JOURNAL OF INDUSTRIAL AND
MANAGEMENT OPTIMIZATION
doi:10.3934/jimo.2022004

OPTIMALITY AND DUALITY FOR E-DIFFERENTIABLE MULTIOBJECTIVE PROGRAMMING PROBLEMS INVOLVING E-TYPE I FUNCTIONS

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(Communicated by Xinmin Yang)

Abstract. In this paper, a new concept of generalized convexity is introduced for not necessarily differentiable multiobjective programming problems with E-differentiable functions. Namely, the concept of E-type I functions is defined for E-differentiable multiobjective programming problem. Based on the introduced concept of generalized convexity, the sufficiency of the so-called E-Karush–Kuhn–Tucker optimality conditions are established for a feasible point to be an E-efficient or a weakly E-efficient solution. Further, the so-called vector Mond-Weir E-dual problem is defined for the considered E-differentiable multiobjective programming problem and several E-duality theorems in the sense of Mond-Weir are derived under appropriate generalized E-type I functions.

1. **Introduction.** During the last few years, generalization of convexity plays a very important role in deriving sufficient conditions in optimization theory. Several classes of generalized convex functions have been defined for the purpose of weakening the limitations of convexity in mathematical programming. Optimality conditions and duality results for differentiable and nondifferentiable multiobjective programming problems have been studied extensively in the literature (see, for example, [4], [5], [6], [2], [7], [11], [13], [12], [10], [16], and others).

One of such important generalizations of the convexity notion is the concept of invexity introduced by Hanson [8] in the case of differentiable scalar optimization problems. After that, Hanson and Mond [9] introduced type I and type II invexities which have been generalized to pseudo type I and quasi type I functions by Reuda and Hanson [18]. Later, Kaul et al. [14] introduced pseudo quasi type I, quasi pseudo type I and strictly pseudo quasi type I functions. Aghezzaf and Hachimi [3] introduced vector-valued generalized type I functions. The definitions of E-convex set and E-convex function were introduced by Youness [19]. This kind of generalized convexity is based on the effect of an operator $E : R^n \to R^n$ on the sets and the domains of functions. Megahed et al. [15] presented the concept of an E-differentiable convex function which transforms a not necessarily differentiable convex function to a differentiable function based on the effect of an operator

2020 *Mathematics Subject Classification.* Primary: 90C26, 90C30, 26B25.
Key words and phrases. E-type I functions, generalized convexity, E-differentiable function, E-optimality conditions, E-Mond-Weir duality.

$E : R^n \to R^n$. Abdulaleem [1] introduced a new concept of generalized convexity as a generalization of the notion of E-differentiable E-convexity and the notion of differentiable invexity. Namely, he defined the concept of E-differentiable E-invexity in the case of (not necessarily differentiable) vector optimization problems with E-differentiable functions.

In this paper, new classes of nonconvex E-differentiable vector optimization problems is considered in which the involved functions are not necessarily differentiable. Namely, the concept of the so-called E-type I functions for E-differentiable vector optimization problems is introduced. Moreover, several classes of generalized vector-valued E-type I functions are also defined for E-differentiable functions as a generalization of the concept of differentiable vector-valued type I functions introduced by Hanson and Mond [9]. Further, the sufficient optimality conditions are derived for the considered E-differentiable vector optimization problem under appropriate E-type I and/or generalized E-type I hypotheses. Optimality results established in the paper are illustrated by examples of E-differentiable multiobjective optimization problems with E-type I functions or with generalized E-type I functions. Furthermore, the so-called vector E-dual problems in the sense of Mond-Weir is defined for E-differentiable vector dual problems. Then, several Mond-Weir E-duality results are established between the considered E-differentiable multicriteria optimization problem and its Mond-Weir vector dual problem also under appropriate generalized E-type I functions.

2. **Preliminaries and definitions.** Let R^n be the n-dimensional Euclidean space and R^n_+ be its nonnegative orthant. The following convention for equalities and inequalities will be used in the paper. For any vectors $x = (x_1, x_2, ..., x_n)^T$ and $y = (y_1, y_2, ..., y_n)^T$ in R^n, we define:

(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, \cdots, n$;
(ii) $x > y$ if and only if $x_i > y_i$ for all $i = 1, 2, \cdots, n$;
(iii) $x \geqq y$ if and only if $x_i \geqq y_i$ for all $i = 1, 2, \cdots, n$;
(iv) $x \geq y$ if and only if $x_i \geqq y_i$ for all $i = 1, 2, \cdots, n$ but $x \neq y$.

We now give the definition of an E-invex set introduced by Abdulaleem [1].

Definition 2.1. [1] Let $E : R^n \to R^n$. A set $M \subseteq R^n$ is said to be an E-invex set if and only if there exists a vector-valued function $\eta : M \times M \to R^n$ such that the relation

$$E(u) + \lambda\eta(E(x), E(u)) \in M$$

holds for all $x, u \in M$ and any $\lambda \in [0, 1]$.

We now recall for a common reader the definition of an E-differentiable function introduced by Megahed et al. [15].

Definition 2.2. [15] Let $E : R^n \to R^n$ and $f : M \to R$ be a (not necessarily) differentiable function at a given point $u \in M$. It is said that f is an E-differentiable function at u if and only if $f \circ E$ is a differentiable function at u (in the usual sense), that is,

$$(f \circ E)(x) = (f \circ E)(u) + \nabla(f \circ E)(u)(x - u) + \theta(u, x - u)\|x - u\|, \quad (1)$$

where $\theta(u, x - u) \to 0$ as $x \to u$.

Thus, $(\overline{x}, \tau, \mu)$ is a feasible solution for (MWD$_E$). If $(\overline{x}, \tau, \mu)$ is not a (weak) efficient solution for (MWD$_E$), then there exists a feasible solution $(\widetilde{x}, \widetilde{\tau}, \widetilde{\mu})$ of (MWD$_E$) such that $f(E(\widetilde{x})) < f(E(\overline{x}))$, which contradicts the Theorem 4.5. Hence $(\overline{x}, \tau, \mu)$ is a (weak) efficient solution for (MWD$_E$).

Moreover, we have, by Lemma 2.7, that $E(\overline{x}) \in \Omega$. Since $\overline{x} \in \Omega_E$ is a weak Pareto solution of the problem (VP$_E$), by Lemma 2.9, it follows that $E(\overline{x})$ is a weak E-Pareto solution in the problem (VP). Then, by the Mond-Weir strong duality between (VP$_E$) and (MWD$_E$), we conclude that also the Mond-Weir strong E-duality holds between the problems (VP) and (MWD$_E$). This means that if $E(\overline{x}) \in \Omega$ is a weak E-Pareto solution of the problem (VP), there exist $\tau \in R^p$, $\mu \in R^m$, $\mu \geqq 0$ such that $(\overline{x}, \tau, \mu)$ is a weakly efficient solution of a maximum type in the Mond-Weir dual problem (MWD$_E$). □

5. **Concluding remarks.** In this paper, new classes of E-differentiable nonconvex multiobjective programming problems has been considered. Namely, the concept of E-type I and/or generalized E-type I has been introduced for E-differentiable multiobjective programming problem. Further, the sufficiency of the so-called E-Karush-Kuhn-Tucker optimality conditions have been established for the considered E-differentiable vector optimization problems under (generalized) E-type I hypotheses. Furthermore, the so-called vector Mond-Weir E-dual problems have been formulated for such E-differentiable multiobjective programming problems. Then, various E-duality theorems between the considered E-differentiable vector optimization problem and its Mond-Weir vector dual problem have been proved under generalized E-type I hypotheses.

However, some interesting topics for further research remain. It would be of interest to investigate whether it is possible to prove similar results for other classes of E-differentiable vector optimization problems. We shall investigate these questions in subsequent papers.

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Received May 2021; revised August 2021; early access February 2022.

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