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# On a Subclass of Meromorphic Univalent Functions Involving Hypergeometric Function

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## ABSTRACT

The main object of the present paper is to, introduce the. class of meromorphic univalent functions Involving! hypergeomatrc function .We obtain~ some interesting geometric properties according to coefficient inequality , growth and distortion theorems , radii of starlikeness and convexity for the" functions in our subclass.

MSC.

## 1 . Introduction

Let  $\Sigma$  denoted be the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \tag{1}$$

which are analytic and meromorphic univalent in punctured unit disk  $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ .

-A function  $f \in \Sigma$  is meromorphic starlike of order  $\alpha$ , ( $0 \leq \alpha < 1$ ) if  $R\left(\frac{zf'(z)}{f(z)}\right) > \alpha$ , ( $z \in U^*$ ).

The class of all such function is denoted by  $\Sigma^*(\alpha)$ . A function  $f \in \Sigma$  is meromorphic convex of order  $\alpha$ , ( $0 \leq \alpha < 1$ )

-  $R\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha$ , ( $z \in U^*$ ). Let  $\Sigma_q$  be the class of function  $f \in \Sigma$ , if

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with  $a_k \geq 0$ . The subclass of  $\Sigma_q$  consisting of starlike functions of order  $\alpha$  is denoted by  $\Sigma_q^*(\alpha)$ , and convex functions of order  $\alpha$  by  $\Sigma_q^k(\alpha)$ . Various subclasses of  $\Sigma$  have been defined and studied by various authors see [1, 2, 3, 4, 5, 6, 7, 10, 11, 12].

For function  $f(z)$  given by (1) and  $g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k$ , we define the Hadamard product or (convolution)  $f$  and  $g$  by

$$f * g = \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^k.$$

For positive real parameters  $(\alpha_1, A_1, \dots, \alpha_\ell, A_\ell, \beta_1, B_1, \dots, \beta_p, B_p)$

$(\ell, p \in \mathbb{N} = \{1, 2, \dots\})$  such that

$$1 + \sum_{n=1}^p B_n - \sum_{n=1}^{\ell} A_n \geq 0, (z \in \mathbb{U}^*) \quad \text{The Wright generalized hypergeometric function}$$

$$= {}_{\ell} \Psi_p [(\alpha_t, A_t)_{1,\ell}, (\beta_t, B_t)_{1,p}; Z], \Psi_p [(\alpha_1, A_1), \dots, (\alpha_\ell, A_\ell); (\beta_1, B_1), \dots, (\beta_p, B_p); Z]^\ell$$

assigned by

$${}_{\ell} \Psi_p [(\alpha_t, A_t)_{1,\ell}, (\beta_t, B_t)_{1,p}; Z] = \sum_{n=0}^{\infty} \left\{ \prod_{t=1}^{\ell} \Gamma(\alpha_t + n A_t) \right\} \left\{ \prod_{t=1}^p \Gamma(\beta_t + n B_t) \right\}^{-1} \frac{z^n}{n!}.$$

If  $A_t = 1, (t = 1, 2, 3, \dots, \ell)$  and  $B_t = 1, (t = 1, 2, 3, \dots, p)$ , then

$$\Omega_{\ell} \Psi_p [(\alpha_t, A_t)_{1,\ell}, (\beta_t, B_t)_{1,p}; z] \equiv F_p(\alpha_1, \dots, \alpha_\ell, \beta_1, \dots, \beta_p; z)$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_\ell)_n z^n}{(\beta_1)_n \dots (\beta_p)_n n!}$$

$(\ell \leq p + 1; \ell, p \in \mathbb{N}_0 = \mathbb{N} = \{0, 1, 2, 3, \dots\}; Z \in \mathbb{U})$ .

That is the generalized hypergeometric function (see [8]). Here  $(\alpha_k)$  is the Pochhammer symbol and  $\Omega = (\prod_{t=1}^{\ell} \Gamma(\alpha_t))^{-1} (\prod_{t=1}^p \Gamma(\beta_t))$ .

When assign the generalized hypergeometric function, we take a Linear operator

$$\mathbf{W} [(\alpha_1, A_1)_{1,\ell}, (\beta_t, B_t)_{1,p}]: \Sigma_q \longrightarrow \Sigma_q.$$

$$\mathbf{W} [(\alpha_t, A_t)_{1,\ell}, (\beta_t, B_t)_{1,p}] f(z) = z^{-1} \Omega_{\ell} \Psi_p [(\alpha_t, A_t)_{1,\ell}, (\beta_t, B_t)_{1,p}; z] * f(z) \quad (2)$$

for convenience, we denote  $\mathbf{W} [(\alpha_t, A_t)_{1,\ell}, (\beta_t, B_t)_{1,p}]$  by  $\mathbf{W} [\alpha_1]$ .

If  $f$  has the form (1) then we obtain

$$\mathbf{W} [\alpha_1] f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \sigma_k(\alpha_1) a_k z^k, \quad (3)$$

where

$$\sigma_k(\alpha_1) = \frac{\Omega \Gamma(\alpha_1 + A_1(k+1)) \dots \Gamma(\alpha_\ell + A_\ell(k+1))}{(k+1)! r(\beta_1 + B_1(k+1)) \dots \Gamma(\beta_\ell + B_\ell(k+1))}. \quad (4)$$

**Definition1.1:** A subclass of  $\Sigma_q$  by utilizing operator  $W[\alpha_1]$  we let  $V(\alpha,\eta)$  denote a subclass of  $\Sigma_q$  consisting of function in (1) satisfying the condition

$$\left| \frac{\frac{z(W(\alpha_1)f(z))''}{(W(\alpha_1)f(z))'} + 2}{\frac{z(W(\alpha_1)f(z))''}{(W(\alpha_1)f(z))'} + 2\alpha} \right| < \eta. \tag{5}$$

$0 < \alpha < 1, 0 < \eta \leq 1$  and  $A_t=1$  ( $t=1,2,3,\dots$ ),  $B_t=1$  ( $t=1,2,3,\dots$ ) Where

Now we must prove the Coefficient Inequality

**2.Coefficient Inequality**

**Theorem 2.1:**  $f$  is a function defined by (1) in the class  $V(\alpha,\eta)$ , if and only if

$$\sum_{k=1}^{\infty} |\sigma_k(\alpha_1)| [k(1 + \eta) + (1 + \eta(2\alpha - 1))] a_k \leq 2\eta(1 - \alpha). \tag{6}$$

*The result is sharp"*

**Proof :** Let the inequity (6) holds true and let  $|z|=1$  by (5). Then we get

$$\left| \frac{z(W(\alpha_1)f(z))''}{(W(\alpha_1)f(z))'} + 2 \right| - \eta \left| \frac{z(W(\alpha_1)f(z))''}{(W(\alpha_1)f(z))'} + 2\alpha \right| < 0,$$

$$|z(W(\alpha_1)f(z))'' + 2(W(\alpha_1)f(z))'| - \eta |z(W(\alpha_1)f(z))'' + 2\alpha(W(\alpha_1)f(z))'|,$$

and by utilizing (3) we have

$$\begin{aligned} (W(\alpha_1)f(z))' &= \frac{-1}{z^2} + \sum_{k=1}^{\infty} n \sigma_k(\alpha_1) a_k z^{k-1} , \\ (W(\alpha_1)f(z))'' &= \frac{+2}{z^3} + \sum_{k=1}^{\infty} k(k-1) \sigma_k(\alpha_1) a_k z^{k-2}, \\ z(w(\alpha_1)f(z))'' &= \frac{2}{z^2} + \sum_{k=1}^{\infty} k(k-1) \sigma_k(\alpha_1) a_k z^{k-1}, \\ &= \left| \frac{2}{z^2} + \sum_{k=1}^{\infty} k(k-1) \sigma_k(\alpha_1) a_k z^{k-1} - \frac{2}{z^2} + 2 \sum_{k=1}^{\infty} k \sigma_k(\alpha_1) a_k z^{k-1} \right| \\ &= \eta \left| \frac{2}{z^2} + \sum_{k=1}^{\infty} k(k-1) \sigma_k(\alpha_1) a_k z^{k-1} - \frac{2\alpha}{z^2} + 2\alpha \sum_{k=1}^{\infty} k \sigma_k(\alpha_1) a_k z^{k-1} \right| \\ &= |\sum_{k=1}^{\infty} k \sigma_k(\alpha_1) a_k z^{k-1} (k-1+2)| - \eta \left| \frac{2}{z^2} (1-\alpha) + \sum_{k=1}^{\infty} k \sigma_k(\alpha_1) a_k z^{k-1} (k-1+2\alpha) \right| \\ &\leq \sum_{k=1}^{\infty} k \sigma_k(\alpha_1) a_k |z|^{k-1} (k+1) \frac{-2\eta}{|z|^2} (1-\alpha) + \eta \sum_{k=1}^{\infty} k \sigma_k(\alpha_1) a_k |z|^{k-1} (k-1+2\alpha) \\ &\leq \sum_{k=1}^{\infty} |\sigma_k(\alpha_1)| [k(1 + \eta) + (1 + \eta(2\alpha - 1))] a_k - 2\eta(1 - \alpha) \leq 0. \end{aligned}$$

Therefore, by the maximum modules theorem we have  $f \in V(\alpha, \eta)$ ,

Conversely, suppose  $f \in V(\alpha, \eta)$ , then

$$\left| \frac{\frac{z(W(\alpha_1)f(z))''}{(W(\alpha_1)f(z))'} + 2}{\frac{z(W(\alpha_1)f(z))''}{(W(\alpha_1)f(z))'} + 2\alpha} \right| < \eta,$$

$$\left| \frac{\sum_{k=1}^{\infty} (k+1) |\sigma_k(\alpha_1)| a_k z^{k-1}}{\frac{2(1-\alpha)}{z^2} + \sum_{k=1}^{\infty} (k-1+2\alpha) |\sigma_k(\alpha_1)| a_k z^{k-1}} \right| < \eta.$$

Since  $|\operatorname{Re}(z)| \leq |z|$  for all  $z$ , we get

$$\operatorname{Re} \left\{ \frac{\sum_{k=1}^{\infty} (k+1) |\sigma_k(\alpha_1)| a_k z^{k-1}}{\frac{2(1-\alpha)}{z^2} + \sum_{k=1}^{\infty} (k-1+2\alpha) |\sigma_k(\alpha_1)| a_k z^{k-1}} \right\} < \eta \quad (7)$$

on the real axis when choosing the value of  $z$  thus the value of

$$\frac{z (W(\alpha_1) f(z))''}{(W(\alpha_1) f(z))'}$$

is real, therefore clearing the denominator of (7) and when  $z \rightarrow 1$  through real axis the result is sharp for the function

$$f_k(z) = \frac{1}{2} + |\sigma_k(\alpha_1)|^{-1} \times \frac{2\eta(1-\alpha)}{[k(1+\eta) + (1+\eta(2\alpha-1))]} z^k, k \geq 1 \quad (8)$$

**Corollary 2.1** : When  $f \in V(\alpha, \eta)$ , then

$$a_k \leq |\sigma_k(\alpha_1)|^{-1} \times \frac{2\eta(1-\alpha)}{[k(1+\eta) + (1+\eta(2\alpha-1))]},$$

where  $0 < \alpha < 1, 0 < \eta \leq 1$ .

### 3. Growth and Distortion Theorems

Distortion and growth Theorems property for the function  $f \in V(\alpha, \eta)$ , is given as follows :

**Theorem 3.1:** Let  $f$  be a function defined by (1) is in the class  $V(\alpha, \eta)$ .

Then for  $0 < |z| = r < 1$  we get

$$\frac{1}{r} - r |\sigma_1(\alpha_1)|^{-1} \times \frac{\eta(1-\alpha)}{(1+\alpha\eta)} \leq |f(z)| \leq \frac{1}{r} + r \frac{\eta(1-\alpha)}{(1+\alpha\eta)} |\sigma_1(\alpha_1)|^{-1},$$

equivalences for

$$f(z) = \frac{1}{z} + |\sigma_1(\alpha_1)|^{-1} \times \frac{\eta(1-\alpha)}{(1+\alpha\eta)} z.$$

**Proof:** Since  $f \in V(\alpha, \eta)$ , then we get by theorem 2.1, then the inequality

$$\sum_{k=1}^{\infty} \sigma_k(\alpha_1) [k(\eta+1) + (1+\eta(2\alpha-1))] a_k \leq 2\eta(1-\alpha).$$

Then

$$|f(z)| \leq \left| \frac{1}{z} \right| + \sum_{k=1}^{\infty} a_k |z|^k,$$

for  $0 < |z| = r < 1$ , we get

$$|f(z)| < \frac{1}{r} + r \sum_{k=1}^{\infty} a_k \leq \frac{1}{r} - |\sigma_1(\alpha_1)|^{-1} \times \frac{\eta(1-\alpha)}{(1+\alpha\eta)} r.$$

In addition to

$$|f(z)| \geq \left| \frac{1}{z} \right| - \sum_{k=1}^{\infty} a_k |z|^k \geq \frac{1}{r} - |\sigma_1(\alpha_1)|^{-1} \times \frac{\eta(1-\alpha)}{(1+\alpha\eta)} r, |z| = r.$$

**Theorem 3.2:** Let A function  $f$  defined by (1) in the class  $f \in V(\alpha, \eta)$ . Then

for  $0 < |z| = r < 1$  we get

$$\frac{1}{r^2} - |\sigma_1(\alpha_1)|^{-1} \times \frac{\eta(1-\alpha)}{(1+\alpha\eta)} \leq |f'(z)| \leq \frac{1}{r^2} + |\sigma_1(\alpha_1)|^{-1} \times \frac{\eta(1-\alpha)}{(1+\alpha\eta)}.$$

Equivalences for

$$f(z) = \frac{1}{z} + |\sigma_1(\alpha_1)|^{-1} \times \frac{\eta(1-\alpha)}{(1+\alpha\eta)} z.$$

**proof:** Form Theorem 2.1, we get

$$\sum_{k=1}^{\infty} |\sigma_k(\alpha_1)| [k(1+\eta) + (1+\eta(2\alpha-1))] a_k \leq 2\eta(1-\alpha).$$

Thus

$$|f'(z)| \leq \left| \frac{-1}{z^2} \right| + \sum_{k=1}^{\infty} k a_k |z|^{k-1},$$

for  $0 < r = |z| < 1$  we get

$$\begin{aligned} |f'(z)| &\leq \left| \frac{-1}{r^2} \right| + \sum_{k=1}^{\infty} k a_k \\ &\leq \frac{1}{r^2} + |\sigma_1(\alpha_1)|^{-1} \times \frac{\eta(1-\alpha)}{(1+\alpha\eta)}. \end{aligned}$$

And

$$\begin{aligned} |f'(z)| &\geq \left| \frac{-1}{z^2} \right| - \sum_{k=1}^{\infty} k a_k |z|^{k-1}, \\ &\geq \left| \frac{1}{r^2} \right| - \sum_{k=1}^{\infty} k a_k \\ &\geq \frac{1}{r^2} - |\sigma_1(\alpha_1)|^{-1} \times \frac{\eta(1-\alpha)}{(1+\alpha\eta)}. \end{aligned}$$

### 4.Hadamard product

**Theorem 4.1:** If the function  $g, f \in V(\alpha, \eta)$ . Then  $(f * g) \in V(\alpha, \eta)$ , for

$$\begin{aligned} f(z) &= \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k, \\ g(z) &= \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k, \end{aligned}$$

and

$$(f * g)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^k,$$

where

$$\delta = \frac{2\eta^2(1-\alpha)(k+1)}{2\eta^2(1-\alpha)(k+2\alpha-1) - |\sigma_n(\alpha_1)| [k(1+\eta) + (1+\eta(2\alpha-1))]^2}$$

**Proof:** Since  $f, g \in V(\alpha, \eta)$ , then by Theorem 2.1, we have

$$\sum_{k=1}^{\infty} |\sigma_k(\alpha_1)| \frac{[k(1+\eta) + (1+\eta(2\alpha-1))]}{2\eta(1-\alpha)} a_k \leq 1,$$

and

$$\sum_{k=1}^{\infty} |\sigma_k(\alpha_1)| \frac{[n(1+\eta) + (1+\eta(2\alpha-1))]}{2\eta(1-\alpha)} b_k \leq 1,$$

we must find the largest  $\delta$  such that

$$\sum_{k=1}^{\infty} |\sigma_k(\alpha_1)| \frac{[k(1 + \eta) + (1 + \eta(2\alpha - 1))]}{2\eta(1 - \alpha)} \sqrt{a_k b_k} \leq 1. \tag{9}$$

To prove the theorem it is over to show that

$$\begin{aligned} & |\sigma_k(\alpha_1)| \frac{[k(1 + \delta) + (1 + \delta(2\alpha - 1))]}{2\delta(1 - \alpha)} a_k b_k \\ & \leq |\sigma_k(\alpha_1)| \frac{[k(1 + \eta) + (1 + \eta(2\alpha - 1))]}{2\eta(1 - \alpha)} \sqrt{a_k b_k} , \end{aligned}$$

which is equivalent to

$$\sqrt{a_k b_k} \leq \frac{\delta [k(1 + \eta) + (1 + \eta(2\alpha - 1))]}{\beta [k(1 + \delta) + (1 + \delta(2\alpha - 1))]} .$$

From (9) we get

$$\sqrt{a_k b_k} \leq |\sigma_r(\alpha_1)| \frac{2\eta(1 - \alpha)}{[k(1 + \eta) + (1 + \eta(2\alpha - 1))]} .$$

We must proof that

$$|\sigma_k(\alpha_1)| \frac{2\eta(1 - \alpha)}{[k(1 + \eta) + (1 + \eta(2\alpha - 1))]} \leq \frac{\delta [k(1 + \eta) + (1 + \eta(2\alpha - 1))]}{\eta [k(1 + \delta) + (1 + \delta(2\alpha - 1))]} ,$$

which gives

$$\delta \leq \frac{2\eta^2(\alpha - 1)(k + 1)}{2\eta^2(1 - \alpha)(k + 2\alpha^{-1}) - |\sigma_k(\alpha_1)| [k(1 + \eta) + (1 + \eta(2\alpha - 1))]^2} .$$

**Theorem 4.2** : If the function  $f_i$  ( $i=1, 2$ ) defined by

$$f_i(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,i} z^k , (a_{k,i} \geq 0 , i = 1, 2)$$

be in the class  $V(\alpha, \eta)$ , then the function defined

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} (a^2_{k,1} + a^2_{k,2}) z^k ,$$

is in the class  $V(\alpha, \eta)$ , where

$$\beta = \frac{4\eta^2(\alpha - 1)(k + 1)}{4\eta^2(\alpha - 1)(k + 2\alpha^{-1}) - |\sigma_k(\alpha_1)| [k(1 + \eta) + (1 + \eta(2\alpha - 1))]^2}$$

**proof:** Since  $f_i \in V(\alpha, \eta)$ , ( $i= 1, 2$ ), then by Theorem 2.1, we get

$$\sum_{k=1}^{\infty} |\sigma_k(\alpha_1)| \frac{[k(1 + \eta) + (1 + \eta(2\alpha - 1))]}{2\eta(1 - \alpha)} a_{k,i} \leq 1, (i = 1, 2) .$$

**Hence**

$$\begin{aligned} & \sum_{k=1}^{\infty} (|\sigma_k(\alpha_1)| \frac{[k(1 + \eta) + (1 + \eta(2\alpha - 1))]}{2\eta(1 - \alpha)})^2 a_{k,i}^2 \\ & \leq (\sum_{k=1}^{\infty} |\sigma_k(\alpha_1)| \frac{[k(1 + \eta) + (1 + \eta(2\alpha - 1))]}{2\eta(1 - \alpha)} a_{k,i})^2 \leq 1, (i = 1, 2). \end{aligned}$$

Thus

$$\sum_{k=1}^{\infty} \frac{1}{2} |\sigma_k(\alpha_1)| \left( \frac{[k(1+\eta) + (1+\eta(2\alpha-1))]}{2\eta(1-\alpha)} \right)^2 (a_{k,1}^2 + a_{k,2}^2) \leq 1,$$

to prove the theorem we must find the largest  $\beta$  such that

$$\frac{[k(\beta+1) + (1+\beta(2\alpha-1))]}{\beta} \leq \frac{|\sigma_k(\alpha_1)| [k(1+\eta) + (1+\eta(2\alpha-1))]^2}{4\eta^2(1-\alpha)}, k \geq 1,$$

so that

$$\beta \leq \frac{4\eta^2(\alpha-1)(k+1)}{4\eta^2(\alpha-1)(k+2\alpha-1) - |\sigma_k(\alpha_1)| [k(1+\eta) + (1+\eta(2\alpha-1))]^2}.$$

**Theorem 4.3:** If  $f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \in V(\alpha, \eta)$ , and

$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k$  with  $|b_k| \leq 1$  is in the class  $V(\alpha, \eta)$   
 then  $f(z) * g(z) \in V(\alpha, \eta)$ .

**Proof :** By Theorem 2.1, we get

$$\sum_{k=1}^{\infty} |\sigma_k(\alpha_1)| [k(1+\eta) + (1+\eta(2\alpha-1))] a_k \leq 2\eta(1-\alpha).$$

Since

$$\begin{aligned} & \sum_{k=1}^{\infty} |\sigma_k(\alpha_1)| \left( \frac{[k(1+\eta) + (1+\eta(2\alpha-1))]}{2\eta(1-\alpha)} \right) |a_k b_k|, \\ &= \sum_{k=1}^{\infty} |\sigma_k(\alpha_1)| \left( \frac{[k(1+\eta) + (1+\eta(2\alpha-1))]}{2\eta(1-\alpha)} \right) a_k |b_k|, \\ &\leq \sum_{k=1}^{\infty} |\sigma_k(\alpha_1)| [k(1+\eta) + (1+\eta(2\alpha-1))] a_k \leq 1. \end{aligned}$$

Thus  $f(z) * g(z) \in V(\alpha, \eta)$ .

**Corollary 4.1:** If  $f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \in V(\alpha, \eta)$ , and  $g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k$  with  $0 \leq b_k \leq 1$  is in the  $V(\alpha, \eta)$ ,  
 then  $f(z) * g(z) \in V(\alpha, \eta)$ .

### 5. Radial of starlikeness and convexity

**Theorem 5.1:** Let  $f(z)$  be the function defined by (1) be in the subclass  $V(\alpha, \eta)$ . Then  $f$  is meromorphically starlike of order  $\delta$  ( $0 \leq \delta < 1$ ) in the disk  $|z| < r_1(\alpha, \eta, \delta)$ , where

$$r_1(\alpha, \eta, \delta) = \inf_k \left\{ |\sigma_k(\alpha_1)| \left( \frac{[k(1+\eta) + (1+\eta(2\alpha-1))]}{2\eta(k+2-\delta)(1-\delta\alpha)} \right)^{\frac{1}{k+1}} \right\}^{-1}$$

the result is sharp for the function given by (8).

**Proof:** We show that

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| \leq 1 - \delta,$$

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| = \left| \frac{\sum_{k=1}^{\infty} (k+1)a_k z^k}{z^{-1} + \sum_{k=1}^{\infty} a_k z^k} \right| \leq \frac{\sum_{k=1}^{\infty} (k+1)a_k |z|^{k+1}}{1 - \sum_{k=1}^{\infty} a_k |z|^{k+1}}.$$

This will be bounded by  $1 - \delta$ ,

$$\frac{\sum_{k=1}^{\infty} (k+1)a_k |z|^{k+1}}{1 - \sum_{k=1}^{\infty} a_k |z|^{k+1}} \leq 1 - \delta,$$

$$\sum_{k=1}^{\infty} (2+k-\delta)a_k |z|^{k+1} \leq 1 - \delta,$$

from Theorem 2.1, we get

$$\sum_{k=1}^{\infty} |\sigma_k(\alpha_1)| \left( \frac{[k(1+\eta) + (1+\eta(2\alpha-1))]}{2\eta(1-\alpha)} \right) a_k \leq 1.$$

Hence

$$|z|^{k+1} \leq |\sigma_k(\alpha_1)| \frac{[k(1+\eta) + (1+\eta(2\alpha-1))](1-\delta)}{2\eta(k+2-\delta)(1-\alpha)},$$

$$|z| \leq \left\{ |\sigma_k(\alpha_1)| \frac{[k(1+\eta) + (1+\eta(2\alpha-1))](1-\delta)}{2\eta(k+2-\delta)(1-\alpha)} \right\}^{\frac{1}{k+1}}.$$

**Theorem 5.2:** Let the function  $f(z)$  defined by (1) be in the subclass  $V(\alpha, \eta)$ .

Then  $f$  is meromorphically convex of order  $\varkappa$  ( $0 \leq \varkappa < 1$ ) in the disk

$|z| < r_2(\eta, \alpha, \varkappa)$ , where

$$r_2(\eta, \alpha, \varkappa) = \inf \left\{ |\sigma_k(\alpha_1)| \frac{[k(1+\eta) + (1+\eta(2\alpha-1))](1-\varkappa)}{2\eta(k+2-\varkappa)(1-\alpha)} \right\}^{\frac{1}{k+1}}.$$

The result is sharp for the function given by (7).

**Proof:** By utilizing the same way in the proof of theorem 5.1 we can get this

$$\left| \frac{zf''(z)}{f'(z)} + 2 \right| \leq 1 - \varkappa, \quad (0 \leq \varkappa < 1).$$

For  $|z| < r_2$  depending on the help of the Theorem 2.1, lead to confirmed of theorem 5.2. □

### References

- [1] M.K.Aouf and H. E.Darwish, Certain meromorphically starlike functions with positive and fixed second coefficient, *Turkish J.Math*, 21(3) (1997), 311-316.
- [2] W. G. Atshan and S. R. Kulkarni, Subclass of Meromorphic Function with Positive Coefficients defined by Ruscheweyh Derivative I, *J. Rajasthan Academy of Physical Sciences*, 6(2)(20), 129-140.



- [3] N. E. Cho ,S. H. Lee and S .Owa,Aclass of meromorphicunivalent functions with positive coefficients, *Kobe J.Math.*,4(1)(1987),43-50.
- [4] J. Clunie ,On meromorphicschlicht functions *J.London Math .Soc.* ,34(1959) ,215-216 .
- [5] J. Dziok ,G. Murugusundaramoorthyand J. Sokol, On certain class of meromorphic functions with positive coefficients, *ActaMath.Sci.Ser .BEngl. Ed.*, 32(4)(2012),1376-1390.
- [6] F. Ghanimand M. Darus, On acertain subclass of meromorphicunivalent functions with fixed second positive coefficients, *Surv.Math. Appl.* ,5(2010) ,49-60.
- [7] F. Ghanimand M. Darus, On class of HypergeometricMeromorphic with fixed second positive coefficients, *GenMath .* 1 7(4)(2009) , 1328.
- [8] F. Ghanim and M. Darus, On acertain subclass of meromorphicunivalent functions with fixed second positive coefficients,*Surv.Math .Appl.*,5 (2010) , 49 - 60 .
- [9] A. R. S. Juma and H. I. Dhayea,On a subclass of Meromorphic univalent function with Fixed Second Coefficient Involving Fox-Wright's HyPergeometricfunction,*Gen.Math*,vol 28 (2015),PP.30-41.
- [10] A. R. S. Juma and H . Zirar,On a Class of Meromorphic univalent function defined by Hypergetric Function, *Gen.Math*,vol 17 (2013),pp.63-73.
- [11] S. Kavitha ,S. Sivasubramanianand K. Muthunagai, Anew subclass of Meromorphicfunction with positive coefficients, *Bull. Math. Anal .Appl.*, 2 (3) (2010) , 109-121.
- [12] J. Ma, Extreme points and minimal outerareaproblem forMeromorphicunivalent functions, *J. Math.Anal.Appl.*,220(2)(1998),769-773.
- [13] N. Magesh, N. B. Gattianand S. Mayilvaganan, On certain subclasses of meromorphic functions with positive and fixed second coefficients involving the Liu-Srivastava linear operator, *ISRNMath. Anal.*, ArticleID 698307 (2012),11pages.
- [14] S. Owaand N. Pascu, Coefficient inequalities for certain classes ofMeromorphicallystarlike and meromorphicallyconvex functions,*JIPAM. J. Inequal. Pure Appl.Math.*,4(1)(2003),6.