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On Differential Subordination of Higher-Order Derivatives of Multivalent Functions

On Differential Subordination of Higher-Order Derivatives of Multivalent Functions

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Abstract. The main objective of this paper is to study the concept of dependency and introduce a new subclass linked to derivatives of higher order for polyvalent functions with a different operator. Thus, the results were important it was obtained with respect to different types of some geometric properties of which coefficient estimate, distortion and growth bounds, radii of starlikeness, convexity, and close-to-convex.

Keywords: Convex functions, Differential subordination, Starlike functions, Multivalent functions, Higher-order derivatives.

1. Motivation and preliminaries

One of the most important notions in the complex analysis, is that the theory of harmonic and analytic [7] univalent functions with (bi or multi-types) [1-6, 8-10, 12-15]. Thus this theory is characterized some special elements to define new interesting certain classes or sub-classes [3-4] of special functions related to various operators [1-2, 4, 9-11] which may be maximized or maximized some real problem by a certain functional family follows from the theory of normal functions via some properties of complex functions [5].

More precisely this field have taken the attention of numerous researchers in the domain of applied science in different situation. Furthermore, these concepts paly a good role to find the exact solution for mathematical modelling, for example many concrete real problems, such as in the study of physical, chemical, engineering domains [6].

For this purpose, assume that $\Delta = \{\boldsymbol{z} \in \boldsymbol{C} : |\boldsymbol{z}| < \mathbf{1}\}$ be the open unit disc and let $\boldsymbol{A}(\boldsymbol{p})$ denote the class of analytic functions of the form:

$f(z) = z^p - \sum_{n=P+1}^{\infty} a_n z^n$, (p ∈ N = {1,2, … }) (1) which are analytic and p-valent in the unit disk Δ, and let $A(1) = A$ see [8-10]. Now, we introduce the differential operator (for nether subordination for higher order derivatives Srivastava-Attiya operator [4]) defined as follows

$$f^{(1)}(z) = pz^{p-1} - \sum_{n=P+1}^{\infty} na_n z^{n-1},$$

⋮

$$f^{(s)}(z) = \frac{p!}{(p-s)!} z^{p-s} - \sum_{n=1}^{\infty} \frac{n!}{(n-s)!}\ a_n\, z^{n-s}, \tag{2}$$

where

$$\gamma(p;s) = \frac{p!}{(p-s)!}\ , \gamma(n;s\ ) = \frac{n!}{(n-s)!} \quad (p \geq s, p \in N, s \in N \cup \{0\}). \tag{3}$$

Using the principle of the dependency properties of polyvalent functions with a different operators [7]. and we have obtained many results regarding these operators as in [9-11].

The functions $f(z)$ and $g(z)$ be analytic functions in Δ, then we say $f(z)$ is subordinate to $g(z)$, if there exists a function $w \in \Psi$, where

$\Psi = \{w \in A{:}\, w(0) = 0$ and $\Psi = \{w\ \in\ A{:}\, w(0)\ = 0$ and $|w(z)|\ <\ 1\}$,

the Schwarz functions, $f(z) = g\big(w(z)\big)$.

We write

$f(z) \prec g(z)$. $(z \in \Delta\ )$

the function g is univalent in Δ, then we get the equivalence [8].

$f(z) \prec g(z) \Leftrightarrow f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$A function $f(z)$ is called starlike (convex) in Δ if

satisfies the following condition:

$\left(Re\left(\frac{zf'(z)}{f(z)}\right)\right) > 0, \left(Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0\right)$, respectively where $z \in \Delta, 0 < r \leq 1$ (See [3]).

Let $f(z) \in A(P)$ in the subclass $K^*(A, B, \gamma, p)$ if satisfying the following

$$1 + \frac{1}{\gamma}\left\{\frac{zf^{s+1}(z)}{f^s(z)} - \gamma\right\} \prec \frac{1+AZ}{1+BZ} \tag{5}$$

$(-1 \leq B < A \leq 1)$ and $(0 \leq \gamma < 1)$ see [12].

2. Main results

The sufficient conditions for a differential subordination for the class $k(A, B, \gamma, p)$ is studied and demonstrated.

Theorem 2.1. A function $f(z)$ given of the form (5) is in $K\ ^*(A, B, \gamma, p)$ if and only if

$$\sum_{n=p+1}^{\infty}\left(\frac{n!}{(n-s)!} - \frac{n!}{(n-s-1)!}\right)|\gamma(A - B) - (1 + B) + \gamma| a_n \tag{6}$$
$$\leq |\gamma(A - B) + \gamma(B\gamma + 1) - B - 1|.$$

Proof. Let $f(z) \in K^*(A, B, \gamma, p)$. Then we get

$$D(z) = 1 + \frac{1}{\gamma}\left\{\frac{zf^{s+1}(z)}{f^s(z)} - \gamma\right\} \prec \frac{1+Az}{1+Bz}, \tag{7}$$

then we have

$$D(z) = \frac{1+AT(z)}{1+BT(z)},$$

where $T(z)$ is Schwarz function see [8]

$$D(z)(1+BT(z)) = 1 + AT(z)$$

$$T(z) = \frac{D(z)-1}{A-BD(z)}.$$

Such that $|T(z)| < 1$, we get

$$\left|\frac{1+\frac{1}{\gamma}\left\{\frac{zf^{s+1}(z)}{f^{s}(z)}-\gamma\right\}-1}{A-B\left\{1+\frac{1}{\gamma}\left\{\frac{zf^{s+1}(z)}{f^{s}(z)}-\gamma\right\}\right\}}\right| < 1$$

$$\left|\frac{zf^{s+1}(z)-\gamma f^{s}(z)}{[A-B(1-\gamma)]\gamma f^{s}(z)-\gamma z f^{s+1}(z)}\right| < 1 \tag{8}$$

$$zf^{s+1}(z) - \gamma f^{s}(z)$$

$$= (1-\gamma)\left[\frac{p!}{(p-s-1)!} - \frac{p!}{(p-s)!}\right] z^{p-s} + \sum_{n=p+1}^{\infty}\left(\frac{n!}{(n-s)!} - \frac{n!}{(n-s-1)!}\right)(\gamma-1)a_n z^{n-s}. \tag{9}$$

Now

$$[A-B(1-\gamma)]\gamma f^{s}(z) - \gamma z f^{s+1}(z)$$

$$= \{[A-B(1-\gamma)]\gamma - B\}\left[\frac{p!}{(p-s-1)!} - \frac{p!}{(p-s)!}\right] z^{p-s} + \sum_{n=p+1}^{\infty}\left(\frac{n!}{(n-s)!} - \frac{n!}{(n-s-1)!}\right)(B(1\gamma) + \quad A\gamma)a_n z^{n-s}.$$

(10)

By substitution (9) and (10) in (8), we get

$$\left|\frac{(1-\gamma)\left[\frac{p!}{(p-s-1)!}-\frac{p!}{(p-s)!}\right]z^{p-s}+\sum_{n=p+1}^{\infty}\left(\frac{n!}{(n-s)!}-\frac{n!}{(n-s-1)!}\right)(\gamma-1)a_n z^{n-s}}{\{[A-B(1-\gamma)]\gamma-B\}\left[\frac{p!}{(p-s-1)!}-\frac{p!}{(p-s)!}\right]z^{p-s}+\sum_{n=p+1}^{\infty}\left(\frac{n!}{(n-s)!}-\frac{n!}{(n-s-1)!}\right)(B(1-\gamma)+A\gamma)a_n z^{n-s}}\right| < 1.$$

When $z \to 1$, we obtain

$$(1-\gamma)\left[\frac{p!}{(p-s-1)!} - \frac{p!}{(p-s)!}\right] z^{p-s} + \sum_{n=p+1}^{\infty}\left(\frac{n!}{(n-s)!} - \frac{n!}{(n-s-1)!}\right)|(\gamma-1)|a_n$$

$$\leq \{[A-B(1-\gamma)]\gamma - B\}\left[\frac{p!}{(p-s-1)!} - \frac{p!}{(p-s)!}\right] z^{p-s} + \sum_{n=p+1}^{\infty}\left(\frac{n!}{(n-s)!} - \frac{n!}{(n-s-1)!}\right)(B(1-\gamma) + \quad A\gamma)a_n.$$

Then

$$\sum_{n=p+1}^{\infty}\left(\frac{n!}{(n-s)!}-\frac{n!}{(n-s-1)!}\right)|\gamma(A-B)-(1+B)+\gamma|a_n \leq |\gamma(A-B)+\gamma(B\gamma+1)-B-1|.$$

By using the above theorem we have.

Corollary 2.2. Let the function f is belonging in $K^*(A,B,\gamma,p)$. Then

$$a_n \leq \frac{|\gamma(A-B)+\gamma(B\gamma+1)-B-1|}{\left(\frac{n!}{(n-s)!}-\frac{n!}{(n-s-1)!}\right)|\gamma(A-B)-(1+B)+\gamma|}, n \geq p+1$$

we have

$$f(z) = z^p - \frac{|\gamma(A-B)+\gamma(B\gamma+1)-B-1|}{\frac{n!}{(n-s)!}-\frac{n!}{(n-s-1)!}|\gamma(A-B)-(1+B)+\gamma|}z^n. \quad (11)$$

Thus we can deduced the following important result.

Theorem 2.3. A function $f(z) = z^p - \sum_{n=P+1}^{\infty} a_n z^n$ is in $K(A,B,\gamma,p)$ if and only if

$$\sum_{n=p+1}^{\infty}\left(\frac{n!(n-s)}{(n-s-m)!}-\frac{n!}{(n-s-m-1)!}\right)|\gamma(A-B)+\gamma-B\gamma|a_n$$

$$\leq |\gamma(A-B)-B\gamma+\gamma|\frac{p!(p-s)}{(p-s-m)(p-s-m-1)!}a_n. \quad (12)$$

Proof. Suppose $g(z) = zf^{s+1}(z)$ since $f(z) \in K(A,B,\gamma,p)$ and $zf^{s+1}(z) \in K(A,B,\gamma,p)$

therefore from (7) we get

$$D(z) = 1 + \frac{1}{\gamma}\left\{\frac{zg^{m+1}(z)}{g^m(z)} - \gamma\right\} \prec \frac{1+Az}{1+Bz}, \quad (13)$$

where

$$g^m(z) = \frac{p!(p-s)}{(p-s-m)!}z^{p-s-m} - \frac{n!(n-s)}{(n-s-m)!}a^n z^{n-s-m}. \quad (14)$$

Such that $|T(z)| < 1$, we get

$$\left|\frac{1+\frac{1}{\gamma}\left\{\frac{zg^{m+1}(z)}{g^m(z)}-\gamma\right\}-1}{A-B\left\{1+\frac{1}{\gamma}\left\{\frac{zg^{m+1}(z)}{g^m(z)}-\gamma\right\}\right\}}\right| < 1$$

$$\left|\frac{zg^{m+1}(z)-\gamma g^m(z)}{[A-B(1-\gamma)]\gamma g^m(z)-\gamma z g^{m+1}(z)}\right| < 1 \quad (15)$$

$$zg^{m+1}(z)-\gamma g^{m}(z)$$

$$=(p-s-m-\gamma)\frac{p!(p-s)}{(p-s-m)(p-s-m-1)!}z^{p-s-m}+\sum_{n=p+1}^{\infty}\left(\frac{n!(n-s)}{(n-s-m)!}-\frac{n!(n-s)}{(n-s--m-1)!}\right)(\gamma-1)a_n. \quad (16)$$

Now

$$[\gamma(A-B)-B\gamma]g^{m}(z)+zg^{m+1}(z)$$

$$=[\gamma(A-B)-B\gamma+p-s-m]\frac{p!(p-s)}{(p-s-m-1)!}a_n z^{n-s-m}$$

$$+\sum_{n=p+1}^{\infty}\left(\frac{n!(n-s)}{(n-s-m)!}-\frac{n!(n-s)}{(n-s--m-1)!}\right)(\gamma(A-B)-B\gamma+1)a_n z^{n-s-m}. \quad (17)$$

From (15), we obtain

$$\left|\frac{(p-s-m-\gamma)\frac{p!(p-s)}{(p-s-m)(p-s-m-1)!}z^{p-s-m}+\sum_{n=p+1}^{\infty}\left(\frac{n!(n-s)}{(n-s-m)!}-\frac{n!(n-s)}{(n-s-m-1)!}\right)(\gamma-1)a_n\, z^{n-s-m}}{[\gamma(A-B)-B\gamma+p-s-m]\frac{p!(p-s)}{(p-s-m-1)!}a_n z^{n-s-m}+\sum_{n=p+1}^{\infty}\left(\frac{n!(n-s)}{(n-s-m)!}-\frac{n!(n-s)}{(n-s-m-1)!}\right)(\gamma(A-B)-B\gamma+1)a_n z^{n-s-m}.}\right|<1.$$

Since $z\to 1$ we get

$$(p-s-m-\gamma)\frac{p!(p-s)}{(p-s-m)(p-s-m-1)!}z^{p-s-m}+\sum_{n=p+1}^{\infty}\left(\frac{n!(n-s)}{(n-s-m)!}-\frac{n!(n-s)}{(n-s--m-1)!}\right)(\gamma-1)a_n$$

$$\leq[\gamma(A-B)-B\gamma+p-s-m]\frac{p!(p-s)}{(p-s-m-1)!}a_n z^{n-s-m}$$
$$+\sum_{n=p+1}^{\infty}\left(\frac{n!(n-s)}{(n-s-m)!}-\frac{n!(n-s)}{(n-s--m-1)!}\right)(\gamma(A-B)-B\gamma+1)a_n.$$

Then

$$\sum_{n=p+1}^{\infty}\left(\frac{n!(n-s)}{(n-s-m)!}-\frac{n!}{(n-s--m-1)!}\right)|\gamma(A-B)+\gamma-B\gamma|a_n$$

$$\leq|\gamma(A-B)-B\gamma+\gamma|\frac{p!(p-s)}{(p-s-m)(p-s-m-1)!}a_n.$$

From the theorem 2.3, we can get.

Corollary 2.4. Let the function f is belonging in $K^*(A,B,\gamma,p)$. Then

$$a_n\leq\frac{|\gamma(A-B)-B\gamma+\gamma|\frac{p!(p-s)}{(p-s-m)(p-s-m-1)!}}{\left(\frac{n!(n-s)}{(n-s-m)!}-\frac{n!}{(n-s-m-1)!}\right)|\gamma(A-B)+\gamma-B\gamma|},n\geq p+1$$

we have

$$f(z) = z^p - \frac{|\gamma(A-B)-B\gamma+\gamma|\frac{p!(p-s)}{(p-s-m)(p-s-m-1)!}}{\frac{n!(n-s)}{(n-s-m)!}-\frac{n!}{(n-s-m-1)!}|\gamma(A-B)+\gamma-B\gamma|} z^n. \quad (18)$$

The following outcome proves the distortion and growth theorem [2] of the related class.

Theorem 2.5. Let $f(z) \in K^*(A,B,\gamma,p)$. Then

$$|z|^p - |z|^{p+1} \frac{|\gamma(A-B)+\gamma(B\gamma+1)-B-1|}{\left(\frac{(p+1)!}{(n-s+1)!}-\frac{(p+1)!}{(p-s)!}\right)|\gamma(A-B)-(1+B)+\gamma|} \le |f(z)|$$

$$\le |z|^p + |z|^{p+1} \frac{|\gamma(A-B)+\gamma(B\gamma+1)-B-1|}{\left(\frac{(p+1)!}{(n-s+1)!}-\frac{(p+1)!}{(p-s)!}\right)|\gamma(A-B)-(1+B)+\gamma|}, \quad (19)$$

we have

$$f(z) = z^p - z^{p+1} \frac{|\gamma(A-B)+\gamma(B\gamma+1)-B-1|}{\left(\frac{(p+1)!}{(n-s+1)!}-\frac{(p+1)!}{(p-s)!}\right)|\gamma(A-B)-(1+B)+\gamma|}. \quad (20)$$

Growth theorem for the considered subclass $K^*(A,B,\gamma,p)$is given by.

Theorem 2.6. Let $f(z) \in K^*(A,B,\gamma,p)$. Then

$$|z|^p - |z|^{p+1} \frac{|\gamma(A-B)+\gamma(B\gamma+1)-B-1|}{\left(\frac{(p+1)!}{(n-s+1)!}-\frac{(p+1)!}{(p-s)!}\right)|\gamma(A-B)-(1+B)+\gamma|} \le |f(z)|$$

$$\le |z|^p - |z|^{p+1} \frac{|\gamma(A-B)-B\gamma+\gamma|\frac{p!(p-s)}{(p-s-m)(p-s-m-1)!}}{\sum_{n=p+1}^{\infty}\left(\frac{(p+1)(p-s+1)}{(p-s-m+1)!}-\frac{(p+1)!(p-s-m+1)}{(p-s-m)!}\right)|\gamma(A-B)+\gamma-B\gamma|}, \quad (21)$$

we have

$$f(z) = z^p - z^{p+1} \frac{|\gamma(A-B)-B\gamma+\gamma|\frac{p!(p-s)}{(p-s-m)(p-s-m-1)!}}{\sum_{n=p+1}^{\infty}\left(\frac{(p+1)(p-s+1)}{(p-s-m+1)!}-\frac{(p+1)!(p-s-m+1)}{(p-s-m)!}\right)|\gamma(A-B)+\gamma-B\gamma|} \quad (22)$$

The following result shown that the function $f(z)$ satisfies the radii of starlikeness, convexity and close-to-convex to convexity properties depending on [1] , *i.e.*

$f(z) \in K^*(A,B,\gamma,p)$.

Theorem 2.7. Let $f(z) \in K^*(A,B,\gamma,p)$. Then f is of starlikeness order η

in $|z| < r_1$

$$r_1((A,B,\gamma,p,\eta) = \inf\left\{\frac{(p+\eta-2)}{(n+\eta-2)}\left(\frac{\left(\frac{n!}{(n-s)!}-\frac{n!}{(n-s-1)!}\right)|\gamma(A-B)-(1+B)+\gamma|}{|\gamma(A-B)+\gamma(B\gamma+1)-B-1|}\right)\right\}^{\frac{1}{n-p}} \quad (23)$$

Proof. We need to proof that

$$\left|\frac{zf'(z)}{f(z)}\right| \leq 1-\eta,$$

$$\frac{(p-1)|z|^p-\sum_{n=P+1}^{\infty}(n-1)|a_n||z^n|}{|z|^p-\sum_{n=P+1}^{\infty}|a_n||z^n|} \leq 1-\eta. \quad (24)$$

From (24) holds if

$$(p-1)|z|^p-\sum_{n=P+1}^{\infty}(n-1)|a_n||z^n| \leq (1-\eta)[\,|z|^p-\sum_{n=P+1}^{\infty}|a_n||z^n|\,].$$

Then

$$\sum_{n=P+1}^{\infty}\frac{(n+\eta-2)}{(p+\eta-2)}|a_n||z|^{n-p} \leq 1. \quad (25)$$

Form theorem (2.1)

$$\sum_{n=P+1}^{\infty}\frac{\left(\frac{n!}{(n-s)!}-\frac{n!}{(n-s-1)!}\right)|\gamma(A-B)-(1+B)+\gamma|}{|\gamma(A-B)+\gamma(B\gamma+1)-B-1|}a_n \leq 1. \quad (26)$$

Using (24) and (25) we get

$$\frac{(p+\eta-2)}{(n+\eta-2)}|z|^{n-p} \leq \frac{\left(\frac{n!}{(n-s)!}-\frac{n!}{(n-s-1)!}\right)|\gamma(A-B)-(1+B)+\gamma|}{|\gamma(A-B)+\gamma(B\gamma+1)-B-1|}$$

$$|z|^{n-p} \leq \frac{(p+\eta-2)}{(n+\eta-2)}\left(\frac{\left(\frac{n!}{(n-s)!}-\frac{n!}{(n-s-1)!}\right)|\gamma(A-B)-(1+B)+\gamma|}{|\gamma(A-B)+\gamma(B\gamma+1)-B-1|}\right)$$

$$|z|^{n-p} \leq \left\{\frac{(p+\eta-2)}{(n+\eta-2)}\left(\frac{\left(\frac{n!}{(n-s)!}-\frac{n!}{(n-s-1)!}\right)|\gamma(A-B)-(1+B)+\gamma|}{|\gamma(A-B)+\gamma(B\gamma+1)-B-1|}\right)\right\}.^{\frac{1}{n-p}}$$

The next theorem shows the convexity property of the considered subclass functions.

Theorem 2.8. Let $f(z) \in K^*(A,B,\gamma,p)$. Then f is of convex function of order η

in $|z| < r_2$

$$r_2\,(A,B,\gamma,p,\eta) = \inf\left\{\frac{(p+\eta-2)}{(n+\eta-2)}\left(\frac{\left(\frac{n!}{(n-s)!}-\frac{n!}{(n-s-1)!}\right)|\gamma(A-B)-(1+B)+\gamma|}{|\gamma(A-B)+\gamma(B\gamma+1)-B-1|}\right)\right\}^{\frac{1}{n-p}}$$

Proof. We have to proof that

$$\left|\frac{zg'(z)}{g(z)}\right| \leq 1-\eta,$$

where $g(z) = zf^{m+1}(z)$

$$\frac{\frac{p!}{(p-s-1)!}(p-s-1)|z|^{p-s} - \sum_{n=p+1}^{\infty}\frac{n!}{(n-s-1)!}(n-s-1)|a_n||z|^{n-s}}{\frac{p!}{(p-s-1)!}|z|^{p-s} - \sum_{n=p+1}^{\infty}\frac{n!}{(n-s-1)!}|a_n||z|^{n-s}} \leq 1-\eta. \quad (27)$$

From (27) holds if

$$\frac{p!}{(p-s-1)!}(p-s-1)|z|^{p-s} - \sum_{n=p+1}^{\infty}\frac{n!}{(n-s-1)!}(n-s-1)|a_n||z|^{n-s}$$

$$\leq (1-\eta)\left[\frac{p!}{(p-s-1)!}|z|^{p-s} - \sum_{n=p+1}^{\infty}\frac{n!}{(n-s-1)!}|a_n||z|^{n-s}\right],$$

$$\sum_{n=P+1}^{\infty}\frac{\frac{n!}{(n-s-1)!}(n-s-\eta)}{\frac{p!}{(p-s-1)!}(p-s-\eta+2)}|a_n||z|^{n-s} \leq 1. \quad (28)$$

From theorem (2.1) we have

$$\frac{\left(\frac{n!}{(n-s)!}-\frac{n!}{(n-s-1)!}\right)|\gamma(A-B)-(1+B)+\gamma|}{|\gamma(A-B)+\gamma(B\gamma+1)-B-1|} \leq 1. \quad (29)$$

By using (28) and (29) we get

$$\frac{\frac{n!}{(n-s-1)!}(n-s-\eta)}{\frac{p!}{(p-s-1)!}(p-s-\eta+2)}|z|^{n-s}$$

$$\leq \frac{\left(\frac{n!}{(n-s)!}-\frac{n!}{(n-s-1)!}\right)|\gamma(A-B)-(1+B)+\gamma|}{|\gamma(A-B)+\gamma(B\gamma+1)-B-1|}$$

$$|z|^{n-s} \leq \frac{\frac{p!}{(p-s-1)!}(p-s-\eta+2)}{\frac{n!}{(n-s-1)!}(n-s-\eta)}\left(\frac{\left(\frac{n!}{(n-s)!}-\frac{n!}{(n-s-1)!}\right)|\gamma(A-B)-(1+B)+\gamma|}{|\gamma(A-B)+\gamma(B\gamma+1)-B-1|}\right)$$

$$f(z) = \left\{\frac{\frac{p!}{(p-s-1)!}(p-s-\eta+2)}{\frac{n!}{(n-s-1)!}(n-s-\eta)}\left(\frac{\left(\frac{n!}{(n-s)!}-\frac{n!}{(n-s-1)!}\right)|\gamma(A-B)-(1+B)+\gamma|}{|\gamma(A-B)+\gamma(B\gamma+1)-B-1|}\right)\right\}^{\frac{1}{n-p}}.$$

Hence the proof completed.

The subsequent theorem shows the close-to-convexity property of the considered subclass functions.

Theorem 2.9. Let $f(z) \in K^*(A,B,\gamma,p)$. Then f is close-to-convex function of order η

in $|z| < r_3$

$$r_3\,(A,B,\gamma,p,\eta) = \inf\left\{\frac{[p|z|^{p-1}-(2-\eta)]\left[\left(\frac{n!}{(n-s)!}-\frac{n!}{(n-s-1)!}\right)|\gamma(A-B)-(1+B)+\gamma|\right]}{n|z|^{p-1}(|\gamma(A-B)+\gamma(B\gamma+1)-B-1|)}\right\}^{\frac{1}{n-p}} \quad (30)$$

Proof. We have to prove that $|f'(z)-1| < 1-\eta$,

that is

$$|f'(z)-1| \le p|z|^{p-1} - \sum_{n=P+1}^{\infty} n|a_n||z|^{n-1} - 1 < 1-\eta$$

$$|f'(z)-1| \le p|z|^{p-1} - \sum_{n=P+1}^{\infty} n|a_n||z|^{n-1} < 2-\eta. \quad (31)$$

From theorem (2.1) we have

$$\sum_{n=p|+1}^{\infty} a_n \le \frac{|\gamma(A-B)+\gamma(B\gamma+1)-B-1|}{\left(\frac{n!}{(n-s)!}-\frac{n!}{(n-s-1)!}\right)|\gamma(A-B)-(1+B)+\gamma|}.\, n \ge p+1$$

$$\sum_{n=p+1}^{\infty} \frac{\left(\frac{n!}{(n-s)!}-\frac{n!}{(n-s-1)!}\right)|\gamma(A-B)-(1+B)+\gamma|}{|\gamma(A-B)+\gamma(B\gamma+1)-B-1|} a_n \le 1. \quad (32)$$

Observe that (32) is true if

$$\frac{p|z|^{n-2-p+p}}{p|z|^{p-1}-(2-\eta)} \le \frac{\left(\frac{n!}{(n-s)!}-\frac{n!}{(n-s-1)!}\right)|\gamma(A-B)-(1+B)+\gamma|}{|\gamma(A-B)+\gamma(B\gamma+1)-B-1|},$$

that is

$$\frac{[p|z|^{p-1}-(2-\eta)]\left[\left(\frac{n!}{(n-s)!}-\frac{n!}{(n-s-1)!}\right)|\gamma(A-B)-(1+B)+\gamma|\right]}{n|z|^{p-1}(|\gamma(A-B)+\gamma(B\gamma+1)-B-1|)}.$$

Therefore

$$|z| \le \left\{\frac{[p|z|^{p-1}-(2-\eta)]\left[\left(\frac{n!}{(n-s)!}-\frac{n!}{(n-s-1)!}\right)|\gamma(A-B)-(1+B)+\gamma|\right]}{n|z|^{p-1}(|\gamma(A-B)+\gamma(B\gamma+1)-B-1|)}\right\}^{\frac{1}{n-p}},$$

which complete the proof.

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4. Conclusion

We have shown that subclass is associated with higher-order derivatives of multivalent functions. Many interesting results concerning the harmonic multivalent functions defined by differential operators. Finally, some geometric properties like a coefficient estimate, distortion and growth bounds, radii of starlikeness, convexity, and close to convexity.

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