(v) - \frac{1}{2}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{32}(v) -$

$\frac{1}{10}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}\partial_{31}(v)$

And

$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(-\frac{1}{10}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}(v)\right)$

$= \sigma_2\left(\frac{1}{10}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(4)}\partial_{32}(v)\right) - \frac{1}{10}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(4)}\partial_{32}(v) +$

$\sigma_2\left(\frac{1}{10}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(4)}\partial_{32}(v)\right) - \frac{1}{10}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(4)}\partial_{32}(v)$

$= -\frac{1}{10}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(4)}\partial_{32}(v) - \frac{1}{10}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(4)}\partial_{32}(v)$

$= -\frac{1}{5}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}^{(2)}(v) - \frac{1}{2}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{32}(v) -$

$\frac{1}{10}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}\partial_{31}(v)$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}y\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(7)}x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_3 \otimes \mathcal{D}_0$

$= \sigma_2\left(3\,\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(7)}x(v) - \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(7)}x\partial_{32}^{(2)}(v) - \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(6)}x\partial_{32}\partial_{31}(v) -\right.$

$\left.\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(5)}x\partial_{31}^{(2)}(v) + \mathcal{Z}_{32}y\,\mathcal{Z}_{32}^{(2)}y\partial_{21}^{(7)}(v)\right)$

$= \frac{3}{30}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}^{(2)}(v) - \frac{3}{35}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}^{(2)}(v) -$

$\frac{6}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{32}(v) - \frac{1}{21}\mathcal{Z}_{32}y\mathcal{Z}_{32}^{(2)}y\partial_{21}^{(5)}\partial_{32}^{(2)}(v) -$

$\frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}\partial_{31}(v) - \frac{1}{10}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}^{(2)}(v)$

$= -\frac{2}{5}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}^{(2)}(v) - \frac{6}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{32}(v) -$

$\frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}\partial_{31}(v)$

And

$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(-\frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{32}(v)\right)$

$= \sigma_2\left(\frac{1}{15}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(5)}\partial_{32}(v)\right) - \frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(5)}\partial_{32}(v) +$

$\sigma_2\left(\frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(5)}\partial_{32}(v)\right) - \frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(5)}\partial_{32}(v)$

$= -\frac{1}{15}\, \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(5)}\partial_{32}(v) - \frac{1}{15}\, \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(5)}\partial_{32}(v)$

$= -\frac{2}{5}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}^{(2)}(v) - \frac{6}{15}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{32}(v) -$

$\frac{1}{15}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}\partial_{31}(v)$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{21}^{(8)}x(v)\right)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_2 \otimes \mathcal{D}_0$

$= \sigma_2\Big(3\, \mathcal{Z}_{32}^{(3)}\mathcal{y}\mathcal{Z}_{21}^{(8)}x(v) - \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(8)}x\partial_{32}^{(2)}(v) - \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(7)}x\partial_{32}\partial_{31}(v) -$

$\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(6)}x\partial_{31}^{(2)}(v) + \mathcal{Z}_{32}\mathcal{y}\, \mathcal{Z}_{32}^{(2)}\mathcal{y}\partial_{21}^{(8)}(v)\Big)$

$= \frac{3}{45}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v) - \frac{15}{252}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v) -$

$\frac{3}{9}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v) - \frac{1}{28}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}^{(2)}\mathcal{y}\partial_{21}^{(6)}\partial_{32}^{(2)}(v) -$

$\frac{1}{21}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v) - \frac{1}{15}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v)$

$= -\frac{2}{21}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v) - \frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v) -$

$\frac{1}{21}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v)$

And

$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(-\frac{1}{21}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{32}(v)\right)$

$= \sigma_2\left(\frac{1}{21}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(6)}\partial_{32}(v)\right) - \frac{1}{21}\, \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(6)}\partial_{32}(v) +$

$\sigma_2\left(\frac{1}{21}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(2)}\partial_{21}^{(6)}\partial_{32}(v)\right) - \frac{1}{21}\, \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(6)}\partial_{32}(v)$

$= -\frac{1}{21}\, \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(6)}\partial_{32}(v) - \frac{1}{21}\, \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(6)}\partial_{32}(v)$

$= -\frac{2}{21}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v) - \frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v) -$

$\frac{1}{21}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v)$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{21}^{(9)}x(v)\right)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$

$= \sigma_2\left(3\, \mathcal{Z}_{32}^{(3)}\mathcal{y}\mathcal{Z}_{21}^{(9)}x(v)\right) - \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(9)}x\partial_{32}^{(2)}(v) - \sigma_2\Big(\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(8)}x\partial_{32}\partial_{31}(v) -$

$\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(7)}x\partial_{31}^{(2)}(v)+\mathcal{Z}_{32}y\,\mathcal{Z}_{32}^{(2)}y\partial_{21}^{(9)}(v)\Big)$

$=\frac{3}{63}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}^{(2)}(v)-\frac{3}{84}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}\partial_{31}(v)-$

$\frac{1}{28}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}\partial_{31}(v)-\frac{1}{21}\mathcal{Z}_{32}y\mathcal{Z}_{32}^{(2)}y\partial_{21}^{(5)}\partial_{32}^{(2)}(v)$

$=0$

- $(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2})\Big(\mathcal{Z}_{32}y\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(10)}x(v)\Big)$; where $v\in\mathcal{D}_{18}\otimes\mathcal{D}_0\otimes\mathcal{D}_0$

$=\sigma_2\Big(3\,\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(10)}x(v)\Big)-\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(10)}x\partial_{32}^{(2)}(v)-\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(9)}x\partial_{32}\partial_{31}(v)-$

$\sigma_2(-\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(8)}x\partial_{31}^{(2)}(v)+\mathcal{Z}_{32}y\,\mathcal{Z}_{32}^{(2)}y\partial_{21}^{(10)}(v)\Big)$

$=\frac{3}{84}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v)-\frac{1}{28}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v)$

$=0$

- $(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2})\Big(\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x(v)\Big)$; where $v\in\mathcal{D}_{13}\otimes\mathcal{D}_5\otimes\mathcal{D}_0$

$=\sigma_2\Big(\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x\partial_{32}^{(2)}(v)+\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{31}(v)+\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}x\partial_{32}\partial_{31}^{(2)}(v)+$

$\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{31}^{(3)}(v)-5\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(5)}x(v)+\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(4)}x\partial_{21}(v)\Big)$

$=-\frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{31}^{(2)}(v)+\frac{7}{18}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}^{(2)}(v)+$

$\frac{10}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(4)}\partial_{32}(v)-\frac{1}{2}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}^{(2)}(v)-$

$\frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}\partial_{31}(v)-\frac{4}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(4)}\partial_{32}(v)-$

$\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(3)}\partial_{31}(v)$

$=-\frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{31}^{(2)}(v)-\frac{1}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}^{(2)}(v)-$

$\frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(4)}\partial_{32}(v)-\frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}\partial_{31}(v)-$

$\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(3)}\partial_{31}(v)$

And

$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(-\tfrac{1}{9}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\mathcal{Z}_{21}\mathcal{x}\partial_{21}^{(2)}\partial_{31}(v) - \tfrac{1}{18}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\mathcal{Z}_{21}\mathcal{x}\partial_{21}^{(3)}\partial_{32}(v)\right)$$

$$= \sigma_2\left(\tfrac{1}{9}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\partial_{32}^{(2)}\partial_{21}^{(2)}\partial_{31}(v)\right) - \tfrac{1}{9}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}\mathcal{x}\partial_{32}\partial_{21}^{(2)}\partial_{31}(v) + \sigma_2\left(\tfrac{1}{9}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(2)}\partial_{21}^{(2)}\partial_{31}(v)\right) - \tfrac{1}{9}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\partial_{21}\partial_{21}^{(2)}\partial_{31}(v) + \sigma_2\left(\tfrac{1}{18}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\partial_{32}^{(2)}\partial_{21}^{(3)}\partial_{32}(v)\right) - \tfrac{1}{18}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}\mathcal{x}\partial_{32}\partial_{21}^{(3)}\partial_{32}(v) + \sigma_2\left(\tfrac{1}{18}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(2)}\partial_{21}^{(3)}\partial_{32}(v)\right) - \tfrac{1}{18}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\partial_{21}\partial_{21}^{(3)}\partial_{32}(v)$$

$$= -\tfrac{1}{9}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}\mathcal{x}\partial_{21}^{(2)}\partial_{32}\partial_{31}(v) - \tfrac{2}{9}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}\mathcal{x}\partial_{21}\partial_{31}^{(2)}(v) - \tfrac{3}{9}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\partial_{21}^{(3)}\partial_{31}(v) - \tfrac{2}{18}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}\mathcal{x}\partial_{21}^{(3)}\partial_{32}^{(2)}(v) - \tfrac{1}{18}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}\mathcal{x}\partial_{21}^{(2)}\partial_{32}\partial_{31}(v) - \tfrac{4}{18}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\partial_{21}^{(4)}\partial_{32}(v)$$

$$= -\tfrac{2}{9}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}\mathcal{x}\partial_{21}\partial_{31}^{(2)}(v) - \tfrac{1}{9}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}\mathcal{x}\partial_{21}^{(3)}\partial_{32}^{(2)}(v) - \tfrac{2}{9}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\partial_{21}^{(4)}\partial_{32}(v) - \tfrac{1}{6}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}\mathcal{x}\partial_{21}^{(2)}\partial_{32}\partial_{31}(v) - \tfrac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\partial_{21}^{(3)}\partial_{31}(v)$$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(3)}\mathcal{y}\mathcal{Z}_{21}^{(5)}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_4 \otimes \mathcal{D}_0$

$$= \sigma_2\left(\mathcal{Z}_{21}^{(5)}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\partial_{32}^{(3)}(v) + \mathcal{Z}_{21}^{(4)}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\partial_{32}^{(2)}\partial_{31}(v) + \mathcal{Z}_{21}^{(3)}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\partial_{32}\partial_{31}^{(2)}(v) + \mathcal{Z}_{21}^{(2)}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\partial_{31}^{(3)}(v) - 6\mathcal{Z}_{32}^{(3)}\mathcal{y}\mathcal{Z}_{21}^{(6)}\mathcal{x}(v) + \mathcal{Z}_{32}^{(3)}\mathcal{y}\mathcal{Z}_{21}^{(5)}\mathcal{x}\partial_{21}(v)\right)$$

$$= -\tfrac{1}{9}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}\mathcal{x}\partial_{21}^{(2)}\partial_{31}^{(2)}(v) + \tfrac{4}{15}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}\mathcal{x}\partial_{21}^{(4)}\partial_{32}^{(2)}(v) + \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\partial_{21}^{(5)}\partial_{32}(v) - \tfrac{28}{90}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}\mathcal{x}\partial_{21}^{(4)}\partial_{32}^{(2)}(v) - \tfrac{7}{90}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}\mathcal{x}\partial_{21}^{(3)}\partial_{32}\partial_{31}(v) - \tfrac{10}{9}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\partial_{21}^{(5)}\partial_{32}(v) - \tfrac{2}{9}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\partial_{21}^{(4)}\partial_{31}(v)$$

$$= -\tfrac{1}{9}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}\mathcal{x}\partial_{21}^{(2)}\partial_{31}^{(2)}(v) - \tfrac{2}{45}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}\mathcal{x}\partial_{21}^{(4)}\partial_{32}^{(2)}(v) -$$

$\frac{1}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{32}(v)-\frac{7}{90}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}\partial_{31}(v)-$
$\frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(4)}\partial_{31}(v)$

And

$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(-\frac{1}{18}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{31}(v)-\frac{1}{45}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}(v)\right)$$

$=\sigma_2\left(\frac{1}{18}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(3)}\partial_{31}(v)\right)-\frac{1}{18}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(3)}\partial_{31}(v)+$
$\sigma_2\left(\frac{1}{18}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(3)}\partial_{31}(v)\right)-\frac{1}{18}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(3)}\partial_{31}(v)+$
$\sigma_2\left(\frac{1}{45}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(4)}\partial_{32}(v)\right)-\frac{1}{45}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(4)}\partial_{32}(v)+$
$\sigma_2\left(\frac{1}{45}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(4)}\partial_{32}(v)\right)-\frac{1}{45}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(4)}\partial_{32}(v)$
$=-\frac{1}{18}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}\partial_{31}(v)-\frac{2}{18}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}^{(2)}(v)-$
$\frac{4}{18}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(4)}\partial_{31}(v)-\frac{2}{45}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}^{(2)}(v)-$
$\frac{1}{45}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}\partial_{31}(v)-\frac{5}{45}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{32}(v)$
$=-\frac{1}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}^{(2)}(v)-\frac{2}{45}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}^{(2)}(v)-$
$\frac{1}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{32}(v)-\frac{7}{90}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}\partial_{31}(v)-$
$\frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(4)}\partial_{31}(v)$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(2)}x(v)\right)$; where $v\in\mathcal{D}_{14}\otimes\mathcal{D}_4\otimes\mathcal{D}_0$

$=\sigma_2\left(\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(2)}x\partial_{32}^{(3)}(v)+\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(2)}x\partial_{32}^{(2)}\partial_{31}(v)+\right.$
$\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{31}^{(2)}(v)+\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x\partial_{31}^{(3)}(v)-15\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(6)}x(v)+$
$\left.\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(4)}x\partial_{21}^{(2)}(v)\right)$
$=-\frac{5}{6}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}^{(2)}(v)+\frac{2}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}^{(2)}(v)+$
$\frac{5}{2}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{32}(v)+\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}^{(2)}(v)-$

$\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}^{(2)}(v) - \frac{1}{2}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}\partial_{31}(v) -$
$\frac{1}{6}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}^{(2)}(v) - \frac{10}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{32}(v) -$
$\frac{4}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(4)}\partial_{31}(v)$
$= -\frac{2}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}^{(2)}(v) - \frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}^{(2)}(v) -$
$\frac{5}{6}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{32}(v) - \frac{1}{2}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}\partial_{31}(v) -$
$\frac{3}{4}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(4)}\partial_{31}(v)$

And

$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\Big(-\frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{31}(v) -$
$\frac{1}{6}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}(v)\Big)$
$= \sigma_2\left(\frac{1}{3}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(3)}\partial_{31}(v)\right) - \frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(3)}\partial_{31}(v) +$
$\sigma_2\left(\frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(2)}\partial_{21}^{(3)}\partial_{31}(v)\right) - \frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(3)}\partial_{31}(v) +$
$\sigma_2\left(\frac{1}{6}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(4)}\partial_{32}(v)\right) - \frac{1}{6}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(4)}\partial_{32}(v) +$
$\sigma_2\left(\frac{1}{6}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(2)}\partial_{21}^{(4)}\partial_{32}(v)\right) - \frac{1}{6}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(4)}\partial_{32}(v)$
$= -\frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}\partial_{31}(v) - \frac{2}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}^{(2)}(v) -$
$\frac{4}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(4)}\partial_{31}(v) - \frac{2}{6}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}^{(2)}(v) -$
$\frac{1}{6}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}\partial_{31}(v) - \frac{5}{6}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{32}(v)$
$= -\frac{2}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}^{(2)}(v) - \frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}^{(2)}(v) -$
$\frac{5}{6}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{32}(v) - \frac{1}{2}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}\partial_{31}(v) -$
$\frac{3}{4}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(4)}\partial_{31}(v)$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(3)}\mathcal{y}\mathcal{Z}_{21}^{(6)}x\mathcal{Z}_{21}x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_3 \otimes \mathcal{D}_0$

$= \sigma_2\Big(\mathcal{Z}_{21}^{(6)}x\mathcal{Z}_{21}x\partial_{32}^{(3)}(v) + \mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{31}(v) + \mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x\partial_{32}\partial_{31}^{(2)}(v) +$

$$\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}x\partial_{31}^{(3)}(v) - 7\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(7)}x(v) + \mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(6)}x\partial_{21}(v)\Big)$$

$$= -\frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}^{(2)}(v) + \frac{1}{5}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}^{(2)}(v) + \frac{14}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{32}(v) - \frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}^{(2)}(v) - \frac{2}{45}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}\partial_{31}(v) - \mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{32}(v) - \frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{31}(v)$$

$$= -\frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}^{(2)}(v) - \frac{1}{45}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}^{(2)}(v) - \frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{32}(v) - \frac{2}{45}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}\partial_{31}(v) - \frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{31}(v)$$

And

$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(-\frac{1}{30}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{31}(v) - \frac{1}{90}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{32}(v)\right)$$

$$= \sigma_2\left(\frac{1}{30}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(4)}\partial_{31}(v)\right) - \frac{1}{30}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(4)}\partial_{31}(v) + \sigma_2\left(\frac{1}{30}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(4)}\partial_{31}(v)\right) - \frac{1}{30}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(4)}\partial_{31}(v) + \sigma_2\left(\frac{1}{90}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(5)}\partial_{32}(v)\right) - \frac{1}{90}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(5)}\partial_{32}(v) + \sigma_2\left(\frac{1}{90}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(5)}\partial_{32}(v)\right) - \frac{1}{90}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(5)}\partial_{32}(v)$$

$$= -\frac{1}{30}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}\partial_{31}(v) - \frac{2}{30}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}^{(2)}(v) - \frac{5}{30}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{31}(v) - \frac{2}{90}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}^{(2)}(v) - \frac{1}{90}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}\partial_{31}(v) - \frac{6}{90}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{32}(v)$$

$$= -\frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}^{(2)}(v) - \frac{1}{45}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}^{(2)}(v) - \frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{32}(v) - \frac{2}{45}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}\partial_{31}(v) - \frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{31}(v)$$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21}^{(2)} x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_3 \otimes \mathcal{D}_0$

$= \sigma_2 \Big(\mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21}^{(2)} x \partial_{32}^{(3)}(v) + \mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(2)} x \partial_{32}^{(2)} \partial_{31}(v) +$

$\mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(2)} x \partial_{32} \partial_{31}^{(2)}(v) + \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(2)} x \partial_{31}^{(3)}(v) - 21 \mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(7)} x(v) +$

$\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(5)} x \partial_{21}^{(2)}(v)\Big)$

$= -\frac{11}{30} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(3)} \partial_{31}^{(2)}(v) + \frac{3}{5} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{32}^{(2)}(v) +$

$\frac{14}{5} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(6)} \partial_{32}(v) - \frac{7}{9} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{32}^{(2)}(v) -$

$\frac{14}{45} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(4)} \partial_{32} \partial_{31}(v) - \frac{7}{90} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(3)} \partial_{31}^{(2)}(v) -$

$\frac{10}{3} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(6)} \partial_{32}(v) - \frac{10}{9} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(5)} \partial_{31}(v)$

$= -\frac{4}{9} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(3)} \partial_{31}^{(2)}(v) - \frac{8}{45} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{32}^{(2)}(v) -$

$\frac{8}{15} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(6)} \partial_{32}(v) - \frac{14}{45} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(4)} \partial_{32} \partial_{31}(v) -$

$\frac{10}{9} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(5)} \partial_{31}(v)$

And

$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\Big(-\frac{2}{9} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(4)} \partial_{31}(v) -$

$\frac{4}{45} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(5)} \partial_{32}(v)\Big)$

$= \sigma_2 \left(\frac{2}{9} \mathcal{Z}_{21} x \mathcal{Z}_{21} x \partial_{32}^{(2)} \partial_{21}^{(4)} \partial_{31}(v)\right) - \frac{2}{9} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{32} \partial_{21}^{(4)} \partial_{31}(v) +$

$\sigma_2 \left(\frac{2}{9} \mathcal{Z}_{32} y \mathcal{Z}_{32} y \partial_{21}^{(2)} \partial_{21}^{(4)} \partial_{31}(v)\right) - \frac{2}{9} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21} \partial_{21}^{(4)} \partial_{31}(v) +$

$\sigma_2 \left(\frac{4}{45} \mathcal{Z}_{21} x \mathcal{Z}_{21} x \partial_{32}^{(2)} \partial_{21}^{(5)} \partial_{32}(v)\right) - \frac{4}{45} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{32} \partial_{21}^{(5)} \partial_{32}(v) +$

$\sigma_2 \left(\frac{4}{45} \mathcal{Z}_{32} y \mathcal{Z}_{32} y \partial_{21}^{(2)} \partial_{21}^{(5)} \partial_{32}(v)\right) - \frac{4}{45} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21} \partial_{21}^{(5)} \partial_{32}(v)$

$= -\frac{2}{9} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(4)} \partial_{32} \partial_{31}(v) - \frac{4}{9} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(3)} \partial_{31}^{(2)}(v) -$

$\frac{10}{9} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(5)} \partial_{31}(v) - \frac{8}{45} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{32}^{(2)}(v) -$

$\frac{4}{45} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(4)} \partial_{32} \partial_{31}(v) - \frac{24}{45} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(6)} \partial_{32}(v)$

$= -\frac{4}{9}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}^{(2)}(v) - \frac{8}{45}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}^{(2)}(v) -$
$\frac{8}{15}Z_{32}yZ_{31}z\partial_{21}^{(6)}\partial_{32}(v) - \frac{14}{45}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}\partial_{31}(v) -$
$\frac{10}{9}Z_{32}yZ_{31}z\partial_{21}^{(5)}\partial_{31}(v)$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(Z_{32}^{(3)}yZ_{21}^{(4)}xZ_{21}^{(3)}x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_3 \otimes \mathcal{D}_0$

$= \sigma_2\Big(Z_{21}^{(4)}xZ_{21}^{(3)}x\partial_{32}^{(3)}(v) + Z_{21}^{(3)}xZ_{21}^{(3)}x\partial_{32}^{(2)}\partial_{31}(v) +$
$Z_{21}^{(2)}xZ_{21}^{(3)}x\partial_{32}\partial_{31}^{(2)}(v) + Z_{21}xZ_{21}^{(3)}x\partial_{31}^{(3)}(v) - 35Z_{32}^{(3)}yZ_{21}^{(7)}x(v) +$
$Z_{32}^{(3)}yZ_{21}^{(4)}x\partial_{21}^{(3)}(v)\Big)$

$= -\frac{5}{6}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}^{(2)}(v) + Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}^{(2)}(v) +$
$\frac{14}{3}Z_{32}yZ_{31}z\partial_{21}^{(6)}\partial_{32}(v) - \frac{5}{3}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}^{(2)}(v) -$
$Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}\partial_{31}(v) - \frac{1}{2}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}^{(2)}(v) -$
$\frac{20}{3}Z_{32}yZ_{31}z\partial_{21}^{(6)}\partial_{32}(v) - \frac{10}{3}Z_{32}yZ_{31}z\partial_{21}^{(5)}\partial_{31}(v)$

$= -\frac{4}{3}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}^{(2)}(v) - \frac{2}{3}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}^{(2)}(v) -$
$2\,Z_{32}yZ_{31}z\partial_{21}^{(6)}\partial_{32}(v) - Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}\partial_{31}(v) -$
$\frac{10}{3}Z_{32}yZ_{31}z\partial_{21}^{(5)}\partial_{31}(v)$

And

$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\Big(-\frac{2}{3}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(4)}\partial_{31}(v) -$
$\frac{1}{3}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(5)}\partial_{32}(v)\Big)$

$= \sigma_2\left(\frac{2}{3}Z_{21}xZ_{21}x\partial_{32}^{(2)}\partial_{21}^{(4)}\partial_{31}(v)\right) - \frac{2}{3}Z_{32}yZ_{21}^{(2)}x\partial_{32}\partial_{21}^{(4)}\partial_{31}(v) +$
$\sigma_2\left(\frac{2}{3}Z_{32}yZ_{32}y\partial_{21}^{(2)}\partial_{21}^{(4)}\partial_{31}(v)\right) - \frac{2}{3}Z_{32}yZ_{31}z\partial_{21}\partial_{21}^{(4)}\partial_{31}(v) +$
$\sigma_2\left(\frac{1}{3}Z_{21}xZ_{21}x\partial_{32}^{(2)}\partial_{21}^{(5)}\partial_{32}(v)\right) - \frac{1}{3}Z_{32}yZ_{21}^{(2)}x\partial_{32}\partial_{21}^{(5)}\partial_{32}(v) +$
$\sigma_2\left(\frac{1}{3}Z_{32}yZ_{32}y\partial_{21}^{(2)}\partial_{21}^{(5)}\partial_{32}(v)\right) - \frac{1}{3}Z_{32}yZ_{31}z\partial_{21}\partial_{21}^{(5)}\partial_{32}(v)$

$= -\frac{2}{3}\ \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}\partial_{31}(v) - \frac{4}{3}\ \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}^{(2)}(v) -$
$\frac{10}{3}\ \mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{31}(v) - \frac{2}{3}\ \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}^{(2)}(v) -$
$\frac{1}{3}\ \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}\partial_{31}(v) - 2\ \mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{32}(v)$

$= -\frac{4}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}^{(2)}(v) - \frac{2}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}^{(2)}(v) -$
$2\ \mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{32}(v) - \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}\partial_{31}(v) -$
$\frac{10}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{31}(v)$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(7)}x\mathcal{Z}_{21}x(v)\right)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_2 \otimes \mathcal{D}_0$

$= \mathcal{Z}_{21}^{(7)}x\mathcal{Z}_{21}x\partial_{32}^{(3)}(v) + \sigma_2\left(\mathcal{Z}_{21}^{(6)}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{31}(v) + \mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}x\partial_{32}\partial_{31}^{(2)}(v) +\right.$
$\left.\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}x\partial_{31}^{(3)}(v) - 8\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(8)}x(v) + \mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(7)}x\partial_{21}(v)\right)$

$= -\frac{2}{45}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v) + \frac{10}{63}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v) +$
$\frac{8}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v) - \frac{6}{35}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v) -$
$\frac{1}{35}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v) - \frac{14}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v) -$
$\frac{2}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{31}(v)$

$= -\frac{2}{45}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v) - \frac{4}{315}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v) -$
$\frac{2}{45}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v) - \frac{1}{35}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v) -$
$\frac{2}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{31}(v)$

And

$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(-\frac{1}{45}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{31}(v) -\right.$
$\left.\frac{2}{315}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{32}(v)\right)$

$= \sigma_2\left(\frac{1}{45}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(5)}\partial_{31}(v)\right) - \frac{1}{45}\ \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(5)}\partial_{31}(v) +$
$\sigma_2\left(\frac{1}{45}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(5)}\partial_{31}(v)\right) - \frac{1}{45}\ \mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(5)}\partial_{31}(v) +$

$\sigma_2\left(\frac{2}{315}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(6)}\partial_{32}(v)\right)-\frac{2}{315}\ \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(6)}\partial_{32}(v)+$

$\sigma_2\left(\frac{2}{315}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(6)}\partial_{32}(v)\right)-\frac{2}{315}\ \mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(6)}\partial_{32}(v)$

$=-\frac{1}{45}\ \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v)-\frac{2}{45}\ \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v)-$

$\frac{2}{15}\ \mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{31}(v)-\frac{4}{315}\ \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v)-$

$\frac{2}{315}\ \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v)-\frac{14}{315}\ \mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v)$

$=-\frac{2}{45}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v)-\frac{4}{315}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v)-$

$\frac{2}{45}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v)-\frac{1}{35}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v)-$

$\frac{2}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{31}(v)$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(6)}x\mathcal{Z}_{21}^{(2)}x(v)\right)$; where $v\in\mathcal{D}_{16}\otimes\mathcal{D}_2\otimes\mathcal{D}_0$

$=\mathcal{Z}_{21}^{(6)}x\mathcal{Z}_{21}^{(2)}x\partial_{32}^{(3)}(v)+\sigma_2\left(\mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}^{(2)}x\partial_{32}^{(2)}\partial_{31}(v)+\right.$

$\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{31}^{(2)}(v)+\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(2)}x\partial_{31}^{(3)}(v)-28\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(8)}x(v)+$

$\left.\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(6)}x\partial_{21}^{(2)}(v)\right)$

$=-\frac{13}{45}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v)+\frac{5}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v)+$

$\frac{28}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v)-\frac{2}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v)-$

$\frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v)-\frac{2}{45}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v)-$

$\frac{7}{2}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v)-\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{31}(v)$

$=-\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v)-\frac{1}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v)-$

$\frac{7}{18}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v)-\frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v)-$

$\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{31}(v)$

And

$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(-\frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{31}(v)-\frac{1}{18}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{32}(v)\right)$$

$$=\sigma_2\left(\frac{1}{6}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(5)}\partial_{31}(v)\right)-\frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(5)}\partial_{31}(v)+\sigma_2\left(\frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(5)}\partial_{31}(v)\right)-\frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(5)}\partial_{31}(v)+\sigma_2\left(\frac{1}{18}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(6)}\partial_{32}(v)\right)-\frac{1}{18}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(6)}\partial_{32}(v)+\sigma_2\left(\frac{1}{18}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(6)}\partial_{32}(v)\right)-\frac{1}{18}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(6)}\partial_{32}(v)$$

$$=-\frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v)-\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v)-\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{31}(v)-\frac{1}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v)-\frac{1}{18}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v)-\frac{7}{18}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v)$$

$$=-\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v)-\frac{1}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v)-\frac{7}{18}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v)-\frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v)-\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{31}(v)$$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}^{(3)}x(v)\right)$; where $v\in\mathcal{D}_{16}\otimes\mathcal{D}_2\otimes\mathcal{D}_0$

$$=\mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}^{(3)}x\partial_{32}^{(3)}(v)+\sigma_2\left(\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(3)}x\partial_{32}^{(2)}\partial_{31}(v)+\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(3)}x\partial_{32}\partial_{31}^{(2)}(v)+\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(3)}x\partial_{31}^{(3)}(v)-56\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(8)}x(v)+\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(5)}x\partial_{21}^{(3)}(v)\right)$$

$$=-\frac{4}{5}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v)+\frac{10}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v)+\frac{56}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v)-\frac{14}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v)-\frac{7}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v)-\frac{28}{90}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v)-\frac{70}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v)-\frac{10}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{31}(v)$$

$= -\frac{10}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v) - \frac{4}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v) -$
$\frac{14}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v) - \frac{7}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v) -$
$\frac{10}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{31}(v)$

And

$(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2})\left(-\frac{5}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{31}(v) -\right.$
$\left.\frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{32}(v)\right)$

$= \sigma_2\left(\frac{5}{9}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(5)}\partial_{31}(v)\right) - \frac{5}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(5)}\partial_{31}(v) +$
$\sigma_2\left(\frac{5}{9}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(5)}\partial_{31}(v)\right) - \frac{5}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(5)}\partial_{31}(v) +$
$\sigma_2\left(\frac{2}{9}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(6)}\partial_{32}(v)\right) - \frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(6)}\partial_{32}(v) +$
$\sigma_2\left(\frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(6)}\partial_{32}(v)\right) - \frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(6)}\partial_{32}(v)$

$= -\frac{5}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v) - \frac{10}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v) -$
$\frac{10}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{31}(v) - \frac{4}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v) -$
$\frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v) - \frac{14}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v)$

$= -\frac{10}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v) - \frac{4}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v) -$
$\frac{14}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v) - \frac{7}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v) -$
$\frac{10}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{31}(v)$

- $(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2})\left(\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(4)}x(v)\right)$; where $v \in \mathcal{D}_{16} \otimes \mathcal{D}_2 \otimes \mathcal{D}_0$

$= \mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(4)}x\partial_{32}^{(3)}(v) + \sigma_2\left(\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(4)}x\partial_{32}^{(2)}\partial_{31}(v) +\right.$
$\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(4)}x\partial_{32}\partial_{31}^{(2)}(v) + \mathcal{Z}_{21}x\mathcal{Z}_{21}^{(4)}x\partial_{31}^{(3)}(v) - 70\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(8)}x(v) +$
$\left.\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(4)}x\partial_{21}^{(4)}(v)\right)$

$$= -\frac{11}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v) + \frac{25}{18}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v) + \frac{70}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v) - \frac{5}{2}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v) - \frac{5}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v) - \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v) - \frac{35}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v) - \frac{20}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{31}(v)$$

$$= -\frac{20}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v) - \frac{10}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v) - \frac{35}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v) - \frac{5}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v) - \frac{20}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{31}(v)$$

And

$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(-\frac{10}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{31}(v) - \frac{5}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{32}(v)\right)$$

$$= \sigma_2\left(\frac{10}{9}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(5)}\partial_{31}(v)\right) - \frac{10}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(5)}\partial_{31}(v) + \sigma_2\left(\frac{10}{9}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(5)}\partial_{31}(v)\right) - \frac{10}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(5)}\partial_{31}(v) + \sigma_2\left(\frac{5}{9}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(6)}\partial_{32}(v)\right) - \frac{5}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(6)}\partial_{32}(v) + \sigma_2\left(\frac{5}{9}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(6)}\partial_{32}(v)\right) - \frac{5}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(6)}\partial_{32}(v)$$

$$= -\frac{10}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v) - \frac{20}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v) - \frac{20}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{31}(v) - \frac{10}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v) - \frac{5}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v) - \frac{35}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v)$$

$$= -\frac{20}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v) - \frac{10}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v) - \frac{35}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v) - \frac{5}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v) - \frac{20}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{31}(v)$$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(8)} x \mathcal{Z}_{21} x(v)\right)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$

$$= \mathcal{Z}_{21}^{(8)} x \mathcal{Z}_{21} x \partial_{32}^{(3)}(v) + \mathcal{Z}_{21}^{(7)} x \mathcal{Z}_{21} x \partial_{32}^{(2)} \partial_{31}(v) + \sigma_2 \Big(\mathcal{Z}_{21}^{(6)} x \mathcal{Z}_{21} x \partial_{32} \partial_{31}^{(2)}(v) + \mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21} x \partial_{31}^{(3)}(v) - 9\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(9)} x(v) + \mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(8)} x \partial_{21}(v)\Big)$$

$$= -\frac{2}{63}\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}^{(2)}(v) - \frac{3}{28}\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(6)} \partial_{32} \partial_{31}(v) - \frac{5}{252}\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(6)} \partial_{32} \partial_{31}(v) - \frac{8}{9}\mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(8)} \partial_{32}(v) - \frac{1}{9}\mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(7)} \partial_{31}(v)$$

$$= -\frac{2}{63}\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}^{(2)}(v) - \frac{8}{63}\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(6)} \partial_{32} \partial_{31}(v) - \frac{8}{9}\mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(8)} \partial_{32}(v) - \frac{1}{9}\mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(7)} \partial_{31}(v)$$

And

$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\Big(-\frac{1}{63}\mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{31}(v) - \frac{1}{9}\mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(7)} \partial_{32}(v)\Big)$$

$$= \sigma_2\left(\frac{1}{63}\mathcal{Z}_{21} x \mathcal{Z}_{21} x \partial_{32}^{(2)} \partial_{21}^{(6)} \partial_{31}(v)\right) - \frac{1}{63}\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{32} \partial_{21}^{(6)} \partial_{31}(v) + \sigma_2\left(\frac{1}{63}\mathcal{Z}_{32} y \mathcal{Z}_{32} y \partial_{21}^{(2)} \partial_{21}^{(6)} \partial_{31}(v)\right) - \frac{1}{63}\mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21} \partial_{21}^{(6)} \partial_{31}(v) + \sigma_2\left(\frac{1}{9}\mathcal{Z}_{21} x \mathcal{Z}_{21} x \partial_{32}^{(2)} \partial_{21}^{(7)} \partial_{32}(v)\right) - \frac{1}{9}\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{32} \partial_{21}^{(7)} \partial_{32}(v) + \sigma_2\left(\frac{1}{9}\mathcal{Z}_{32} y \mathcal{Z}_{32} y \partial_{21}^{(2)} \partial_{21}^{(7)} \partial_{32}(v)\right) - \frac{1}{9}\mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21} \partial_{21}^{(7)} \partial_{32}(v)$$

$$= -\frac{1}{63}\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(6)} \partial_{32} \partial_{31}(v) - \frac{2}{63}\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}^{(2)}(v) - \frac{1}{9}\mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(7)} \partial_{31}(v) - \frac{1}{9}\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(6)} \partial_{32} \partial_{31}(v) - \frac{8}{9}\mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(8)} \partial_{32}(v)$$

$$= -\frac{2}{63}\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}^{(2)}(v) - \frac{8}{63}\mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(6)} \partial_{32} \partial_{31}(v) - \frac{8}{9}\mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(8)} \partial_{32}(v) - \frac{1}{9}\mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(7)} \partial_{31}(v)$$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(7)} x \mathcal{Z}_{21}^{(2)} x(v)\right)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$

$$= \mathcal{Z}_{21}^{(7)} x \mathcal{Z}_{21}^{(2)} x \partial_{32}^{(3)}(v) + \mathcal{Z}_{21}^{(6)} x \mathcal{Z}_{21}^{(2)} x \partial_{32}^{(2)} \partial_{31}(v) + \sigma_2 \Big(\mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21}^{(2)} x \partial_{32} \partial_{31}^{(2)}(v) + \mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(2)} x \partial_{31}^{(3)}(v) - 36 \mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(9)} x(v) + \mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(7)} x \partial_{21}^{(2)}(v)\Big)$$

$$= -\tfrac{5}{21} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}^{(2)}(v) - \tfrac{3}{7} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(6)} \partial_{32} \partial_{31}(v) - \tfrac{6}{35} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(6)} \partial_{32} \partial_{31}(v) - \tfrac{1}{35} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}^{(2)}(v) - \tfrac{56}{15} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(8)} \partial_{32}(v) - \tfrac{14}{15} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(7)} \partial_{31}(v)$$

$$= -\tfrac{4}{15} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}^{(2)}(v) - \tfrac{3}{5} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(6)} \partial_{32} \partial_{31}(v) - \tfrac{56}{15} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(8)} \partial_{32}(v) - \tfrac{14}{15} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(7)} \partial_{31}(v)$$

And

$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\Big(-\tfrac{2}{15} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{31}(v) - \tfrac{7}{15} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(7)} \partial_{32}(v)\Big)$$

$$= \sigma_2\left(\tfrac{2}{15} \mathcal{Z}_{21} x \mathcal{Z}_{21} x \partial_{32}^{(2)} \partial_{21}^{(6)} \partial_{31}(v)\right) - \tfrac{2}{15} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{32} \partial_{21}^{(6)} \partial_{31}(v) + \sigma_2\left(\tfrac{2}{15} \mathcal{Z}_{32} y \mathcal{Z}_{32} y \partial_{21}^{(2)} \partial_{21}^{(6)} \partial_{31}(v)\right) - \tfrac{2}{15} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21} \partial_{21}^{(6)} \partial_{31}(v) + \sigma_2\left(\tfrac{7}{15} \mathcal{Z}_{21} x \mathcal{Z}_{21} x \partial_{32}^{(2)} \partial_{21}^{(7)} \partial_{32}(v)\right) - \tfrac{7}{15} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{32} \partial_{21}^{(7)} \partial_{32}(v) + \sigma_2\left(\tfrac{7}{15} \mathcal{Z}_{32} y \mathcal{Z}_{32} y \partial_{21}^{(2)} \partial_{21}^{(7)} \partial_{32}(v)\right) - \tfrac{7}{15} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21} \partial_{21}^{(7)} \partial_{32}(v)$$

$$= -\tfrac{2}{15} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(6)} \partial_{32} \partial_{31}(v) - \tfrac{4}{15} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}^{(2)}(v) - \tfrac{14}{15} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(7)} \partial_{31}(v) - \tfrac{7}{15} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(6)} \partial_{32} \partial_{31}(v) - \tfrac{56}{15} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(8)} \partial_{32}(v)$$

$$= -\tfrac{4}{15} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}^{(2)}(v) - \tfrac{3}{5} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(6)} \partial_{32} \partial_{31}(v) - \tfrac{56}{15} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(8)} \partial_{32}(v) - \tfrac{14}{15} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(7)} \partial_{31}(v)$$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(6)}x\mathcal{Z}_{21}^{(3)}x(v)\right)$; where $v\in\mathcal{D}_{17}\otimes\mathcal{D}_1\otimes\mathcal{D}_0$

$=\mathcal{Z}_{21}^{(6)}x\mathcal{Z}_{21}^{(3)}x\partial_{32}^{(3)}(v)+\mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}^{(3)}x\partial_{32}^{(2)}\partial_{31}(v)+\sigma_2\left(\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(3)}x\partial_{32}\partial_{31}^{(2)}(v)+\right.$
$\left.\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(3)}x\partial_{31}^{(3)}(v)-84\,\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(9)}x(v)+\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(6)}x\partial_{21}^{(3)}(v)\right)$

$=-\frac{7}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}^{(2)}(v)-\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}\partial_{31}(v)-$
$\frac{2}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}\partial_{31}(v)-\frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}^{(2)}(v)-$
$\frac{28}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}\partial_{32}(v)-\frac{7}{2}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{31}(v)$

$=-\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}^{(2)}(v)-\frac{5}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}\partial_{31}(v)-$
$\frac{28}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}\partial_{32}(v)-\frac{7}{2}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{31}(v)$

And

$\left(\delta_{\mathcal{L}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(-\frac{1}{2}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{31}(v)-\right.$
$\left.\frac{7}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(7)}\partial_{32}(v)\right)$

$=\sigma_2\left(\frac{1}{2}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(6)}\partial_{31}(v)\right)-\frac{1}{2}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(6)}\partial_{31}(v)+$
$\sigma_2\left(\frac{1}{2}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(6)}\partial_{31}(v)\right)-\frac{1}{2}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(6)}\partial_{31}(v)+$
$\sigma_2\left(\frac{7}{6}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(7)}\partial_{32}(v)\right)-\frac{7}{6}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(7)}\partial_{32}(v)+$
$\sigma_2\left(\frac{7}{6}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(7)}\partial_{32}(v)\right)-\frac{7}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(7)}\partial_{32}(v)$

$=-\frac{1}{2}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}\partial_{31}(v)-\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}^{(2)}(v)-$
$\frac{7}{2}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{31}(v)-\frac{7}{6}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}\partial_{31}(v)-$
$\frac{28}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}\partial_{32}(v)$

$=-\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}^{(2)}(v)-\frac{5}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}\partial_{31}(v)-$
$\frac{28}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}\partial_{32}(v)-\frac{7}{2}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{31}(v)$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21}^{(4)} x(v)\right)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$

$$= \mathcal{Z}_{21}^{(5)} x \mathcal{Z}_{21}^{(4)} x \partial_{32}^{(3)}(v) + \mathcal{Z}_{21}^{(4)} x \mathcal{Z}_{21}^{(4)} x \partial_{32}^{(2)} \partial_{31}(v) + \sigma_2\Big(\mathcal{Z}_{21}^{(3)} x \mathcal{Z}_{21}^{(4)} x \partial_{32} \partial_{31}^{(2)}(v) + \mathcal{Z}_{21}^{(2)} x \mathcal{Z}_{21}^{(4)} x \partial_{31}^{(3)}(v) - 12\, \mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(9)} x(v) + \mathcal{Z}_{32}^{(3)} y \mathcal{Z}_{21}^{(5)} x \partial_{21}^{(4)}(v)\Big)$$

$$= -\frac{13}{9} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}^{(2)}(v) - \frac{3}{2} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(6)} \partial_{32} \partial_{31}(v) - \frac{14}{9} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(6)} \partial_{32} \partial_{31}(v) - \frac{7}{9} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}^{(2)}(v) - \frac{140}{9} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(8)} \partial_{32}(v) - \frac{70}{9} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(7)} \partial_{31}(v)$$

$$= -\frac{20}{9} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}^{(2)}(v) - \frac{55}{18} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(6)} \partial_{32} \partial_{31}(v) - \frac{140}{9} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(8)} \partial_{32}(v) - \frac{70}{9} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(7)} \partial_{31}(v)$$

And

$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\Big(-\frac{10}{9} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(6)} \partial_{31}(v) - \frac{35}{18} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(7)} \partial_{32}(v)\Big)$$

$$= \sigma_2\left(\frac{10}{9} \mathcal{Z}_{21} x \mathcal{Z}_{21} x \partial_{32}^{(2)} \partial_{21}^{(6)} \partial_{31}(v)\right) - \frac{10}{9} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{32} \partial_{21}^{(6)} \partial_{31}(v) + \sigma_2\left(\frac{10}{9} \mathcal{Z}_{32} y \mathcal{Z}_{32} y \partial_{21}^{(2)} \partial_{21}^{(6)} \partial_{31}(v)\right) - \frac{10}{9} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21} \partial_{21}^{(6)} \partial_{31}(v) + \sigma_2\left(\frac{35}{18} \mathcal{Z}_{21} x \mathcal{Z}_{21} x \partial_{32}^{(2)} \partial_{21}^{(7)} \partial_{32}(v)\right) - \frac{35}{18} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{32} \partial_{21}^{(7)} \partial_{32}(v) + \sigma_2\left(\frac{35}{18} \mathcal{Z}_{32} y \mathcal{Z}_{32} y \partial_{21}^{(2)} \partial_{21}^{(7)} \partial_{32}(v)\right) - \frac{35}{18} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21} \partial_{21}^{(7)} \partial_{32}(v)$$

$$= -\frac{10}{9} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(6)} \partial_{32} \partial_{31}(v) - \frac{20}{9} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}^{(2)}(v) - \frac{70}{9} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(7)} \partial_{31}(v) - \frac{35}{18} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(6)} \partial_{32} \partial_{31}(v) - \frac{140}{9} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(8)} \partial_{32}(v)$$

$$= -\frac{20}{9} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(5)} \partial_{31}^{(2)}(v) - \frac{55}{18} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(6)} \partial_{32} \partial_{31}(v) - \frac{140}{9} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(8)} \partial_{32}(v) - \frac{70}{9} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(7)} \partial_{31}(v)$$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(5)}x(v)\right)$; where $v\in\mathcal{D}_{17}\otimes\mathcal{D}_1\otimes\mathcal{D}_0$

$=\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(5)}x\partial_{32}^{(3)}(v)+\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(5)}x\partial_{32}^{(2)}\partial_{31}(v)+\sigma_2\left(\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(5)}x\partial_{32}\partial_{31}^{(2)}(v)+\right.$

$\left.\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(5)}x\partial_{31}^{(3)}(v)-126\,\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(9)}x(v)+\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(4)}x\partial_{21}^{(5)}(v)\right)$

$=-\frac{5}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}^{(2)}(v)-\frac{3}{2}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}\partial_{31}(v)-$

$\frac{5}{2}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}\partial_{31}(v)-\frac{5}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}^{(2)}(v)-$

$\frac{56}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}\partial_{32}(v)-\frac{35}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{31}(v)$

$=-\frac{10}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}^{(2)}(v)-4\,\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}\partial_{31}(v)-$

$\frac{56}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}\partial_{32}(v)-\frac{35}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{31}(v)$

And

$\left(\delta_{\mathcal{L}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(-\frac{5}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{31}(v)-\right.$

$\left.\frac{7}{3}\,\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(7)}\partial_{32}(v)\right)$

$=\sigma_2\left(\frac{5}{3}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(6)}\partial_{31}(v)\right)-\frac{5}{3}\,\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(6)}\partial_{31}(v)+$

$\sigma_2\left(\frac{5}{3}\,\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(6)}\partial_{31}(v)\right)-\frac{5}{3}\,\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(6)}\partial_{31}(v)+$

$\sigma_2\left(\frac{7}{3}\,\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(7)}\partial_{32}(v)\right)-\frac{7}{3}\,\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(7)}\partial_{32}(v)+$

$\sigma_2\left(\frac{7}{3}\,\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(7)}\partial_{32}(v)\right)-\frac{7}{3}\,\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(7)}\partial_{32}(v)$

$=-\frac{5}{3}\,\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}\partial_{31}(v)-\frac{10}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}^{(2)}(v)-$

$\frac{35}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{31}(v)-\frac{7}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}\partial_{31}(v)-$

$\frac{56}{3}\,\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}\partial_{32}(v)$

$=-\frac{10}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}^{(2)}(v)-4\,\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}\partial_{31}(v)-$

$\frac{56}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}\partial_{32}(v)-\frac{35}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{31}(v)$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(3)}\mathcal{y}\mathcal{Z}_{21}^{(9)}x\mathcal{Z}_{21}x(v)\right)$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$

$$= \mathcal{Z}_{21}^{(9)}x\mathcal{Z}_{21}x\partial_{32}^{(3)}(v) + \mathcal{Z}_{21}^{(8)}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{31}(v) + \mathcal{Z}_{21}^{(7)}x\mathcal{Z}_{21}x\partial_{32}\partial_{31}^{(2)}(v) +$$
$$\sigma_2\left(\mathcal{Z}_{21}^{(6)}x\mathcal{Z}_{21}x\partial_{31}^{(3)}(v) - 10\, \mathcal{Z}_{32}^{(3)}\mathcal{y}\mathcal{Z}_{21}^{(10)}x(v) + \mathcal{Z}_{32}^{(3)}\mathcal{y}\mathcal{Z}_{21}^{(9)}x\partial_{21}(v)\right)$$
$$= -\frac{1}{42}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v) + \frac{1}{84}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}\partial_{31}(v) +$$
$$\frac{1}{84}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}\partial_{31}(v)$$
$$= 0$$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(3)}\mathcal{y}\mathcal{Z}_{21}^{(8)}x\mathcal{Z}_{21}^{(2)}x(v)\right)$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$

$$= \mathcal{Z}_{21}^{(8)}x\mathcal{Z}_{21}^{(2)}x\partial_{32}^{(3)}(v) + \mathcal{Z}_{21}^{(7)}x\mathcal{Z}_{21}^{(2)}x\partial_{32}^{(2)}\partial_{31}(v) + \mathcal{Z}_{21}^{(6)}x\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{31}^{(2)}(v) +$$
$$\sigma_2\left(\mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}^{(2)}x\partial_{31}^{(3)}(v) - 45\, \mathcal{Z}_{32}^{(3)}\mathcal{y}\mathcal{Z}_{21}^{(10)}x(v) + \mathcal{Z}_{32}^{(3)}\mathcal{y}\mathcal{Z}_{21}^{(8)}x\partial_{21}^{(2)}(v)\right)$$
$$= -\frac{17}{84}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v) - \frac{5}{252}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v) -$$
$$\frac{1}{9}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{21}\partial_{31}(v)$$
$$= -\frac{2}{9}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v) - \frac{8}{9}\, \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(8)}\partial_{31}(v)$$

And

$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(-\frac{1}{9}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(7)}\partial_{31}(v)\right)$$
$$= \sigma_2\left(\frac{1}{9}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(7)}\partial_{31}(v)\right) - \frac{1}{9}\, \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(7)}\partial_{31}(v) +$$
$$\sigma_2\left(\frac{1}{9}\, \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(2)}\partial_{21}^{(7)}\partial_{31}(v)\right) - \frac{1}{9}\, \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(7)}\partial_{31}(v)$$
$$= -\frac{2}{9}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v) - \frac{8}{9}\, \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(8)}\partial_{31}(v)$$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(3)}\mathcal{y}\mathcal{Z}_{21}^{(7)}x\mathcal{Z}_{21}^{(3)}x(v)\right)$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$

$$= \mathcal{Z}_{21}^{(7)}x\mathcal{Z}_{21}^{(3)}x\partial_{32}^{(3)}(v) + \mathcal{Z}_{21}^{(6)}x\mathcal{Z}_{21}^{(3)}x\partial_{32}^{(2)}\partial_{31}(v) + \mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}^{(3)}x\partial_{32}\partial_{31}^{(2)}(v) +$$
$$\sigma_2\left(\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(3)}x\partial_{31}^{(3)}(v) - 120\, \mathcal{Z}_{32}^{(3)}\mathcal{y}\mathcal{Z}_{21}^{(10)}x(v) + \mathcal{Z}_{32}^{(3)}\mathcal{y}\mathcal{Z}_{21}^{(7)}x\partial_{21}^{(3)}(v)\right)$$
$$= -\frac{16}{21}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v) - \frac{1}{35}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{21}\partial_{31}^{(2)}(v) -$$

$\frac{2}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{21}^{(2)}\partial_{31}(v)$

$= -\frac{98}{105}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v) - \frac{56}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}\partial_{31}(v)$

And

$(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2})\left(-\frac{7}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(7)}\partial_{31}(v)\right)$

$= \sigma_2\left(\frac{7}{15}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(7)}\partial_{31}(v)\right) - \frac{7}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(7)}\partial_{31}(v) +$

$\sigma_2\left(\frac{7}{15}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(7)}\partial_{31}(v)\right) - \frac{7}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(7)}\partial_{31}(v)$

$= -\frac{98}{105}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v) - \frac{56}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}\partial_{31}(v)$

- $(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2})\left(\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(6)}x\mathcal{Z}_{21}^{(4)}x(v)\right)$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$

$= \mathcal{Z}_{21}^{(6)}x\mathcal{Z}_{21}^{(4)}x\partial_{32}^{(3)}(v) + \mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}^{(4)}x\partial_{32}^{(2)}\partial_{31}(v) + \mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(4)}x\partial_{32}\partial_{31}^{(2)}(v) +$

$\sigma_2\left(\mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(4)}x\partial_{31}^{(3)}(v) - 210\,\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(10)}x(v) + \mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(6)}x\partial_{21}^{(4)}(v)\right)$

$= -\frac{5}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v) - \frac{2}{45}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{21}^{(2)}\partial_{31}^{(2)}(v) -$

$\frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{21}^{(3)}\partial_{31}(v)$

$= -\frac{7}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v) - \frac{28}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}\partial_{31}(v)$

And

$(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2})\left(-\frac{7}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(7)}\partial_{31}(v)\right)$

$= \sigma_2\left(\frac{7}{6}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(7)}\partial_{31}(v)\right) - \frac{7}{6}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(7)}\partial_{31}(v) +$

$\sigma_2\left(\frac{7}{6}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(7)}\partial_{31}(v)\right) - \frac{7}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(7)}\partial_{31}(v)$

$= -\frac{7}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v) - \frac{28}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}\partial_{31}(v)$

- $(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2})\left(\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}^{(5)}x(v)\right)$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$

$= \mathcal{Z}_{21}^{(5)}x\mathcal{Z}_{21}^{(5)}x\partial_{32}^{(3)}(v) + \mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(5)}x\partial_{32}^{(2)}\partial_{31}(v) + \mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(5)}x\partial_{32}\partial_{31}^{(2)}(v) +$

$$\sigma_2\left(\mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(5)}x\partial_{31}^{(3)}(v) - 252\,\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(10)}x(v) + \mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(5)}x\partial_{21}^{(5)}(v)\right)$$

$$= -\frac{7}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v) - \frac{7}{90}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{21}^{(3)}\partial_{31}^{(2)}(v) -$$

$$\frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(4)}\partial_{21}^{(4)}\partial_{31}(v)$$

$$= -\frac{35}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v) - \frac{140}{9}\,\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}\partial_{31}(v)$$

And

$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(-\frac{35}{18}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(7)}\partial_{31}(v)\right)$$

$$= \sigma_2\left(\frac{35}{18}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(7)}\partial_{31}(v)\right) - \frac{35}{18}\,\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(7)}\partial_{31}(v) +$$

$$\sigma_2\left(\frac{35}{18}\,\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(7)}\partial_{31}(v)\right) - \frac{35}{18}\,\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(7)}\partial_{31}(v)$$

$$= -\frac{35}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v) - \frac{140}{9}\,\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}\partial_{31}(v)$$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(6)}x(v)\right)$; where $v \in \mathcal{D}_{18} \otimes \mathcal{D}_0 \otimes \mathcal{D}_0$

$$= \mathcal{Z}_{21}^{(4)}x\mathcal{Z}_{21}^{(6)}x\partial_{32}^{(3)}(v) + \mathcal{Z}_{21}^{(3)}x\mathcal{Z}_{21}^{(6)}x\partial_{32}^{(2)}\partial_{31}(v) + \mathcal{Z}_{21}^{(2)}x\mathcal{Z}_{21}^{(6)}x\partial_{32}\partial_{31}^{(2)}(v) +$$

$$\sigma_2\left(\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(6)}x\partial_{31}^{(3)}(v) - 210\,\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(10)}x(v) + \mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(4)}x\partial_{21}^{(6)}(v)\right)$$

$$= -\frac{13}{6}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v) - \frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{21}^{(4)}\partial_{31}^{(2)}(v) -$$

$$\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(3)}\partial_{21}^{(5)}\partial_{31}(v)$$

$$= -\frac{14}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v) - \frac{56}{3}\,\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}\partial_{31}(v)$$

And

$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(-\frac{7}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(7)}\partial_{31}(v)\right)$$

$$= \sigma_2\left(\frac{7}{3}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(7)}\partial_{31}(v)\right) - \frac{7}{3}\,\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(7)}\partial_{31}(v) +$$

$$\sigma_2\left(\frac{7}{3}\,\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(7)}\partial_{31}(v)\right) - \frac{7}{3}\,\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(7)}\partial_{31}(v)$$

$$= -\frac{14}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v) - \frac{56}{3}\,\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}\partial_{31}(v)$$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(2)}x(v)\right)$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_6 \otimes \mathcal{D}_1$

$= \sigma_2 \left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(3)}x(v) - \mathcal{Z}_{21}x\mathcal{Z}_{21}^{(2)}x\partial_{32}^{(2)}(v) + \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(3)}x\partial_{32}(v) -\right.$

$\left.\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(3)}(v)\right) + \mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(2)}(v)$

$= \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{31}(v) - \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(2)}(v) + \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{32}(v) +$

$\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(2)}(v)$

$= \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{31}(v) + \frac{2}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(2)}(v) + \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{32}(v)$

And

$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(\frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}(v)\right)$

$= \sigma_2\left(-\frac{1}{3}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}(v)\right) + \frac{1}{3}\ \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}(v) -$

$\sigma_2\left(\frac{1}{3}\ \mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}(v)\right) + \frac{1}{3}\ \mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}(v)$

$= \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{31}(v) + \frac{2}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(2)}(v) + \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{32}(v)$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(3)}x(v)\right)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_5 \otimes \mathcal{D}_1$

$= \sigma_2 \left(2\ \mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(4)}x(v) - \mathcal{Z}_{21}x\mathcal{Z}_{21}^{(3)}x\partial_{32}^{(2)}(v) + \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(4)}x\partial_{32}(v) -\right.$

$\left.\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(4)}(v)\right) + \mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(3)}(v)$

$= \frac{2}{12}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{31}(v) - \frac{2}{4}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(3)}(v) + \frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}(v) +$

$\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(3)}(v)$

$= \frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{31}(v) + \frac{1}{2}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(3)}(v) + \frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}(v)$

And

$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(\frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(2)}(v)\right)$

$= \sigma_2\left(-\frac{1}{6}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(2)}(v)\right) + \frac{1}{6}\ \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(2)}(v) -$

$\sigma_2\left(\frac{1}{6}\ \mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(2)}(v)\right) + \frac{1}{6}\ \mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(2)}(v)$

$= \frac{1}{6} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21} \partial_{31}(v) + \frac{1}{2} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(3)}(v) + \frac{1}{6} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(2)} \partial_{32}(v)$

- $(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}) \left(\mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21}^{(4)} x(v)\right)$; where $v \in \mathcal{D}_{13} \otimes \mathcal{D}_4 \otimes \mathcal{D}_1$

$= \sigma_2 \left(3\, \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(5)} x(v) - \mathcal{Z}_{21} x \mathcal{Z}_{21}^{(4)} x \partial_{32}^{(2)}(v) + \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(5)} x \partial_{32}(v) - \mathcal{Z}_{32} y \mathcal{Z}_{32} y \partial_{21}^{(5)}(v)\right) + \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(4)}(v)$

$= \frac{3}{30} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(2)} \partial_{31}(v) - \frac{3}{5} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(4)}(v) + \frac{1}{10} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(3)} \partial_{32}(v) + \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(4)}(v)$

$= \frac{1}{10} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(2)} \partial_{31}(v) + \frac{2}{5} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(4)}(v) + \frac{1}{10} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(3)} \partial_{32}(v)$

And

$(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}) \left(\frac{1}{10} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21} x \partial_{21}^{(3)}(v)\right)$

$= \sigma_2 \left(-\frac{1}{10} \mathcal{Z}_{21} x \mathcal{Z}_{21} x \partial_{32}^{(2)} \partial_{21}^{(3)}(v)\right) + \frac{1}{10} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{32} \partial_{21}^{(3)}(v) - \sigma_2 \left(\frac{1}{10} \mathcal{Z}_{32} y \mathcal{Z}_{32} y \partial_{21}^{(2)} \partial_{21}^{(3)}(v)\right) + \frac{1}{10} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21} \partial_{21}^{(3)}(v)$

$= \frac{1}{10} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(2)} \partial_{31}(v) + \frac{2}{5} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(4)}(v) + \frac{1}{10} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(3)} \partial_{32}(v)$

- $(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}) \left(\mathcal{Z}_{32} y \mathcal{Z}_{31} z \mathcal{Z}_{21}^{(5)} x(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_3 \otimes \mathcal{D}_1$

$= \sigma_2 \left(4\, \mathcal{Z}_{32}^{(2)} y \mathcal{Z}_{21}^{(6)} x(v) - \mathcal{Z}_{21} x \mathcal{Z}_{21}^{(5)} x \partial_{32}^{(2)}(v) + \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(6)} x \partial_{32}(v) - \mathcal{Z}_{32} y \mathcal{Z}_{32} y \partial_{21}^{(6)}(v)\right) + \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(5)}(v)$

$= \frac{4}{60} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(3)} \partial_{31}(v) - \frac{4}{6} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(5)}(v) + \frac{1}{15} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(4)} \partial_{32}(v) + \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(5)}(v)$

$= \frac{1}{15} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(3)} \partial_{31}(v) + \frac{1}{3} \mathcal{Z}_{32} y \mathcal{Z}_{31} z \partial_{21}^{(5)}(v) + \frac{1}{15} \mathcal{Z}_{32} y \mathcal{Z}_{21}^{(2)} x \partial_{21}^{(4)} \partial_{32}(v)$

And

$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(\tfrac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(4)}(v)\right)$$

$$=\sigma_2\left(-\tfrac{1}{15}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(4)}(v)\right)+\tfrac{1}{15}\ \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(4)}(v)-$$

$$\sigma_2\left(\tfrac{1}{15}\ \mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(4)}(v)\right)+\tfrac{1}{15}\ \mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(4)}(v)$$

$$=\tfrac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}(v)+\tfrac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}(v)+\tfrac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}(v)$$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(6)}x(v)\right)$; where $v\in\mathcal{D}_{15}\otimes\mathcal{D}_2\otimes\mathcal{D}_1$

$$=\sigma_2\left(5\ \mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(7)}x(v)-\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(6)}x\partial_{32}^{(2)}(v)+\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(7)}x\partial_{32}(v)-\right.$$

$$\left.\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(7)}(v)\right)+\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}(v)$$

$$=\tfrac{5}{105}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}(v)-\tfrac{5}{7}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}(v)+$$

$$\tfrac{1}{21}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}(v)+\ \mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}(v)$$

$$=\tfrac{1}{21}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}(v)+\tfrac{2}{7}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}(v)+\tfrac{1}{21}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}(v)$$

And

$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(\tfrac{1}{21}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(5)}(v)\right)$$

$$=\sigma_2\left(-\tfrac{1}{21}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(5)}(v)\right)+\tfrac{1}{21}\ \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(5)}(v)-$$

$$\sigma_2\left(\tfrac{1}{21}\ \mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(5)}(v)\right)+\tfrac{1}{21}\ \mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(5)}(v)$$

$$=\tfrac{1}{21}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}(v)+\tfrac{2}{7}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}(v)+\tfrac{1}{21}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}(v)$$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(7)}x(v)\right)$; where $v\in\mathcal{D}_{16}\otimes\mathcal{D}_1\otimes\mathcal{D}_1$

$$=\sigma_2\left(6\ \mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(8)}x(v)\right)-\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(7)}x\partial_{32}^{(2)}(v)+\sigma_2\left(\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(8)}x\partial_{32}(v)-\right.$$

$$\left.\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(8)}(v)\right)+\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}(v)$$

$$=\tfrac{6}{168}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}(v)-\tfrac{6}{8}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}(v)+$$

$\frac{1}{28}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}(v)+\ \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(7)}(v)$

$=\frac{1}{28}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}(v)+\frac{1}{4}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(7)}(v)+\frac{1}{28}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}(v)$

And

$\left(\delta_{\mathcal{L}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(\frac{1}{28}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(6)}(v)\right)$

$=\sigma_2\left(-\frac{1}{28}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(6)}(v)\right)+\frac{1}{28}\ \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(6)}(v)-$

$\sigma_2\left(\frac{1}{28}\ \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(2)}\partial_{21}^{(6)}(v)\right)+\frac{1}{28}\ \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(6)}(v)$

$=\frac{1}{28}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}(v)+\frac{1}{4}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(7)}(v)+\frac{1}{28}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}(v)$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(8)}x(v)\right)$; where $v\in\mathcal{D}_{17}\otimes\mathcal{D}_0\otimes\mathcal{D}_1$

$=\sigma_2\left(7\ \mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{21}^{(9)}x(v)\right)-\mathcal{Z}_{21}x\mathcal{Z}_{21}^{(8)}x\partial_{32}^{(2)}(v)+\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(9)}x\partial_{32}(v)-$

$\sigma_2\left(\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(9)}(v)\right)+\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(8)}(v)$

$=\frac{7}{252}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}(v)-\frac{7}{9}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(8)}(v)+\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(8)}(v)$

$=\frac{1}{36}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}(v)+\frac{2}{9}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(8)}(v)$

And

$\left(\delta_{\mathcal{L}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(\frac{1}{36}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(7)}(v)\right)$

$=\sigma_2\left(-\frac{1}{36}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(7)}(v)\right)+\frac{1}{36}\ \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(7)}(v)-$

$\sigma_2\left(\frac{1}{36}\ \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(2)}\partial_{21}^{(7)}(v)\right)+\frac{1}{36}\ \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(7)}(v)$

$=\frac{1}{36}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}(v)+\frac{2}{9}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z\partial_{21}^{(8)}(v)$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z(v)\right)$; where $v\in\mathcal{D}_9\otimes\mathcal{D}_9\otimes\mathcal{D}_0$

$=\sigma_2\left(2\ \mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{31}z(v)-2\ \mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}z(v)+\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\partial_{31}(v)\right)$

$=0$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\mathcal{Z}_{21}^{(2)}\mathcal{x}(v)\right)$; where $v \in \mathcal{D}_{11} \otimes \mathcal{D}_7 \otimes \mathcal{D}_0$

$= \sigma_2\left(-\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}^{(2)}\mathcal{x}\partial_{32}^{(2)}(v) + \mathcal{Z}_{21}^{(2)}\mathcal{x}\mathcal{Z}_{21}^{(3)}\mathcal{x}\partial_{32}(v) + \mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(3)}(v) + \right.$

$\left.\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\partial_{21}^{(2)}(v)\right)$

$= \frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}\mathcal{x}\partial_{31}\partial_{32}(v) - \frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\partial_{21}^{(2)}\partial_{32}(v) + \frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\partial_{32}\partial_{21}^{(2)}(v)$

$= \frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}\mathcal{x}\partial_{32}\partial_{31}(v) + \frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\partial_{21}\partial_{31}(v)$

And

$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(\frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\mathcal{Z}_{21}\mathcal{x}\partial_{31}(v)\right)$

$= \sigma_2\left(-\frac{1}{3}\mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}\mathcal{x}\partial_{32}^{(2)}\partial_{31}(v)\right) + \frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}\mathcal{x}\partial_{32}\partial_{31}(v) -$

$\sigma_2\left(\frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(2)}\partial_{31}(v)\right) + \frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\partial_{21}\partial_{31}(v)$

$= \frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}\mathcal{x}\partial_{32}\partial_{31}(v) + \frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\partial_{21}\partial_{31}(v)$

• $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\mathcal{Z}_{21}^{(3)}\mathcal{x}(v)\right)$; where $v \in \mathcal{D}_{12} \otimes \mathcal{D}_6 \otimes \mathcal{D}_0$

$= \sigma_2\left(\mathcal{Z}_{32}^{(3)}\mathcal{y}\mathcal{Z}_{21}^{(4)}\mathcal{x}(v) - \mathcal{Z}_{21}\mathcal{x}\mathcal{Z}_{21}^{(3)}\mathcal{x}\partial_{32}^{(3)}(v) + \mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{21}^{(4)}\mathcal{x}\partial_{32} - \right.$

$\left.\mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{32}\mathcal{y}\partial_{21}^{(4)}(v) + \mathcal{Z}_{32}^{(2)}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\partial_{21}^{(3)}(v)\right)$

$= \frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}\mathcal{x}\partial_{31}^{(2)}(v) - \frac{1}{6}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}\mathcal{x}\partial_{21}^{(2)}\partial_{32}^{(2)}(v) - \frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\partial_{21}^{(3)}\partial_{32}(v) +$

$\frac{1}{12}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}\mathcal{x}\partial_{21}\partial_{31}\partial_{32}(v) - \frac{1}{4}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\partial_{21}^{(3)}\partial_{32}(v) +$

$\frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\partial_{32}\partial_{21}^{(3)}(v)$

$= \frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}\mathcal{x}\partial_{31}^{(2)}(v) - \frac{1}{6}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}\mathcal{x}\partial_{21}^{(2)}\partial_{32}^{(2)}(v) - \frac{1}{4}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\partial_{21}^{(3)}\partial_{32}(v) +$

$\frac{1}{12}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{21}^{(2)}\mathcal{x}\partial_{21}\partial_{32}\partial_{31}(v) + \frac{1}{3}\mathcal{Z}_{32}\mathcal{y}\mathcal{Z}_{31}\mathcal{z}\partial_{21}^{(2)}\partial_{31}(v)$

And

$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(\tfrac{1}{6}Z_{32}yZ_{31}zZ_{21}x\partial_{21}\partial_{31}(v)-\right.$$
$$\left.\tfrac{1}{12}\,Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(2)}\partial_{32}(v)\right)$$
$$=\sigma_2\left(-\tfrac{1}{6}Z_{21}xZ_{21}x\partial_{32}^{(2)}\partial_{21}\partial_{31}(v)\right)+\tfrac{1}{6}\,Z_{32}yZ_{21}^{(2)}x\partial_{32}\partial_{21}\partial_{31}(v)-$$
$$\sigma_2\left(\tfrac{1}{6}Z_{32}yZ_{32}y\partial_{21}^{(2)}\partial_{21}\partial_{31}(v)\right)+\tfrac{1}{6}\,Z_{32}yZ_{31}z\partial_{21}\partial_{21}\partial_{31}(v)+$$
$$\sigma_2\left(\tfrac{1}{12}Z_{21}xZ_{21}x\partial_{32}^{(2)}\partial_{21}^{(2)}\partial_{32}(v)\right)-\tfrac{1}{12}\,Z_{32}yZ_{21}^{(2)}x\partial_{32}\partial_{21}^{(2)}\partial_{32}(v)+$$
$$\sigma_2\left(\tfrac{1}{12}Z_{32}yZ_{32}y\partial_{21}^{(2)}\partial_{21}^{(2)}\partial_{32}(v)\right)-\tfrac{1}{12}\,Z_{32}yZ_{31}z\partial_{21}\partial_{21}^{(2)}\partial_{32}(v)$$
$$=\tfrac{1}{6}\,Z_{32}yZ_{21}^{(2)}x\partial_{21}\partial_{32}\partial_{31}(v)+\tfrac{2}{6}\,Z_{32}yZ_{21}^{(2)}x\partial_{31}^{(2)}(v)+$$
$$\tfrac{2}{6}\,Z_{32}yZ_{31}z\partial_{21}^{(2)}\partial_{31}(v)-\tfrac{2}{12}\,Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}^{(2)}(v)-$$
$$\tfrac{1}{12}\,Z_{32}yZ_{21}^{(2)}x\partial_{21}\partial_{32}\partial_{31}(v)-\tfrac{3}{12}\,Z_{32}yZ_{31}z\partial_{21}^{(3)}\partial_{32}(v)$$
$$=\tfrac{1}{3}Z_{32}yZ_{21}^{(2)}x\partial_{31}^{(2)}(v)-\tfrac{1}{6}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}^{(2)}(v)-\tfrac{1}{4}Z_{32}yZ_{31}z\partial_{21}^{(3)}\partial_{32}(v)+$$
$$\tfrac{1}{12}Z_{32}yZ_{21}^{(2)}x\partial_{21}\partial_{32}\partial_{31}(v)+\tfrac{1}{3}Z_{32}yZ_{31}z\partial_{21}^{(2)}\partial_{31}(v)$$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(Z_{32}^{(2)}yZ_{31}zZ_{21}^{(4)}x(v)\right)$; where $v\in\mathcal{D}_{13}\otimes\mathcal{D}_5\otimes\mathcal{D}_0$

$$=\sigma_2\left(2\,Z_{32}^{(3)}yZ_{21}^{(5)}x(v)-Z_{21}xZ_{21}^{(4)}x\partial_{32}^{(3)}(v)+Z_{32}^{(2)}yZ_{21}^{(5)}x\partial_{32}-\right.$$
$$\left.Z_{32}^{(2)}yZ_{32}y\partial_{21}^{(5)}(v)+Z_{32}^{(2)}yZ_{31}z\partial_{21}^{(4)}(v)\right)$$
$$=\tfrac{2}{9}Z_{32}yZ_{21}^{(2)}x\partial_{21}\partial_{31}^{(2)}(v)-\tfrac{14}{90}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}^{(2)}(v)-$$
$$\tfrac{4}{9}Z_{32}yZ_{31}z\partial_{21}^{(4)}\partial_{32}(v)+\tfrac{1}{30}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}\partial_{32}(v)-$$
$$\tfrac{1}{5}Z_{32}yZ_{31}z\partial_{21}^{(4)}\partial_{32}(v)+\tfrac{1}{3}Z_{32}yZ_{31}z\partial_{32}\partial_{21}^{(4)}(v)$$
$$=\tfrac{2}{9}Z_{32}yZ_{21}^{(2)}x\partial_{21}\partial_{31}^{(2)}(v)-\tfrac{7}{45}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}^{(2)}(v)-$$
$$\tfrac{14}{45}Z_{32}yZ_{31}z\partial_{21}^{(4)}\partial_{32}(v)+\tfrac{1}{30}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}\partial_{31}(v)+$$
$$\tfrac{1}{3}Z_{32}yZ_{31}z\partial_{21}^{(3)}\partial_{31}(v)$$

And

$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(\frac{1}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(2)}\partial_{31}(v) - \frac{7}{90}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{32}(v)\right)$$

$$= \sigma_2\left(-\frac{1}{9}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(2)}\partial_{31}(v)\right) + \frac{1}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(2)}\partial_{31}(v) - \sigma_2\left(\frac{1}{9}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(2)}\partial_{31}(v)\right) + \frac{1}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(2)}\partial_{31}(v) + \sigma_2\left(\frac{7}{90}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(3)}\partial_{32}(v)\right) - \frac{7}{90}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(3)}\partial_{32}(v) + \sigma_2\left(\frac{7}{90}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(3)}\partial_{32}(v)\right) - \frac{7}{90}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(3)}\partial_{32}(v)$$

$$= \frac{1}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}\partial_{31}(v) + \frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{31}^{(2)}(v) + \frac{3}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(3)}\partial_{31}(v) - \frac{14}{90}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}^{(2)}(v) - \frac{7}{90}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}\partial_{31}(v) - \frac{28}{90}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(4)}\partial_{32}(v)$$

$$= \frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}\partial_{31}^{(2)}(v) - \frac{7}{45}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}^{(2)}(v) - \frac{14}{45}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(4)}\partial_{32}(v) + \frac{1}{30}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{32}\partial_{31}(v) + \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(3)}\partial_{31}(v)$$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(5)}x(v)\right)$; where $v \in \mathcal{D}_{14} \otimes \mathcal{D}_4 \otimes \mathcal{D}_0$

$$= \sigma_2\left(3\,\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(6)}x(v) - \mathcal{Z}_{21}x\mathcal{Z}_{21}^{(5)}x\partial_{32}^{(3)}(v) + \mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(6)}x\partial_{32} - \mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{32}y\partial_{21}^{(6)}(v) + \mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{31}z\partial_{21}^{(5)}(v)\right)$$

$$= \frac{3}{18}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}^{(2)}(v) - \frac{6}{45}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}^{(2)}(v) - \frac{3}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{32}(v) + \frac{1}{60}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}\partial_{32}(v) - \frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{32}(v) + \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{32}\partial_{21}^{(5)}(v)$$

$$= \frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}^{(2)}(v) - \frac{2}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}^{(2)}(v) - \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{32}(v) + \frac{1}{60}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}\partial_{31}(v) + \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(4)}\partial_{31}(v)$$

And

$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(\frac{1}{12}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(3)}\partial_{31}(v) - \frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(4)}\partial_{32}(v)\right)$$

$$= \sigma_2\left(-\frac{1}{12}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(3)}\partial_{31}(v)\right) + \frac{1}{12}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(3)}\partial_{31}(v) - \sigma_2\left(\frac{1}{12}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(3)}\partial_{31}(v)\right) + \frac{1}{12}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(3)}\partial_{31}(v) + \sigma_2\left(\frac{1}{15}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(4)}\partial_{32}(v)\right) - \frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(4)}\partial_{32}(v) + \sigma_2\left(\frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(4)}\partial_{32}(v)\right) - \frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(4)}\partial_{32}(v)$$

$$= \frac{1}{12}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}\partial_{31}(v) + \frac{2}{12}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}^{(2)}(v) + \frac{4}{12}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(4)}\partial_{31}(v) - \frac{2}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}^{(2)}(v) - \frac{1}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}\partial_{31}(v) - \frac{5}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{32}(v)$$

$$= \frac{1}{6}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(2)}\partial_{31}^{(2)}(v) - \frac{2}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}^{(2)}(v) - \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{32}(v) + \frac{1}{60}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{32}\partial_{31}(v) + \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(4)}\partial_{31}(v)$$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(6)}x(v)\right)$; where $v \in \mathcal{D}_{15} \otimes \mathcal{D}_3 \otimes \mathcal{D}_0$

$$= \sigma_2\left(4\,\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(7)}x(v) - \mathcal{Z}_{21}x\mathcal{Z}_{21}^{(6)}x\partial_{32}^{(3)}(v) + \mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(7)}x\partial_{32} - \mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{32}y\partial_{21}^{(7)}(v) + \mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{31}z\partial_{21}^{(6)}(v)\right)$$

$$= \frac{4}{30}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}^{(2)}(v) - \frac{4}{35}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}^{(2)}(v) - \frac{8}{15}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{32}(v) + \frac{1}{105}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}\partial_{32}(v) - \frac{1}{7}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{32}(v) + \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{32}\partial_{21}^{(6)}(v)$$

$$= \frac{2}{15}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}^{(2)}(v) - \frac{4}{35}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}^{(2)}(v) - \frac{12}{35}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{32}(v) + \frac{1}{105}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}\partial_{31}(v) + \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(5)}\partial_{31}(v)$$

And

$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(\frac{1}{15}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(4)}\partial_{31}(v)-\frac{2}{35}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(5)}\partial_{32}(v)\right)$$

$$=\sigma_2\left(-\frac{1}{15}Z_{21}xZ_{21}x\partial_{32}^{(2)}\partial_{21}^{(4)}\partial_{31}(v)\right)+\frac{1}{15}Z_{32}yZ_{21}^{(2)}x\partial_{32}\partial_{21}^{(4)}\partial_{31}(v)-$$
$$\sigma_2\left(\frac{1}{15}Z_{32}yZ_{32}y\partial_{21}^{(2)}\partial_{21}^{(4)}\partial_{31}(v)\right)+\frac{1}{15}Z_{32}yZ_{31}z\partial_{21}\partial_{21}^{(4)}\partial_{31}(v)+$$
$$\sigma_2\left(\frac{2}{35}Z_{21}xZ_{21}x\partial_{32}^{(2)}\partial_{21}^{(5)}\partial_{32}(v)\right)-\frac{2}{35}Z_{32}yZ_{21}^{(2)}x\partial_{32}\partial_{21}^{(5)}\partial_{32}(v)+$$
$$\sigma_2\left(\frac{2}{35}Z_{32}yZ_{32}y\partial_{21}^{(2)}\partial_{21}^{(5)}\partial_{32}(v)\right)-\frac{1}{15}Z_{32}yZ_{31}z\partial_{21}\partial_{21}^{(5)}\partial_{32}(v)$$

$$=\frac{1}{15}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}\partial_{31}(v)+\frac{2}{15}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}^{(2)}(v)+$$
$$\frac{5}{15}Z_{32}yZ_{31}z\partial_{21}^{(5)}\partial_{31}(v)-\frac{4}{35}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}^{(2)}(v)-$$
$$\frac{2}{35}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}\partial_{31}(v)-\frac{12}{35}Z_{32}yZ_{31}z\partial_{21}^{(6)}\partial_{32}(v)$$

$$=\frac{2}{15}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(3)}\partial_{31}^{(2)}(v)-\frac{4}{35}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}^{(2)}(v)-$$
$$\frac{12}{35}Z_{32}yZ_{31}z\partial_{21}^{(6)}\partial_{32}(v)+\frac{1}{105}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(4)}\partial_{32}\partial_{31}(v)+\frac{1}{3}Z_{32}yZ_{31}z\partial_{21}^{(5)}\partial_{31}(v)$$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(Z_{32}^{(2)}yZ_{31}zZ_{21}^{(7)}x(v)\right)$; where $v\in\mathcal{D}_{16}\otimes\mathcal{D}_2\otimes\mathcal{D}_0$

$$=\sigma_2\left(5\,Z_{32}^{(3)}yZ_{21}^{(8)}x(v)\right)-Z_{21}xZ_{21}^{(7)}x\partial_{32}^{(3)}(v)+\sigma_2\left(Z_{32}^{(2)}yZ_{21}^{(8)}x\partial_{32}-\right.$$
$$\left.Z_{32}^{(2)}yZ_{32}y\partial_{21}^{(8)}(v)+Z_{32}^{(2)}yZ_{31}z\partial_{21}^{(7)}(v)\right)$$

$$=\frac{5}{45}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v)-\frac{25}{252}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v)-$$
$$\frac{5}{9}Z_{32}yZ_{31}z\partial_{21}^{(7)}\partial_{32}(v)+\frac{1}{168}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}\partial_{32}(v)-$$
$$\frac{1}{8}Z_{32}yZ_{31}z\partial_{21}^{(7)}\partial_{32}(v)+\frac{1}{3}Z_{32}yZ_{31}z\partial_{32}\partial_{21}^{(7)}(v)$$

$$=\frac{1}{9}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v)-\frac{25}{252}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v)-$$
$$\frac{25}{72}Z_{32}yZ_{31}z\partial_{21}^{(7)}\partial_{32}(v)+\frac{1}{168}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v)+$$
$$\frac{1}{3}Z_{32}yZ_{31}z\partial_{21}^{(6)}\partial_{31}(v)$$

And

$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(\frac{1}{18}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(5)}\partial_{31}(v) - \frac{25}{504}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x\partial_{21}^{(6)}\partial_{32}(v)\right)$$

$$= \sigma_2\left(-\frac{1}{18}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(5)}\partial_{31}(v)\right) + \frac{1}{18}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(5)}\partial_{31}(v) - \sigma_2\left(\frac{1}{18}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(5)}\partial_{31}(v)\right) + \frac{1}{18}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(5)}\partial_{31}(v) + \sigma_2\left(\frac{25}{504}\mathcal{Z}_{21}x\mathcal{Z}_{21}x\partial_{32}^{(2)}\partial_{21}^{(6)}\partial_{32}(v)\right) - \frac{25}{504}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}\partial_{21}^{(6)}\partial_{32}(v) + \sigma_2\left(\frac{25}{504}\mathcal{Z}_{32}y\mathcal{Z}_{32}y\partial_{21}^{(2)}\partial_{21}^{(6)}\partial_{32}(v)\right) - \frac{25}{504}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}\partial_{21}^{(6)}\partial_{32}(v)$$

$$= \frac{1}{18}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v) + \frac{2}{18}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v) + \frac{6}{18}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{31}(v) - \frac{50}{504}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v) - \frac{25}{504}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v) - \frac{175}{504}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v)$$

$$= \frac{1}{9}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(4)}\partial_{31}^{(2)}(v) - \frac{25}{252}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}^{(2)}(v) - \frac{25}{72}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{32}(v) + \frac{1}{168}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{32}\partial_{31}(v) + \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(6)}\partial_{31}(v)$$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2} + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{31}z\mathcal{Z}_{21}^{(8)}x(v)\right)$; where $v \in \mathcal{D}_{17} \otimes \mathcal{D}_1 \otimes \mathcal{D}_0$

$$= \sigma_2\left(6\,\mathcal{Z}_{32}^{(3)}y\mathcal{Z}_{21}^{(9)}x(v)\right) - \mathcal{Z}_{21}x\mathcal{Z}_{21}^{(8)}x\partial_{32}^{(3)}(v) + \sigma_2\left(\mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{21}^{(9)}x\partial_{32} - \mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{32}y\partial_{21}^{(9)}(v) + \mathcal{Z}_{32}^{(2)}y\mathcal{Z}_{31}z\partial_{21}^{(8)}(v)\right)$$

$$= \frac{6}{63}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}^{(2)}(v) + \frac{6}{84}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}\partial_{31}(v) + \frac{1}{252}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}\partial_{32}(v) - \frac{1}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}\partial_{32}(v) + \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{32}\partial_{21}^{(8)}(v)$$

$$= \frac{2}{21}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}^{(2)}(v) + \frac{19}{252}\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}\partial_{31}(v) + \frac{2}{9}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(8)}\partial_{32}(v) + \frac{1}{3}\mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}^{(7)}\partial_{31}(v)$$

And

$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(\frac{1}{21}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(6)}\partial_{31}(v)+\frac{1}{36}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(7)}\partial_{32}(v)\right)$$

$$=\sigma_2\left(-\frac{1}{21}Z_{21}xZ_{21}x\partial_{32}^{(2)}\partial_{21}^{(6)}\partial_{31}(v)\right)+\frac{1}{21}Z_{32}yZ_{21}^{(2)}x\partial_{32}\partial_{21}^{(6)}\partial_{31}(v)-\sigma_2\left(\frac{1}{21}Z_{32}yZ_{32}y\partial_{21}^{(2)}\partial_{21}^{(6)}\partial_{31}(v)\right)+\frac{1}{21}Z_{32}yZ_{31}z\partial_{21}\partial_{21}^{(6)}\partial_{31}(v)-\sigma_2\left(\frac{1}{36}Z_{21}xZ_{21}x\partial_{32}^{(2)}\partial_{21}^{(7)}\partial_{32}(v)\right)+\frac{1}{36}Z_{32}yZ_{21}^{(2)}x\partial_{32}\partial_{21}^{(7)}\partial_{32}(v)-\sigma_2\left(\frac{1}{36}Z_{32}yZ_{32}y\partial_{21}^{(2)}\partial_{21}^{(7)}\partial_{32}(v)\right)+\frac{1}{36}Z_{32}yZ_{31}z\partial_{21}\partial_{21}^{(7)}\partial_{32}(v)$$

$$=\frac{1}{21}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}\partial_{31}(v)+\frac{2}{21}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}^{(2)}(v)+\frac{7}{21}Z_{32}yZ_{31}z\partial_{21}^{(7)}\partial_{31}(v)+\frac{1}{36}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}\partial_{31}(v)+\frac{8}{36}Z_{32}yZ_{31}z\partial_{21}^{(8)}\partial_{32}(v)$$

$$=\frac{2}{21}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(5)}\partial_{31}^{(2)}(v)+\frac{19}{252}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(6)}\partial_{32}\partial_{31}(v)+\frac{2}{9}Z_{32}yZ_{31}z\partial_{21}^{(8)}\partial_{32}(v)+\frac{1}{3}Z_{32}yZ_{31}z\partial_{21}^{(7)}\partial_{31}(v)$$

- $\left(\delta_{\mathcal{M}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{M}_3\mathcal{M}_2}\right)\left(Z_{32}^{(2)}yZ_{31}zZ_{21}^{(9)}x(v)\right)$; where $v\in\mathcal{D}_{18}\otimes\mathcal{D}_0\otimes\mathcal{D}_0$

$$=\sigma_2\left(7\,Z_{32}^{(3)}yZ_{21}^{(10)}x(v)\right)-Z_{21}xZ_{21}^{(9)}x\partial_{32}^{(3)}(v)+Z_{32}^{(2)}yZ_{21}^{(10)}x\partial_{32}-\sigma_2\left(Z_{32}^{(2)}yZ_{32}y\partial_{21}^{(10)}(v)+Z_{32}^{(2)}yZ_{31}z\partial_{21}^{(9)}(v)\right)$$

$$=\frac{7}{84}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v)+\frac{1}{3}Z_{32}yZ_{31}z\partial_{32}\partial_{21}^{(9)}(v)$$

$$=\frac{1}{12}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v)+\frac{1}{3}Z_{32}yZ_{31}z\partial_{21}^{(8)}\partial_{31}(v)$$, and

$$\left(\delta_{\mathcal{L}_3\mathcal{L}_2}+\sigma_2\circ\delta_{\mathcal{L}_3\mathcal{M}_2}\right)\left(\frac{1}{24}Z_{32}yZ_{31}zZ_{21}x\partial_{21}^{(7)}\partial_{31}(v)\right)$$

$$=\sigma_2\left(-\frac{1}{24}Z_{21}xZ_{21}x\partial_{32}^{(2)}\partial_{21}^{(7)}\partial_{31}(v)\right)+\frac{1}{24}Z_{32}yZ_{21}^{(2)}x\partial_{32}\partial_{21}^{(7)}\partial_{31}(v)-\sigma_2\left(\frac{1}{24}Z_{32}yZ_{32}y\partial_{21}^{(2)}\partial_{21}^{(7)}\partial_{31}(v)\right)+\frac{1}{24}Z_{32}yZ_{31}z\partial_{21}\partial_{21}^{(7)}\partial_{31}(v)$$

$$=\frac{1}{12}Z_{32}yZ_{21}^{(2)}x\partial_{21}^{(6)}\partial_{31}^{(2)}(v)+\frac{1}{3}Z_{32}yZ_{31}z\partial_{21}^{(8)}\partial_{31}(v). \quad \blacksquare$$

Eventually, we define the boundary maps in the complex

$$0 \longrightarrow \mathcal{L}_3 \xrightarrow{\partial_3} \mathcal{L}_2 \xrightarrow{\partial_2} \mathcal{L}_1 \xrightarrow{\partial_1} \mathcal{L}_0 \; ; \qquad \ldots(3.3.4)$$

where ∂_1 is the operation of indicated polarization operators, ∂_1, ∂_2 and ∂_3 defined as follows:

- $\partial_1(\mathcal{Z}_{21}x(v)) = \partial_{21}(v)$; where $v \in \mathcal{D}_9 \otimes \mathcal{D}_6 \otimes \mathcal{D}_3$
- $\partial_1(\mathcal{Z}_{32}y(v)) = \partial_{32}(v)$; where $v \in \mathcal{D}_8 \otimes \mathcal{D}_8 \otimes \mathcal{D}_2$
- $\partial_2(\mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x(v)) = \frac{1}{2}\, \mathcal{Z}_{21}x\partial_{21}\partial_{32}(v) + \mathcal{Z}_{21}x\partial_{31}(v) - \mathcal{Z}_{32}y\partial_{21}^{(2)}(v)$; where $v \in \mathcal{D}_{10} \otimes \mathcal{D}_6 \otimes \mathcal{D}_2$
- $\partial_2(\mathcal{Z}_{32}y\mathcal{Z}_{31}z(v)) = \frac{1}{2}\, \mathcal{Z}_{32}y\partial_{32}\partial_{21}(v) - \mathcal{Z}_{21}x\partial_{32}^{(2)}(v) - \mathcal{Z}_{32}y\partial_{31}(v)$; where $v \in \mathcal{D}_9 \otimes \mathcal{D}_8 \otimes \mathcal{D}_1$
- $\partial_3(\mathcal{Z}_{32}y\mathcal{Z}_{31}z\mathcal{Z}_{21}x(v)) = \mathcal{Z}_{32}y\mathcal{Z}_{21}^{(2)}x\partial_{32}(v) + \mathcal{Z}_{32}y\mathcal{Z}_{31}z\partial_{21}(v)$; where $v \in \mathcal{D}_{10} \otimes \mathcal{D}_7 \otimes \mathcal{D}_1$

Theorem (3.3.5):

The complex (3.3.4) is exact and in characteristic-zero gives a resolution of $K_{(8,7,3)}\,(\mathcal{F})$.

Proof:

First, we prove the exactness of the complex

$$0 \longrightarrow \mathcal{L}_3 \xrightarrow{\partial_3} \mathcal{L}_2 \xrightarrow{\partial_2} \mathcal{L}_1$$

Since one component of the map ∂_3 is a diagonalization of $\mathcal{D}_2$ into $\mathcal{D}_1 \otimes \mathcal{D}_1$ it is clear that ∂_3 is injective. To prove the exactness at $\mathcal{L}_2$.

For this, we need to show that:

If $v \in \ker(\partial_2)$ then $\exists\, w \in \mathcal{L}_3$ such that $\partial_3(w) = v$

If $\partial_2(v) = 0$ then $\exists\, (a,b) \in \mathcal{L}_3 \oplus \mathcal{M}_3$ such that

$\delta(a,b) = (v,0) \in \mathcal{L}_2 \oplus \mathcal{M}_2$, but

$\delta(a,b) = \delta_{\mathcal{L}_3\mathcal{L}_2}(a) + \delta_{\mathcal{L}_3\mathcal{M}_2}(a) + \delta_{\mathcal{M}_2\mathcal{L}_2}(b) + \delta_{\mathcal{M}_3\mathcal{M}_2}(b)$. So we get

$\delta_{\mathcal{L}_3\mathcal{L}_2}(a) + \delta_{\mathcal{M}_3\mathcal{L}_2}(b) = v$ …(1),

and

$\delta_{\mathcal{L}_3\mathcal{M}_2}(a)+\delta_{\mathcal{M}_3\mathcal{M}_2}(b) = 0$ …(2)

Now if $w = a + \sigma_3(b)$ we can see that $\partial_3(w) = v$ in fact

$\partial_3(a) = \delta_{\mathcal{L}_3\mathcal{L}_2}(a) + \sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}(a)$, and

$\partial_3\big(\sigma_3(b)\big) = \delta_{\mathcal{M}_3\mathcal{L}_2}(b) + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}(b)$, so

$$\partial_3\big(a + \sigma_3(b)\big) = \partial_3(a) + \partial_3\big(\sigma_3(b)\big)$$

$$= \delta_{\mathcal{L}_3\mathcal{L}_2}(a)+\sigma_2 \circ \delta_{\mathcal{L}_3\mathcal{M}_2}(a) + \delta_{\mathcal{M}_2\mathcal{L}_2}(b) + \sigma_2 \circ \delta_{\mathcal{M}_3\mathcal{M}_2}(b)$$

$$= \delta_{\mathcal{L}_3\mathcal{L}_2}(a)+\delta_{\mathcal{M}_3\mathcal{L}_2}(b) + \sigma_2 \circ \Big((\delta_{\mathcal{L}_3\mathcal{M}_2}(a) + \delta_{\mathcal{M}_3\mathcal{M}_2}(b)\Big)$$

Hence from (1) and (2), we get $\partial_3(w) = v$; where $w = a + \partial_3(b)$.

This proves the exactness at $\mathcal{L}_2$.

As the same way we can prove the exactness at $\mathcal{L}_1$.

Eventually, from Theorem (1.2.7) we get the complex

$$0 \longrightarrow \mathcal{L}_3 \xrightarrow{\partial_3} \mathcal{L}_2 \xrightarrow{\partial_2} \mathcal{L}_1 \xrightarrow{\partial_1} \mathcal{L}_0 \longrightarrow \mathcal{K}_{(8,7,3)}(\mathcal{F}) \longrightarrow 0,$$

is exact. ■

3.4 Characteristic-zero resolution of Weyl module with mapping Cone in the case of partition (8,7,3)

This section illustrates the resolution of Weyl module for characteristic-zero in the case of partition (8,7,3) by using mapping Cone which enables us to get the results without depending on the resolution of Weyl module in characteristic-free for the same partition and prove it to be exact.

In this section before we study the resolution of Weyl module for characteristic-zero in isolation of characteristic-free, we need the mapping Cone [32]

Consider the following commute diagram

$$
\begin{array}{llllllll}
C_0: & C_{n-1} & \xrightarrow{d_{n-1}} & C_n & \xrightarrow{d_n} & C_{n+1} & \xrightarrow{d_{n+1}} & C_{n+2}\ \cdots \\
 & \downarrow f_{n-1} & & \downarrow f_n & & \downarrow f_{n+1} & & \downarrow f_{n+2} \\
D_0: & D_{n-1} & \xrightarrow{d'_{n-1}} & D_n & \xrightarrow{d'_n} & D_{n+1} & \xrightarrow{d'_{n+1}} & D_{n+2}\ \cdots
\end{array}
$$

If the rows sequence are exact and

$\partial_{n-1}: C_n \otimes D_{n-1} \longrightarrow C_{n+1} \otimes D_n$ defined by

$(\alpha,b) \mapsto (-d_n(\alpha), d'_{n-1}(b)\ t + f_n(\alpha))$ such hat $\partial_{n-1} \circ \partial_n = 0; \forall\ n \in \mathbb{Z}^+$

Then the sequence

$$C_{n-1} \xrightarrow{\partial_{n-1}} C_n \otimes D_{n-1} \xrightarrow{\partial_n} C_{n+1} \otimes D_n \xrightarrow{\partial_{n+1}} C_{n+2} \otimes D_{n+1} \xrightarrow{\partial_{n+2}} \ldots,$$

is exact.

Consider the complex of Lascoux in our partition (8,7,3) as the following diagram:

$$
\begin{array}{llcl}
0 \longrightarrow & \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_1\mathcal{F} & \xrightarrow{h_1} & \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_1\mathcal{F} \\
 & f_1 \downarrow & A & \downarrow g_1 \\
0 \longrightarrow & \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} & \xrightarrow{h_2} & \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} \\
 & f_2 \downarrow & B & \downarrow g_2 \\
0 \longrightarrow & \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} & \xrightarrow{h_3} & \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} \\
 & & & \downarrow d' \\
 & & & \mathcal{K}_{(8,7,3)}(\mathcal{F}) \\
 & & & \downarrow \\
 & & & 0
\end{array}
$$

Diagram (3.1)

Where $\hbar_1(v) = \partial_{21}(v)$; $v \in \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_1\mathcal{F}$

$f_1(v) = \partial_{32}(v)$; $v \in \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_1\mathcal{F}$

$\hbar_2(v) = \partial_{21}^{(2)}(v)$; $v \in \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_2\mathcal{F}$

$\hbar_3(v) = \partial_{21}(v)$; $v \in \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_3\mathcal{F}$ and

$g_2(v) = \partial_{32}(v)$; $v \in \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_2\mathcal{F}$

So we need to define g_1 which make the diagram A commute, i.e

$(\partial_{21}^{(2)}\partial_{32})(v) = (g_1 \circ \partial_{21})(v)$

From Capelli identities, we know that

$\partial_{21}^{(2)}\partial_{32} = \partial_{32}\partial_{21}^{(2)} - \partial_{21}\partial_{31}$ and $\partial_{21}\partial_{31} = \partial_{31}\partial_{21}$

Then

$$\partial_{21}^{(2)}\partial_{32} = \tfrac{1}{2}\partial_{32}\partial_{21}\partial_{21} - \partial_{21}\partial_{31}$$

$$= \tfrac{1}{2}\partial_{32}\partial_{21}\partial_{21} - \partial_{31}\partial_{21}$$

$$= \left(\tfrac{1}{2}\partial_{32}\partial_{21} - \partial_{31}\right)\partial_{21}$$

So we get $g_1(v) = \left(\tfrac{1}{2}\partial_{32}\partial_{21} - \partial_{31}\right)(v)$; $v \in \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_1\mathcal{F}$

To find f_2 which make the diagram B commute, i.e.

$(g_2 \circ \hbar_2)(v) = (\hbar_3 \circ f_2)(v)$

$\partial_{32}\partial_{21}^{(2)}(v) = (\partial_{21} \circ f_2)(v)$

$$\partial_{32}\partial_{21}^{(2)}(v) = \partial_{21}^{(2)}\partial_{32} + \partial_{21}\partial_{31}$$

$$= \tfrac{1}{2}\partial_{21}\partial_{21}\partial_{32} - \partial_{21}\partial_{31}$$

$$= \partial_{21}\left(\tfrac{1}{2}\partial_{21}\partial_{32} - \partial_{31}\right)$$

So we get $f_2(v) = \left(\tfrac{1}{2}\partial_{21}\partial_{32} - \partial_{31}\right)(v)$; $v \in \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_2\mathcal{F}$

Now if we use the mapping Cone to the following diagram

$$\begin{array}{ccc} 0 \longrightarrow \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_1\mathcal{F} & \xrightarrow{\hbar_1} & \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_1\mathcal{F} \\ \downarrow f_1 & & \downarrow g_1 \\ \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} & \xrightarrow{\hbar_2} & \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} \end{array}$$

We get the subcomplex

$$0 \longrightarrow \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_1\mathcal{F} \xrightarrow{\varphi_3} \begin{array}{c} \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_1\mathcal{F} \\ \oplus \\ \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} \end{array} \xrightarrow{\delta_1} \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_2\mathcal{F}$$

…(3.4.1)

where $\varphi_3(x) = (-\partial_{21}(x), \partial_{32}(x))$ and

$$\delta_1(x_1, x_2) = (\partial_{21}^{(2)}(x_2) + \left(\tfrac{1}{2}\partial_{32}\partial_{21} - \partial_{31}\right)(x_1)$$

Proposition (3.4.1):

$$\delta_1 \circ \varphi_3 = 0$$

Proof:

$$\delta_1 \circ \varphi_3(\mathscr{b}) = \delta_1(-\partial_{21}(\mathscr{b}), \partial_{32}(\mathscr{b}))$$

$$= \partial_{21}^{(2)}(\partial_{32}(\mathscr{b})) + \left(\tfrac{1}{2}\partial_{32}\partial_{21} - \partial_{31}\right)(-\partial_{21}(\mathscr{b}))$$

$$\delta_1 \circ \varphi_3(\mathscr{b}) = \left(\partial_{21}^{(2)}\partial_{32}\right)(\mathscr{b}) - \left(\tfrac{1}{2}\partial_{32}\partial_{21}\partial_{21}\right)(\mathscr{b}) + (\partial_{31}\partial_{21})(\mathscr{b})$$

$$= \left(\partial_{21}^{(2)}\partial_{32}\right)(\mathscr{b}) - \left(\partial_{32}\,\partial_{21}^{(2)}\right)(\mathscr{b}) + (\partial_{31}\partial_{21})(\mathscr{b})$$

But from Capelli identities we have

$\partial_{21}^{(2)}\,\partial_{32} = \partial_{32}\,\partial_{21}^{(2)} - \partial_{21}\,\partial_{31}$ and $\partial_{31}\,\partial_{21} = \partial_{21}\,\partial_{31}$

Then

$$\delta_1 \circ \varphi_3(\mathscr{b}) = \left(\partial_{32}\,\partial_{21}^{(2)}\right)(\mathscr{b}) - (\partial_{21}\partial_{31})(\mathscr{b}) - \left(\partial_{32}\,\partial_{21}^{(2)}\right)(\mathscr{b}) + (\partial_{21}\partial_{31})(\mathscr{b})$$

$$= 0 \quad \blacksquare$$

By employing a mapping Cone again on the subcomplex (3.4.1) and the rest of diagram (3.1) we have

$$
\begin{array}{ccccccc}
0 \longrightarrow \mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_{7}\mathcal{F}\otimes\mathcal{D}_{1}\mathcal{F} & \xrightarrow{\varphi_3} & \begin{matrix}\mathcal{D}_{9}\mathcal{F}\otimes\mathcal{D}_{8}\mathcal{F}\otimes\mathcal{D}_{1}\mathcal{F}\\ \oplus \\ \mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_{6}\mathcal{F}\otimes\mathcal{D}_{2}\mathcal{F}\end{matrix} & \xrightarrow{\delta_1} & \mathcal{D}_{8}\mathcal{F}\otimes\mathcal{D}_{8}\mathcal{F}\otimes\mathcal{D}_{2}\mathcal{F} \\
 & & \delta_2 \Big\downarrow & \mathrm{C} & \Big\downarrow g_2 \\
 & & \mathcal{D}_{9}\mathcal{F}\otimes\mathcal{D}_{6}\mathcal{F}\otimes\mathcal{D}_{3}\mathcal{F} & \xrightarrow{h_3} & \mathcal{D}_{8}\mathcal{F}\otimes\mathcal{D}_{7}\mathcal{F}\otimes\mathcal{D}_{3}\mathcal{F} \\
 & & & & \Big\downarrow \\
 & & & & \mathcal{K}_{(8,7,3)}(\mathcal{F}) \\
 & & & & \Big\downarrow \\
 & & & & 0
\end{array}
$$

Diagram (3.2)

Now we define

$$
\delta_2 : \begin{matrix}\mathcal{D}_{9}\mathcal{F}\otimes\mathcal{D}_{8}\mathcal{F}\otimes\mathcal{D}_{1}\mathcal{F}\\ \oplus \\ \mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_{6}\mathcal{F}\otimes\mathcal{D}_{2}\mathcal{F}\end{matrix} \longrightarrow \mathcal{D}_{9}\mathcal{F}\otimes\mathcal{D}_{6}\mathcal{F}\otimes\mathcal{D}_{3}\mathcal{F} \quad \text{by}
$$

$$
\delta_2(a,b) = \partial_{32}^{(2)}(a) + \left(\tfrac{1}{2}\partial_{21}\partial_{32} + \partial_{31}\right)(b)
$$

Proposition (3.4.2):

The diagram C is commute.

Proof:

To prove the diagram is commute it is sufficient to prove that

$$
(g_2 \circ \delta_1)(a,b) = (h_3 \circ \delta_2)(a,b)
$$

$$
\begin{aligned}
(g_2 \circ \delta_1)(a,b) &= g_2\left(\partial_{21}^{(2)}(b) + \left(\tfrac{1}{2}\partial_{32}\partial_{21} - \partial_{31}\right)(a)\right) \\
&= \partial_{32}\left(\partial_{21}^{(2)}(b) + \left(\tfrac{1}{2}\partial_{32}\partial_{21} - \partial_{31}\right)(a)\right) \\
&= \left(\partial_{32}\,\partial_{21}^{(2)}\right)(b) + \left(\tfrac{1}{2}\partial_{32}\partial_{32}\partial_{21} - \partial_{32}\partial_{31}\right)(a) \\
&= \left(\partial_{32}\,\partial_{21}^{(2)}\right)(b) + \left(\partial_{32}^{(2)}\partial_{21} - \partial_{32}\partial_{31}\right)(a)
\end{aligned}
$$

But from Capelli identities we have

$\partial_{32}^{(2)}\partial_{21} = \partial_{21}\,\partial_{32}^{(2)} + \partial_{32}\partial_{31}$ and $\partial_{32}\,\partial_{21}^{(2)} = \partial_{21}^{(2)}\,\partial_{32} + \partial_{21}\,\partial_{31}$

So we get

$$\begin{aligned}(g_2 \circ \delta_1)(a, b) &= \left(\partial_{21}^{(2)}\,\partial_{32} + \partial_{21}\,\partial_{31}\right)(b) + \left(\partial_{21}\,\partial_{32}^{(2)} + \partial_{32}\partial_{31} - \partial_{32}\partial_{31}\right)(a)\\ &= \left(\tfrac{1}{2}\,\partial_{21}\,\partial_{21}\,\partial_{32} + \partial_{21}\,\partial_{31}\right)(b) + \left(\partial_{21}\,\partial_{32}^{(2)}\right)(a)\\ &= \partial_{21}\left[\left(\tfrac{1}{2}\,\partial_{21}\,\partial_{32} + \partial_{31}\right)(b) + \partial_{32}^{(2)}(a)\right]\\ &= (h_3 \circ \delta_2)(a, b) \quad \blacksquare\end{aligned}$$

Hence from the mapping Cone, we have the following complex

$$0 \longrightarrow \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_1\mathcal{F} \xrightarrow{\varphi_3} \begin{matrix}\mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_1\mathcal{F}\\ \oplus\\ \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_2\mathcal{F}\end{matrix} \xrightarrow{\varphi_2} \begin{matrix}\mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_2\mathcal{F}\\ \oplus\\ \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_3\mathcal{F}\end{matrix} \xrightarrow{\varphi_1} \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} \xrightarrow{d'_{(8,7,3)}(\mathcal{F})} \mathcal{K}_{(8,7,3)}(\mathcal{F}) \longrightarrow 0$$

where

$$\varphi_2(a, b) = (-\,\delta_1(a, b), \delta_2(a, b))$$

$$= \left(-\,\partial_{21}^{(2)}(b) - \left(\tfrac{1}{2}\,\partial_{32}\,\partial_{21} - \partial_{31}\right)(a), \partial_{32}^{(2)}(a) + \left(\tfrac{1}{2}\,\partial_{21}\,\partial_{32} + \partial_{31}\right)(b)\right)$$

$$\varphi_1(a, b) = \partial_{32}(a) + \partial_{21}(b)$$

Proposition (3.4.3):

$$\varphi_2 \circ \varphi_3 = 0$$

Proof:

$(\varphi_2 \circ \varphi_3)(a) = \varphi_2(-\,\partial_{21}(a), \partial_{32}(a))$; $a \in \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_1\mathcal{F}$

$$= \Big(\left(-\,\partial_{21}^{(2)}\,\partial_{32}\right)(a) + \left(\tfrac{1}{2}\,\partial_{32}\,\partial_{21}\,\partial_{21} - \partial_{31}\,\partial_{21}\right)(a),$$

$$\left(-\,\partial_{32}^{(2)}\,\partial_{21}\right)(a) + \left(\tfrac{1}{2}\,\partial_{21}\,\partial_{32}\,\partial_{32} + \partial_{31}\,\partial_{32}\right)(a)\Big)$$

$$= \Big(\left(-\,\partial_{21}^{(2)}\,\partial_{32}\right)(a) + \left(\partial_{32}\,\partial_{21}^{(2)} - \partial_{31}\,\partial_{21}\right)(a), \left(-\,\partial_{32}^{(2)}\,\partial_{21}\right)(a) +$$

$$\left(\partial_{21}\,\partial_{32}^{(2)} + \partial_{31}\,\partial_{32}\right)(a)\Big)$$

But from Capelli identities we have

$\partial_{32}\,\partial_{21}^{(2)} = \partial_{21}^{(2)}\,\partial_{32} + \partial_{21}\,\partial_{31}$, $\partial_{21}\,\partial_{32}^{(2)} = \partial_{32}^{(2)}\,\partial_{21} - \partial_{32}\,\partial_{31}$,

$\partial_{21}\,\partial_{31} = \partial_{31}\,\partial_{21}$ and $\partial_{32}\,\partial_{31} = \partial_{31}\,\partial_{32}$

Which implies that

$(\varphi_2 \circ \varphi_3)(a)$

$$= \Big(\Big(-\partial_{21}^{(2)}\,\partial_{32}\Big)(a) + \Big(\partial_{21}^{(2)}\,\partial_{32}\Big)(a) + (\partial_{21}\,\partial_{31})(a) - (\partial_{21}\,\partial_{31})(a),$$

$$\Big(-\partial_{32}^{(2)}\,\partial_{21}\Big)(a) + \Big(\partial_{32}^{(2)}\,\partial_{21}\Big)(a) - (\partial_{32}\,\partial_{31})(a) + (\partial_{32}\,\partial_{31})(a)\Big)$$

= (0,0) ■

Proposition (3.4.4):

$$\varphi_1 \circ \varphi_2 = 0$$

Proof:

$$(\varphi_1 \circ \varphi_2)(a,b) = \varphi_1\Big(-\partial_{21}^{(2)}(b) - \Big(\tfrac{1}{2}\,\partial_{32}\,\partial_{21} - \partial_{31}\Big)(a), \partial_{32}^{(2)}(a) + \Big(\tfrac{1}{2}\,\partial_{21}\,\partial_{32} + \partial_{31}\Big)(b)\Big)$$

$$= \Big(-\partial_{32}\,\partial_{21}^{(2)}\Big)(b) - \Big(\tfrac{1}{2}\,\partial_{32}\,\partial_{32}\,\partial_{21}\Big)(a) - (\partial_{32}\,\partial_{31})(a) + \Big(\partial_{21}\,\partial_{32}^{(2)}\Big)(a) + \Big(\tfrac{1}{2}\,\partial_{21}\,\partial_{21}\,\partial_{32}\Big)(b) + (\partial_{21}\,\partial_{31})(b)$$

$$(\varphi_1 \circ \varphi_2)(a,b) = \Big(-\partial_{32}\,\partial_{21}^{(2)}\Big)(b) - \Big(\partial_{32}^{(2)}\,\partial_{21}\Big)(a) - (\partial_{32}\,\partial_{31})(a) + \Big(\partial_{21}\,\partial_{32}^{(2)}\Big)(a) + \Big(\partial_{21}^{(2)}\,\partial_{32}\Big)(b) + (\partial_{21}\,\partial_{31})(b)$$

Again from Capelli identities we get

$(\varphi_1 \circ \varphi_2)(a,b) =$

$$-\partial_{21}^{(2)}\,\partial_{32}(b) - (\partial_{21}\,\partial_{31})(b) - \Big(\partial_{21}\,\partial_{32}^{(2)}\Big)(a) - (\partial_{32}\,\partial_{31})(a) +$$

$$(\partial_{32}\,\partial_{31})(a) + \Big(\partial_{21}\,\partial_{32}^{(2)}\Big)(a) + \Big(\partial_{21}^{(2)}\,\partial_{32}\Big)(b) + (\partial_{21}\,\partial_{31})(b)$$

= 0 ■

Finally, we present the following theorem which shows that the complex of Lascoux in the case of partition (8,7,3) is exact.

Theorem (3.4.5):

The complex

$$0 \longrightarrow \mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_7\mathcal{F}\otimes\mathcal{D}_1\mathcal{F} \xrightarrow{\varphi_3} \begin{array}{c}\mathcal{D}_9\mathcal{F}\otimes\mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_1\mathcal{F}\\ \oplus \\ \mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_6\mathcal{F}\otimes\mathcal{D}_2\mathcal{F}\end{array} \xrightarrow{\varphi_2} \begin{array}{c}\mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_2\mathcal{F}\\ \oplus \\ \mathcal{D}_9\mathcal{F}\otimes\mathcal{D}_6\mathcal{F}\otimes\mathcal{D}_3\mathcal{F}\end{array} \xrightarrow{\varphi_1} \mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_7\mathcal{F}\otimes\mathcal{D}_3\mathcal{F} \xrightarrow{d'_{(8,7,3)}(\mathcal{F})} \mathcal{K}_{(8,7,3)}(\mathcal{F}) \longrightarrow 0$$

is exact.

Proof:

Since the diagrams, A and B in a diagram (3.1) are commutes and each of the maps

$\hbar_1 : \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_1\mathcal{F} \longrightarrow \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_1\mathcal{F}$; where $\hbar_1(v) = \partial_{21}(v)$, and

$\hbar_2 : \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} \longrightarrow \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_2\mathcal{F}$; where $\hbar_2(v) = \partial_{21}^{(2)}(v)$,

are injective [15], then we have a commuting diagram with an exact row. But from Proposition (3.4.1) we have $\delta_1 \circ \varphi_3 = 0$ which implies that the mapping Cone conditions are satisfied and the complex

$$0 \longrightarrow \mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_7\mathcal{F}\otimes\mathcal{D}_1\mathcal{F} \xrightarrow{\varphi_3} \begin{array}{c}\mathcal{D}_9\mathcal{F}\otimes\mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_1\mathcal{F}\\ \oplus \\ \mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_6\mathcal{F}\otimes\mathcal{D}_2\mathcal{F}\end{array} \xrightarrow{\delta_1} \mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_2\mathcal{F}$$

is exact.

Now consider the diagram (3.2), since diagram C is commute and

$\hbar_3 : \mathcal{D}_9\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} \longrightarrow \mathcal{D}_8\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_3\mathcal{F}$; where $\hbar_3(v) = \partial_{21}(v)$ is injective [18], so we have diagram (3.2) commute with exact rows. But $\varphi_2 \circ \varphi_3 = 0$ (Proposition (3.4.3)) and $\varphi_1 \circ \varphi_2 = 0$ (Proposition (3.4.4)) then again the mapping Cone conditions are satisfied, so we get the complex

$$0 \longrightarrow \mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_7\mathcal{F}\otimes\mathcal{D}_1\mathcal{F} \xrightarrow{\varphi_3} \begin{array}{c}\mathcal{D}_9\mathcal{F}\otimes\mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_1\mathcal{F}\\ \oplus \\ \mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_6\mathcal{F}\otimes\mathcal{D}_2\mathcal{F}\end{array} \xrightarrow{\varphi_2} \begin{array}{c}\mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_2\mathcal{F}\\ \oplus \\ \mathcal{D}_9\mathcal{F}\otimes\mathcal{D}_6\mathcal{F}\otimes\mathcal{D}_3\mathcal{F}\end{array} \xrightarrow{\varphi_1} \mathcal{D}_8\mathcal{F}\otimes\mathcal{D}_7\mathcal{F}\otimes\mathcal{D}_3\mathcal{F} \xrightarrow{d'_{(8,7,3)}(\mathcal{F})} \mathcal{K}_{(8,7,3)}(\mathcal{F}) \longrightarrow 0$$

is exact. ■

References

[1] N.T. Abdul Razak, “The reduction of resolution of Weyl module from characteristic-free to Lascoux resolution in case (6,5,3)”, M.Sc. Thesis, Mustansiriyah University, 2016.

[2] K. Akin, D.A. Buchsbaum and J.Weymen, “Schur functors and Schur complexes”, Adv. Math., Vol.44, pp.207-278, 1982.

[3] K. Akin and D.A. Buchsbaum, “Characteristic-free representation theory of the general linear group”, Adv. Math., Vol.58, pp.149-200, 1985.

[4] K. Akin, “On complexes relating the Jacobi-Trudi identity with the Brnstein-Gelfand-Gelfand resolution”, Journal of Algebra, Vol.177, pp.494-503, 1988.

[5] K. Akin and D.A. Buchsbaum “, Charasteristic-free representation theory of the general group II”, Homological Consideration, Adv. Math., Vol.72, 1988.

[6] K. Akin and D.A. Buchsbaum, “A note on the poincare resolution of the coordinate ring of the Grassmannian”, Journal of Algebra, Vol.152, No.2, pp.427-433, 1992.

[7] K. Akin and D.A. Buchsbaum, “Characteristic-free realizations of the Giambelli and Jacoby-Trudi derterminatal identites”, Proc. of K.I.T., Workshop on Algebra and Topology, Springer-Verlag, 1993.

[8] M. Artale and G. Boffi, “On a subcomplex of the Schur complex”, Journal of Algebra, Vol.176, pp.762-785, 1995.

[9] A.O. Azziz, “Resolution of Weyl module and Lascoux resolution in the case of the partition (3,3,2)”, M.Sc. Thesis, Mustansiriyah University, 2015.

[10] G. Boffi and D.A. Buchsbaum, “Threading homology through algebra: selected patterns”, Clorendon Press, Oxford, 2006.

[11] D.A. Buchsbaum, “Jacobi-Trudi and Giabelli identities in characteristic-free form ”, Contemporary Mathematics, Vol.88, 1989.

[12] D.A. Buchsbaum and G.C. Rota, “Projective resolution of Weyl modules”, Natl. Acad. Sci. USA, Vol.90, pp.2448-2450, 1993.

[13] D.A Buchsbaum and G.C. Rota, “A new constructruction in homological algebra”, Nati. Acad. Sci. USA, Vol.91, Issue 10, pp.4115-4119, 1994.

[14] D.A Buchsbaum, “Letter place methods and homotopy”, Birkhauser, pp.41-62, 1998.

[15] D.A. Buchsbaum and G.C. Rota, “Approaches to resolution of Weyl modules”, Adv. In Applied Math., Vol.27, pp.182-191, 2001.

[16] D.A. Buchsbaum, “Resolution of Weyl modules: the rota touch”, Algebraic Combinatorics and Computer Science, Springer-Verlag, Italian, Milano, pp.97-109, 2001.

[17] D.A. Buchsbaum and B.D. Taylor, “Homotopies for resolution of skew-hook shapes”, Adv. In Applied Math., Vol.30, pp.26-43, 2003.

[18] D.A. Buchsbaum, “A characteristic-free example of Lascoux resolution, and letter place methods for intertwining numbers”, European Journal of Gombinatorics, Vol.25, pp.1169-1179, 2004.

[19] C. De Concini, D. Eisenbud and C. Procesi, “Young diagrams and determintal varieties”, Inveent. Math., Vol.59, pp.129-165, 1980.

[20] J. Desarmenien, J.P.S. Kung and G.C. Rota, “Invariant theory”, Young Bitableaux and Combinatories, Adv. Math., Vol.27, pp.63-92, 1978.

[21] A. Eiichi, “Hopf algebra”, Hisae Kinoshita and Hiroko Tonaka, Cambridge University Pree USA, 1977.

[22] F.D. Grosshans, G.C. Rota and J.A. Stein, “Invariant theory and super algebra national science foundation”, No.69, 1987.

[23] H.R. Hassan, “Application of the characteristic-free resolution of Weyl module to the Lascoux resolution in the case (3,3,3)”, Ph.D. Thesis, Universitá di Roma "Tor Vergata", 2005.

[24] H.R. Hassan, “On the resolution of Wely module in the case of two-rowed skew-shape (p + t,q)/(t,0)”, Mustansiriyah J. Sci., Vol.21, No.5, pp.470-473, 2010.

[25] H.R. Hassan, “The reduction of Wely module from characteristic-free to Lascoux resolution in case (4,4,3)”, Ibn Al-Haitham J. for Pure and Applied Sci., Vol.25, No.3, pp.341-355, 2012.

[26] H.R. Hassan, “Complex of Lascoux in partition (4,4,4)”, Iraqi J. Sci., Vol.54, No.1, pp.170-173, 2013.

[27] A. Lascoux, “Polynomes symetriques”, Foncteurs de Schur et Grassmanniennes, Thése Université de Paris, VII, 1977.

[28] M.M. Mohammed, “Application of the characteristic-free of Weyl module to the Lascoux resolution in case (6,6,3)”, M.Sc. Thesis, Mustansiriyah University, 2016.

[29] N.M. Mustafa, “Resolution of Weyl module in the case of the partition (7,6,3)”, M.Sc. Thesis, Mustansiriyah University, 2017.

[30] G.C. Rota and J.A. Stein, “Standard basis in super simpleton algebra”, Natl. Acad.Sci. USA, Vol.86, pp.2521-2524, 1989.

[31] J.J. Rotman, “Introduction to homological algebra”, Academic Prees, INC, 1979.

[32] L.R. Vermani, “An elementary approach to homogical algebra”, Chapman and Hall/CRC, Monographs and Surveys in pure and Applied mathematics, Vol.130, 2003.

Suggestions for Future Works

Suggestions for future works

Based on the present work, the following topics are put forward for future works

1. Study the resolution of Weyl module of skew partition (8,7,3)/(2,1)

   (i.e. the resolution of Weyl module in our case with two-overlap).

2. Study the Lascoux resolution of the skew partition (8,7,3)/(3,1)

   (i.e. the resolution of Lascoux in our case with triple-overlap).

Published Papers

1- Haytham R. Hassan , Niran Sabah Jasim, Application of Weyl module in the case of two rows, Journal of physics:Conference series, IOP science, Vol.1003 (012051), pp.1-15, 2018.

2- Haytham R. Hassan , Niran Sabah Jasim, A complex of characteristic zero in the case of the partition (8,7,3), Science international-Lahore, Vol.30(4), pp.639-641, 2018.

3- Haytham R. Hassan , Niran Sabah Jasim, On free resolution of Weyl module and zero characteristic resolution in the case of partition (8,7,3), Baghdad science journal, Vol.15(4), pp.455-465, 2018.

4- Haytham R. Hassan , Niran Sabah Jasim, Characteristic zero resolution of Weyl module in the case of the partition (8,7,3), to appear.

المستخلص

ليكن $\mathcal{F}$ مقاس حر مُعرف على الحلقة الإبدالية ذات المحايد $\mathcal{R}$ و $\mathcal{D}_n\mathcal{F}$ القوى الجبرية المقسمة من الدرجة n .

بإستخدام تقنيات معقدة من النمط بار و جبر حروف المكان مع مشخصات كابلي، بوكسباوم درس تحلل مقاس وايل وبيّن أن الصف الاكبر من مقاسات-$GL_n(\mathcal{F})$ عُرفت خلالها جميع مقاسات وايل $\mathcal{K}_{\lambda/\mu}(\mathcal{F})$ حيث أن λ/μ شبه تجزئة و $\mathcal{K}_{\lambda/\mu}(\mathcal{F})$ هو صور لتطبيق وايل .$d'_{\lambda/\mu}(\mathcal{F})$.

في هذه الاطروحة ناقشنا تطبيق لتحلل مقاس وايل بصفين في حالة التجزئة (8,7) لإيجاد حدود ذلك التحلل وبرهان إنه تام. أيضا كتطبيق لتحلل مقاس وايل بثلاثة صفوف وجدنا حدود تحلل المميز– الحر في حالة التجزئة (8,7,3)، حدود معقدة لاسكو للتجزئة ذاتها و مخططات معقدة لاسكو أيضاً للتجزئة ذاتها. كتعميم للفكرة ذاتها التي إستخدمها بوكسباوم وجدنا الاختزال من تحلل المميز– الحر الى تحلل المميز– الصفري (تحلل لاسكو) مع إستخدام تطبيقات الحدود المستخدمة في المميز– الصفري في حالة التجزئة (8,7,3). وأخيراً، بإستخدام تطبيق كون ومخططاته درسنا تحلل المميز– الصفري (تحلل لاسكو) وبرهنا إنه تام دون الاعتماد على تطبيقات الحدود للتجزئة ذاتها.

جمهورية العراق

وزارة التعليم العالي والبحث العلمي

الجامعة المستنصرية

كلية العلوم

تطبيق التحلل للمميز-الحر لمقاس وايل على تحلل لاسكو في حالة التجزئة (8,7,3)

اطروحة

مقدمة الى مجلس كلية العلوم – الجامعة المستنصرية

وهي جزء من متطلبات نيل

درجة دكتوراه فلسفة علوم في الرياضيات

مـــــن قبـل

نيران صباح جاسم

بإشــــراف

الاستاذ المساعد الدكتور هيثم رزوقي حسن

١٤٤٠هـ ٢٠١٩م