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ORIGINAL PAPER

Findings of Fractional Iterative Differential Equations Involving First Order Derivative

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Abstract In this paper, we deal with the existence outcomes for a fractional iterative differential equation involving first order derivative in a Banach space. The appliance utilized in this study, is the non-expansive operator method and BGK (Browder–Ghode–Kirk) fixed point theorem. The fractional differential operator is taken in the sense of Riemann–Liouville.

Keywords Iterative differential equations · Fractional iterative differential equation · BGK fixed theorem

Mathematics Subject Classification 37N25 · 39A12 · 35A05 · 35E15

Introduction

In this study, we concern about mathematical models in biology. It refers to the growth of bacteria [1]. We generalize and extend the classical model based on the concept of the fractional calculus [2–7]. The best area of this concept is the fractional iterative differential equations. Various authors have studied particular kinds of differential equations called iterative differential equations [8–12]. Few authors had studied fractional iterative differential equations [13–17].

The existence of solutions, regarding this class, has different entryway, such as Schauder's fixed point, Picard's successive approximation, contraction principle, etc. The fundamental

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form of this equation is as follows:

$$z'(u) = z(z(u)) \tag{1}$$

then extended the Eq. (1) to the form

$$z'(u) = g(z(z(u))) \tag{2}$$

and

$$u'(s) = g(s, u(s), u(u(s))). \tag{3}$$

The fractional iterative differential equation is suggested in [14] for the equation

$$D^{\gamma} u(s) = g(s, u(s), u(u(s))).$$

Ibrahim and Darus [16] studied the existence of an infective disease processes

$$D^{\gamma} u(s) = h(s, u(s), u(\beta s), u(u(s))), \ \ \beta \in (0, 1]. \tag{4}$$

Recently, the authors investigated the existence of the general iterative fractional differential equation [17]

$$D^{\gamma} z(u) = g(u, z^{[1]}(u), z^{[2]}(u), \ldots, z^{[j]}(u)) \tag{5}$$

linked with $z(u_0) = z_0$, where $z^{[m]}(u) := z(z^{[m-1]}(u))$ indicates the $m - th$ iterate of self-mapping z, with $m = 1, 2, \ldots, j$. The stability of the above equation is investigated in [18].

In this note, we found an existence outcome for a fractional iterative differential equation involving first order derivative in a Banach space

$$\begin{aligned} D^{\gamma} v(s) &= f(s, v(s), v(\xi_1 s), v'(\xi_2 s)) \\ v(s_0) &= v_0. \end{aligned} \tag{6}$$

Consequently, as an application of (2), we establish the outcome of the equation

$$\begin{aligned} D^{\gamma} v(s) &= f(s, v(s), v(v(s)), v'(v(s))) \\ v(s_0) &= v_0. \end{aligned} \tag{7}$$

Preliminaries

In this section, we introduce some important concepts, which are useful in the sequel.

Definition 1 The fractional derivative in the **Caputo** sense is defined as:

$$\begin{gathered} D_a^{\beta} g(\tau) = \frac{1}{\Gamma(m-\beta)} \int_a^{\tau} \frac{g^{(m)}(\mu)}{(t-\mu)^{\gamma-m+1}} d\mu \\ (m-1) \leq \beta < m, \end{gathered} \tag{8}$$

where m is integer and β is real number.

Definition 2 The fractional derivative in the Riemann and Liouville sense is defined as:

$$\begin{gathered} Dt^{\beta} g(u) = \frac{1}{\Gamma(m-\beta)} \left[\frac{d}{du}\right]^m \int_a^u \frac{g(\mu)}{(u-\zeta)^{\beta-m+1}} d\mu \\ (m-1) \leq \beta < m, \end{gathered} \tag{9}$$

in which m is integer and β is real number.

Definition 3 The integral operator corresponding to the above operators is formulated by

$$D_a^\gamma g(s) = \int_a^s \frac{(s-\zeta)^{\gamma-1}}{\Gamma(\gamma)} g(\zeta)d\zeta \tag{10}$$

where $\gamma > 0$. Next, we present the descriptions and a fixed point theorem for non-expansive mappings which will be the main role in this note [20].

Definition 4 Let (Z, d) be a metric space and $H : Z \to Z$ is a mapping said to be an γ-contraction if there is $\gamma \in [0, 1)$ so that

$$d(Hz, Hw) \leq \gamma d(z, w), \forall z, w \in Z.$$

When $\gamma = 1$, it is said that the application is non-expansive. Let Q be a not empty of real normed linear space A and $H : Q \to Q$ be a map. In this case, H is not expansive if

$$\|Hz - Hw\| \to \|z - w\|, \quad \forall z, w \in Q.$$

Remark The fixed points of H can be approximated by Krasnoselskij sequence, demarcated as follows:

Let Q be a convex form subset of a normed linear space A and let $H : Q \to Q$ be an self mapping. In view of $s_0 \in Q$ and the real numbers $\xi \in [0, 1]$, s_m is sequence introduced by the formula

$$s_{m+1} = (1 - \xi)s_m + \xi H s_m, \quad m = 0, 1, 2, \ldots$$

is generally called iteration of Krasnoselskij or iteration of Krasnoselskij–Mann. For $s_0 \in Q$, s_m is sequence introduced by the formula:

$$(*)\ s_{m+1} = (1 - \xi_m)s_m + \xi_m H s_m, \quad m = 0, 1, 2, \ldots$$

in which ξ_m is a sequence of real numbers and subset from $[0, 1]$ satisfying an appropriate condition is known iteration of Krasnoselskij–Mann.

The following results will be particularly important for the application section of our note.

Theorem 5 ([21]) *A is Banach space and $Q \subset A$, and $H : Q \to Q$ is a non expansive mapping. For arbitrary $s_0 \in Q$, consider the iteration of Mann process s_m presented by (*) under the following hypotheses:*

(i) *$s_m \in Q$ for each non negative integers m,*
(ii) *$0 \leq \xi_m \leq b < 1$ for each non negative integers m,*
(iii) *$\sum_{m=0}^{\infty} \xi_m = \infty$. If s_m is bounded, next $s_m - Hs_m \to 0$ as $m \to \infty$.*

Corollary 6 ([22]) *Let A be Banach space and Q is a compact, convex and subset of A and let $H : Q \to Q$ be a non expansive mapping. Whether process hypotheses Mann iteration S_m satisfied $(i) - (ii)$ in Theorem 5, therefor S_m is a strongly converges to a fixed point H.*

Corollary 7 ([22]) *Let A be a real normed space and Q is a convex, closed bounded and subset of A and let $H : Q \to Q$ be a non expansive mapping. Whether $I - H$ maps closed bounded subsets of A into closed subsets of A and S_m is iteration of Mann, with ξ_m is satisfied hypotheses $(i) - (ii)$ in Theorem 5, therefor S_m is a strongly converges to a fixed point of H in Q.*

Main Findings

From Eq. (19), have the following problem being studied

$$\begin{aligned} D^{\beta} v(s) &= g(s, v(s), v(\xi_1 s), v'(\xi_2 s)) \\ v(s_0) &= v_0, \end{aligned} \tag{11}$$

in which $s_0, v_0 I, \xi \in (0,1)$ and $g \in C(I \times I \times I \times I),\ where\ I = [a,b]$. This issue Eq. (19) it extends. We make the first result of there are solutions to the problem of initial value (11). For $s \in I$ denoting

$$\begin{aligned} C_s &= \max\{s-a, b-s\} \\ C_{\ell,\beta} &= \left\{ v \in C(I, I) : |v(s_1) - v(s_2)| \le \ell \cdot \frac{|s_1 - s_2|^{\beta}}{\Gamma(\beta+1)},\ \forall s_1, s_1 \in I \right\}, \end{aligned} \tag{12}$$

in which $\ell > 0$. in which $\ell > 0$. It is obvious which $C_{\ell,\beta}$ in non empty compact and convex subset of the space of Banach $C[I]$, $\|.\|$, in which $\|z\| = \sup_{s\in I} |z(t)|$.

Theorem 8 *Suppose the following conditions for the problem of initial value (11) are satisfied*

(1) $g \in C(I \times I \times I \times I)$,
(2) *there is* $\ell_1 > 0$ *so that*

$$|g(\chi, \Upsilon_1, z_1, \varpi_1) - g(\chi, \Upsilon_2, z_2, \varpi_2)| \le \ell_1 [|\Upsilon_1 - \Upsilon_2| + |z_1 - z_2| + |\varpi_1 - \varpi_2|] \tag{13}$$

for every $\chi, \Upsilon_j, z_j, \varpi_j \in I,\ j = 1, 2$,
(3) *if* ℓ *is the constant of Lipschitz involved in (13), therefore*

$$M = \max\{|g(\chi, \Upsilon, z, \varpi)| : (\chi, \Upsilon, z, \varpi) \in [a,b]\} \le \frac{\ell}{2},$$

(4) *One of these conditions are achieved:*

$$(i)\ \frac{M.B^{\beta}}{\Gamma(\beta+1)}.C_{s_0} \le C_{v_0},\quad where\ \ B = \max[a,b]$$

$$(ii)\ s_0 = 0,\ \ M\frac{(B)^{\beta}}{\Gamma(\beta+1)} \le b - v_0,\ \ g(\chi, \Upsilon, z, \varpi) \ge 0,\ \ \forall(\chi, \Upsilon, z, \varpi) \in I,$$

$$(iii)\ s_0 = b,\ \ M\frac{(B)^{\beta}}{\Gamma(\beta+1)} \le v_0 - a,\ \ g(\chi, \Upsilon, z, \varpi) \ge 0,\ \ \forall(\chi, \Upsilon, z, \varpi) \in I,$$

(5) $3\frac{B^{\beta}}{\Gamma(\beta+1)}\ell_1 \cdot C_{s_0} \le 1$.

Next the problem (11) having at least one solution in C_{ℓ}, that may be approximated by iteration Krasnoselskij

$$\begin{aligned} v_{m+1}(u) &= (1-\eta)v_m(u) + \eta v_0 \\ &\quad + \eta \int_{s0}^{u} \frac{(u-\mu)^{\beta-1}}{\Gamma(\beta)} g(\mu, v_m(\mu), v_m(\xi\mu), v_m'(\mu))d\mu,\ \ u \in I,\ \ m \ge 1,\ \ u > \mu \end{aligned}$$

where $\eta\ in(0,1)$ and $v_1, v_1' \in C_{\ell}$ is arbitrary.

Proof As a result of Arzela–Ascoli, let $(C[I], \|.\|)$ Banach space and $C_\ell \subset (C[I], \|.\|)$ is not empty, convex, and compact such that $\|s\| = \sup_{u\in[a,b]} |s(u)|$.

Consider the operator of integral $G : C_\ell \to C[I]$ introduced by

$$(Gv)(u) = v_0 + \int_{s_0}^{u} \frac{(u-\mu)^{\beta-1}}{\Gamma(\beta)} g(\mu, v(\mu), v(\xi\mu), v'(\mu))d\mu, \quad u \in I, u > \mu$$

Let $v = Gv$ is a solution of the initial value problem (11) for Any fixed point. show that C_ℓ is an invariant set with regard to G, i.e, $G(C_\ell)$ subset form C_ℓ.

Whether the condition (1) achieves, therefor for any $v \in C_\ell$ and $u \in C[I]$ we get

$$|(Gv)(u)| \leq |v_0| + \left| \int_{s_0}^{u} \frac{(u-\mu)^{\beta-1}}{\Gamma(\beta)} g(\mu, v(\mu), v(\xi\mu), v'(\mu))d\beta \right| \leq |v_0| + M\frac{(s_0-u)^{\gamma}}{\Gamma(\gamma+1)} \leq b$$

$$|(Gv)(u)| \geq |v_0| - \left| \int_{s_0}^{u} \frac{(u-\mu)^{\beta-1}}{\Gamma(\beta)} g(\mu, v(\mu), v(\xi\mu), v'(\mu))d\beta \right| \geq |v_0| - M\frac{(s_0-u)^{\beta}}{\Gamma(\beta+1)}$$

$$\geq |v_0| - M.\frac{(s_0-u)^{\beta}}{\Gamma(\beta+1)} C_{s_0} \geq |v_0| - C_{v0} \geq a$$

Thus, $Gv \in [a, b]$ for every $v \in C_\ell$.

Currently, for every $u_1, u_2 \in I$, we get

$$\begin{aligned}
|(Gv)(u_1) - (Gv)(u_2)| &\leq \left| \int_{s_0}^{u_1} \frac{(u-\mu)^{\beta-1}}{\Gamma(\beta)} g(\mu, v(\mu), v(\xi\mu), v'(\mu))d\mu \right| \\
&\quad - \left| \int_{s_0}^{u_2} \frac{(u-\mu)^{\beta-1}}{\Gamma(\beta)} g(\mu, v(\mu), v(\xi\mu), v'(\mu))d\mu \right| \\
&\leq \left| \int_{u_1}^{u_2} \frac{(u-\mu)^{\beta-1}}{\Gamma(\beta)} g(\mu, v(\mu), v(\xi\mu), v'(\mu))d\mu \right| \\
&\leq M.\frac{|u_1^{\beta} - u_2^{\beta}| + 2|u_1-u_2|^{\beta}}{\Gamma(\beta+1)} \\
&\leq 2M.\frac{|u_1-u_2|^{\beta}}{\Gamma(\beta+1)} \leq \ell.\frac{|u_1-u_2|^{\beta}}{\Gamma(\beta+1)}
\end{aligned}$$

Hence, $Gv \in C_\ell$ for every $v \in C_\ell$. In the same way which we treat cases (2) and (3).

Then G is a self-mapping of C_ℓ (i.e. $G : C_\ell \to C_\ell$).

We proceed to show that G is non-expansive operator. Let $v, w \in C_\ell$ and $u \in [a, b]$, therefor

$$\begin{aligned}
|(Gv)(u) - (Gw)(u)| &\leq \left| \int_{s_0}^{u} \frac{(u-\mu)^{\beta-1}}{\Gamma(\beta)} [g(\mu, v(\mu), v(\xi\mu), v'(\mu)) \right. \\
&\quad \left. - g(\mu, w(\mu), w(\xi\mu), w'(\mu))]d\mu \right| \\
&\leq \left| \int_{s_0}^{u} \ell_1 \frac{(u-\mu)^{\beta-1}}{\Gamma(\beta)} [|v(\mu) - w(\mu)| + |v(\xi\mu) - w(\xi\mu)| \right. \\
&\quad \left. + \left|v'(\mu) - w'(\mu)\right|]d\mu \right| \\
&\leq 3\frac{(B)^{\beta}}{\Gamma(\beta+1)}.\ell_1.C_{s_0}.\, \|v - w\|.
\end{aligned}$$

Presently taking the norm, we obtain

$$\|Gv - Gw\| \leq 3\frac{(B)^{\beta}}{\Gamma(\beta+1)}.\ell_1.C_{s_0}.\|v - w\|,$$

which in virtue of the condition (5), shows that G is the non expansive operator therefore continues. It is to apply the fixed point theorem of Browder–Ghode–Kirk and obtain the first part of the conclusion and the Corollary 6 or 7 for the second one.

Currently, applying that the same technique in theorem (3.1) with additional iterative differential equation extending problem (7), and we put one assumption for Eq. (7) is $v(s) = \eta.s$ in (19), we have

$$\begin{aligned} D^{\beta}v(s) &= g(s, v(s), v(v(s)), v'(s)) \\ v(s_0) &= v_0 \end{aligned} \tag{14}$$

in which $s_0, v_0 \in I$, $\eta \in (0, 1)$ and $g \in C(I \times I \times I \times I)$are given to. We make the second output of the existence solutions of the initial value problem (14) in the C_{ℓ}.

Theorem 9 *Suppose the following conditions for the initial value problem* (14) *are satisfied*

(1) $g \in C(I \times I \times I \times I)$,
(2) *there is* $\ell_1 > 0$ *so that*

$$g(\chi, \Upsilon_1, z_1, \varpi_1) - g(\chi, \Upsilon_2, z_2, \varpi_2)| \leq \ell_1[|\Upsilon_1 - \Upsilon_2| + |z_1 - z_2| + |\varpi_1 - \varpi_2|] \tag{15}$$

for every $\chi, \Upsilon_j, z_j, \varpi_j \in I$, $j = 1, 2$,
(3) *if* ℓ *is the constant of Lipschitz involved in* (15)*, therefore*

$$M = \max\{|g(\chi, \Upsilon, z, \varpi)| : (\chi, \Upsilon, z, \varpi) \in [a, b]\} \leq \frac{\ell}{2},$$

(4) *One of these conditions are achieved:*

$$(i)\frac{M.B^{\beta}}{\Gamma(\beta+1)}.C_{s_0} \leq C_{v_0}, \quad where \;\; B = \max[a, b]$$

$$(ii)\, s_0 = 0, \;\; M\frac{(B)^{\beta}}{\Gamma(\beta+1)} \leq b - v_0, \;\; g(\chi, \Upsilon, z, \varpi) \geq 0, \;\; \forall(\chi, \Upsilon, z, \varpi) \in I,$$

$$(iii)\, s_0 = b, \;\; M\frac{(B)^{\beta}}{\Gamma(\beta+1)} \leq v_0 - a, \;\; g(\chi, \Upsilon, z, \varpi) \geq 0, \;\; \forall(\chi, \Upsilon, z, \varpi) \in I,$$

(5) $\ell_1 \cdot \frac{(B)^{\beta}}{\Gamma(\beta+1)} \cdot [3 + \ell].C_{s_0} \leq 1$

Next the problem (14) having at least one solution in C_{ℓ}, that may be approximated by iteration Krasnoselskij

$$v_{m+1}(u) = (1-\eta)v_m(u) + \eta v_0 + \eta\int_{s0}^{u}\frac{(u-\mu)^{\beta-1}}{\Gamma(\beta)}g(\mu, v_m(\mu), v(v_m(\mu)), v_m'(\mu))d\mu,$$

$u \in [a, b]$, $\;m \geq 1$, $\;u > \mu$ and where $\eta \in (0, 1)$ and $v_1, v_1' \in C_{\ell}$ is arbitrary.

Proof Consider the operator of integral $G : C_{\ell} \to C[I]$ introduced by

$$(Gv)(u) = v_0 + \int_{s_0}^{u}\frac{(u-\mu)^{\beta-1}}{\Gamma(\beta)}g(\mu, v(\mu), v(v(\mu)), v'(\mu))d\mu, \quad u \in I, u > \mu$$

Similarly as in the Theorem 8, we show which C_ℓ is an invariable set with regard to G, meaning that $G(C_\ell) \subset C_\ell$. We conclude

$$|(Gv)(u)| \leq |v_0| + \left| \int_{s_0}^{u} \frac{(u-\mu)^{\beta-1}}{\Gamma(\beta)} g(\mu, v(\mu), v(v(\mu)), v'(\mu)) d\beta \right| \leq |v_0| + M \frac{(s_0-u)^{\mu}}{\Gamma(\mu+1)} \leq b$$

$$|(Gv)(u)| \geq |v_0| - \left| \int_{s_0}^{u} \frac{(u-\mu)^{\beta-1}}{\Gamma(\beta)} g(\mu, v(\mu), v(v(\mu)), v'(\mu)) d\mu \right| \geq |v_0| - M \frac{(s_0-u)^{\beta}}{\Gamma(\beta+1)}$$
$$\geq |v_0| - M.\frac{(s_0-u)^{\beta}}{\Gamma(\beta+1)} C_{s_0} \geq |v_0| - C_{v0} \geq a$$

Thus, $Gv \in I$ for every $v, v' \in C_\ell$.

Currently, for every $u_1, u_2 \in I$ we get

$$\begin{aligned} |(Gv)(u_1) - (Gv)(u_2)| &\leq \left| \int_{s_0}^{u_1} \frac{(u-\mu)^{\beta-1}}{\Gamma(\beta)} g(\mu, v(\mu), v(v(\mu)), v'(\mu)) d\mu \right| \\ &\quad - \left| \int_{s_0}^{u_2} \frac{(u-\mu)^{\beta-1}}{\Gamma(\beta)} g(\mu, v(\mu), v(v(\mu)), v'((\mu))) d\mu \right| \\ &\leq \left| \int_{u_1}^{u_2} \frac{(u-\mu)^{\beta-1}}{\Gamma(\beta)} g(\mu, v(\mu), v(v(\mu)), v'(\mu)) d\mu \right| \\ &\leq M.\frac{|u_1^{\beta} - u_2^{\beta} + 2(u_1-u_2)^{\beta}|}{\Gamma(\beta+1)} \\ &\leq 2M.\frac{|u_1-u_2|^{\beta}}{\Gamma(\beta+1)} \leq \ell.\frac{|u_1-u_2|^{\beta}}{\Gamma(\beta+1)} \end{aligned}$$

Hence, $Gv \in C_\ell$ for every $v, v' \in C_\ell$. In the same way which we treat cases (2) and (3).

Then G is a self-mapping of C_ℓ(i.e. $G : C_\ell \to C_\ell$).

We show that G is non-expansive operator. Let $v, w \in C_\ell$ and $u \in I$. Therefore

$$\begin{aligned} |(Gv)(u) - (Gw)(u)| &\leq \int_{s_0}^{u} \left| \frac{(u-\mu)^{\beta-1}}{\Gamma(\beta)} [g(\mu, v(\mu), v(v(\mu)), v'(\mu)) \right. \\ &\qquad \left. - g(\mu, w(\mu), w(w(\mu)), w'(\mu))] \right| d\mu \\ &\leq \int_{s_0}^{u} \ell_1 \frac{(u-\mu)^{\beta-1}}{\Gamma(\beta)} [|v(\mu) - w(\mu)| + |v(v(\mu)) - w(w(\mu))| + \left|v'(\mu) - w'(\mu)\right| d\mu \\ &\leq \int_{s_0}^{u} \ell_1 \frac{(u-\mu)^{\beta-1}}{\Gamma(\beta)} [|v(\mu) - w(\mu)| + |v(v(\mu)) - w(w(\mu)) + v(w(\mu)) - v(w(\mu))| \\ &\quad + \left|v'(\mu) - w'(\mu)\right| d\mu \\ &\leq \int_{s_0}^{u} \ell_1 \frac{(u-\mu)^{\beta-1}}{\Gamma(\beta)} [|v(\mu) - w(\mu)| + |v(v(\mu)) - v(w(\mu))| + |v(w(\mu)) - w(w(\mu))| \\ &\quad + \left|v'(\mu) - w'(\mu)\right| d\mu \\ &\leq \ell_1 \int_{s_0}^{u} \frac{(u-\mu)^{\beta-1}}{\Gamma(\beta)} [|v(\mu) - w(\mu)| + \ell. |v(\mu) - w(\mu)| + |v(w(\mu)) - w(w(\mu))| \\ &\quad + \left|v'(\mu) - w'(\mu)\right| d\mu \\ &\leq \ell_1 \frac{(B)^{\beta}}{\Gamma(\beta+1)}.[2+\ell].\frac{|s_0-u|^{\beta}}{\Gamma(\beta+1)}. \|v-w\| \leq \ell_1.\frac{(B)^{\beta}}{\Gamma(\beta+1)}.[3+\ell].C_{s_0}. \|v-w\| \end{aligned}$$

Presently taking the norm, we obtain

$$\|Gv - Gw\| \leq \ell_1 . \frac{(B)^{\beta}}{\Gamma(\beta+1)} .[3+\ell].C_{s_0} . \|v - w\|$$

that, considering the condition (5), shows that G is the non expansive operator therefore continues. In view of the fixed point theorem of Browder–Ghode–Kirk and have the first part of the conclusion and the Corollary 6 or 7 for the second one.

Applications

Consider the problem of initial value as a result liked with an fractional first iterative order differential equation involving first derivatives

Example

Consider the initial value problem

$$\begin{aligned} D^{\frac{1}{2}} v(s) &= -3 + v(s) + v(v(s)) + v'(s) \\ v(1/2) &= 1 \end{aligned} \tag{16}$$

in which $s \in [0, 1]$, and $v, v' \in C^{3,\frac{1}{2}}([0, 1] \times [0, 1] \times [0, 1]$.

Consider $v, v' \in C^{3,\frac{1}{2}}([0, 1] \times [0, 1] \times [0, 1]$ belonging to the set

$$C_{3,\frac{1}{2}} = \left\{ v \in C^{3,}([0, 1] \times [0, 1] \times [0, 1] : |v(u_1) - v(u_2)| \leq 3 \frac{|u_1 - u_2|^{0.5}}{\Gamma(1.5)} \right\}$$

for any $u_1, u_2 \in [0, 1]$ that, in given our notes, that means $\ell = 3$, we obtain

$$a = 0, b = 1, s_0 = 1 \;\; hence \;\; C_{s_0} = \max\{s_0 - a, b - s_0\} = 1, \;\; \max\{0, 1\} = 1, \\ \Gamma(1.5) = 0.8862269255.$$

The function

$$g(s, y, z, w) = -3 + 0.1477044876\,[y + z + w]$$

is Lipschitzian with the Lipschitz constant $\ell_1 = 0.1477044876$. Then, we obtain

$$\ell_1 \frac{1}{0.8862269255}\,[3+\ell]\,C_{s_0} = 1,$$

therefore the condition (5) in Theorem 9 is satisfied. It is also noted that $v(s) = 1, v', (s) = 1 \;\; s \in [0, 1]$ is a solution to the initial- value problem (16). From Theorem 9 initial- value problem (16) having at least one solution in C_3, can be approximated as iteration Krasnoselskji

$$v_{m+1}(u) = (1-\sigma)v_m(u) + \sigma v_0 + \sigma \int_{s0}^{u} \frac{(u-\beta)^{\gamma-1}}{\Gamma(\gamma)} (-3 + v_m(\beta) \\ + v_m(v_m(\beta)) + v_m'(\beta))d\beta,$$

$u \in [a, b], \;\; m \geq 1, \;\; u > \beta$ and where $\sigma \in (0, 1)$ and $v_1, v_1' \in C_\ell$ is arbitrary.

Example

Consider the initial value problem

$$cD^{\frac{2}{3}}v(s) = -1 + v(s) - v(v(s)) + v'(s)$$
$$v\left(\frac{1}{2}\right) = \frac{1}{2}, \tag{17}$$

in which $s \in [0, 1]$, and $v, v' \in C^{4,\frac{2}{3}}([0, 1] \times [0, 1] \times [0, 1]$.

Assume that $v, v' \in C^{4,\frac{2}{3}}([0, 1] \times [0, 1] \times [0, 1])$ belonging to the set

$$C_{4,\frac{2}{3}} = \left\{ v \in C^{4,\frac{2}{3}}([0, 1] \times [0, 1] \times [0, 1] : |v(u_1) - v(u_2)| \le 4.\frac{|u_1 - u_2|^{\frac{2}{3}}}{\Gamma(\frac{2}{3} + 1)} \right\}$$

for any $u_1, u_2 \in [0, 1]$ that, in given our notes, that means $\ell = 4$, we lead to

$$a = 0, b = 1, s_0 = 0.5 \;\; hence \;\; C_{s_0} = \max\{s_0 - a, b - s_0\} = 0.5, \;\; \max\{0, 1\} = 1,$$
$$\Gamma\left(\frac{2}{3} + 1\right) = \frac{2}{3}.\Gamma\left(\frac{2}{3}\right).$$

The function

$$g(s, y, z, w) = -1 + \frac{2\Gamma\left(\frac{2}{3}\right)}{9}[y + z + w]$$

is Lipschitzian with the Lipschitz constant $\ell_1 = \frac{2\Gamma(\frac{2}{3})}{9}$. Then, we have

$$\ell_1 \frac{1}{\frac{2}{3}.\Gamma\left(\frac{2}{3}\right)}[3 + \ell].C_{s_0} = 1,$$

thus, the condition (5) in Theorem 9 is satisfied. It is, also noted that $v(s) = \frac{1}{2}, v'(s) = -1, \;\; s \in [0, 1]$ is a solution to the initial- value problem (17). By Theorem 9 initial- value problem (17) having at least one solution inC_5, can be approximated as iteration Krasnoselskji

$$v_{m+1}(u) = (1 - \sigma)v_m(u) + \sigma v_0 + \sigma \int_{s0}^{u} \frac{(u - \beta)^{\gamma - 1}}{\Gamma(\gamma)}(-1 + v_m(\beta) - v_m(v_m(\beta)) + v'_m(\beta))d\beta,$$

$u \in [a, b], \;\; m \ge 1, \;\; u > \beta$ and where $\sigma \in (0, 1)$ and $v_1, v'_1 \in C_\ell$ is arbitrary.

Conclusion

From above, we conclude that the iterative differential equations can be extended into fractional differential equation including the first order derivative. Moreover, as future work, one can investigate the existence of second order derivative such as

$$\begin{matrix} D^\gamma v(s) = f(s, v(s), v(\xi_1 s), v'(\xi_2 s), v''(\xi_3 s)) \\ v(s_0) = v_0, v'(s_0) = v'_0, v''(s_0) = v''_0. \end{matrix}, \tag{18}$$

or

$$\begin{matrix} D^\gamma v(s) = f(s, v(s), v(\xi_1 s), v'(\xi_2 s), v''(\xi_3 s))) \\ v(s_0) = v_0, v'(s_0) = v'_0, v''(s_0) = v''_0. \end{matrix} \tag{19}$$

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Compliance with Ethical Standards

Conflicts of Interest The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions All the authors jointly worked on deriving the results and approved the final manuscript.

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