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E-Duality Results For E-Differentiable E-Invex Multiobjective Programming Problems

Najeeb Abdulaleem^{1,2}

¹ Department of Mathematics, Hadhramout University, P.O. BOX : (50511-50512), Al-Mahrah, Yemen.

² Faculty of Mathematics and Computer Science, University of Łódź , Banacha 22, 90-238 Łódź , Poland.

E-mail: nabbas985@gmail.com

Abstract. In this paper, a class of E-differentiable multiobjective programming problems with both inequality and equality constraints is considered. The so-called vector Mond-Weir E-dual problem and vector mixed E-duality problem are defined for the considered E-differentiable multiobjective programming problem with both inequality and equality constraints and several Mond-Weir and mixed E-duality theorems are established under (generalized) E-invexity hypotheses.

1. Introduction

In recent years, attempts are made by several authors to define various classes of differentiable and nondifferentiable generalized convex functions in optimization theory and to study their optimality criteria and duality results for optimization problems in which the involved functions belong to such classes of nonconvex functions (see, for example, [4], [5], [6], [7], [9], [10], [13], and others).

One of the notions of generalized convexity introduced into optimization theory is the concept of E-convexity. The definitions of E-convex set and E-convex function were introduced by Youness [21]. This kind of generalized convexity is based on the effect of an operator $E: \mathbb{R}^n \rightarrow \mathbb{R}^n$ on the sets and the domain of functions. Megahed et al. [15] presented the concept of an E-differentiable convex function which transforms a (not necessarily) differentiable convex function to a differentiable function based on the effect of an operator $E: \mathbb{R}^n \rightarrow \mathbb{R}^n$. Recently, Abdulaleem [1] introduced a new concept of generalized convexity as a generalization of the notion of E-differentiable E-convexity. Namely, he defined the concept of E-differentiable E-invexity in the case of (not necessarily) differentiable vector optimization problems with E-differentiable functions.

In this paper, a class of nonconvex E-differentiable vector optimization problems with both inequality and equality constraints is considered in which the involved functions are E-invex. For such a (not necessarily differentiable) multiobjective programming problem, its vector Mond-Weir E-dual problem and vector mixed E-dual problem are defined. Then, several Mond-Weir and mixed E-duality results are established between the considered E-differentiable multicriteria optimization problem and its vector



E-duals under appropriate E-invexity hypotheses.

2. Preliminaries

Let R^n be the n -dimensional Euclidean space and R_+^n be its nonnegative orthant. The following convention for equalities and inequalities will be used in the paper. For any vectors $x = (x_1, x_2, \dots, x_n)^T$ and $y = (y_1, y_2, \dots, y_n)^T$ in R^n , we define:

- (i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, \dots, n$;
- (ii) $x > y$ if and only if $x_i > y_i$ for all $i = 1, 2, \dots, n$;
- (iii) $x \geq y$ if and only if $x_i \geq y_i$ for all $i = 1, 2, \dots, n$;
- (iv) $x \succ y$ if and only if $x \geq y$ and $x \neq y$.

Definition 1 [1] Let $E: R^n \rightarrow R^n$. A set $M \subseteq R^n$ is said to be an E-invex set if and only if there exists a vector-valued function $\eta: M \times M \rightarrow R^n$ such that the relation

$$E(u) + \lambda \eta(E(x), E(u)) \in M$$

holds for all $x, u \in M$ and any $\lambda \in [0, 1]$.

Remark 2 If η is a vector-valued function defined by $\eta(z, y) = z - y$, then the definition of an E-invex set reduces to the definition of an E-convex set (see Youness [21]).

Remark 3 If $E(a) \equiv a$, then the definition of an E-invex set with respect to the function η reduces to the definition of an invex set with respect to η (see Mohan and Neogy [17]).

Definition 4 [15] Let $E: R^n \rightarrow R^n$ and $f: M \rightarrow R$ be a (not necessarily) differentiable function at a given point $u \in M$. It is said that f is an E-differentiable function at u if and only if $f \circ E$ is a differentiable function at u (in the usual sense) and, moreover,

$$(f \circ E)(x) = (f \circ E)(u) + \nabla(f \circ E)(u)(x - u) + \theta(u, x - u)\|x - u\|, \quad (1)$$

where $\theta(u, x - u) \rightarrow 0$ as $x \rightarrow u$.

Definition 5 [1] Let $E: R^n \rightarrow R^n$, $M \subseteq R^n$ be a nonempty open E-invex set with respect to the vector-valued function $\eta: R^n \times R^n \rightarrow R^n$ and $f: R^n \rightarrow R^k$ be an E-differentiable function on M . It is said that f is a vector-valued E-invex function with respect to η at u on M if, for all $x \in M$,

$$f_i(E(x)) - f_i(E(u)) \geq \nabla f_i(E(u))\eta(E(x), E(u)), \quad i = 1, \dots, k. \quad (2)$$

If inequalities (2) hold for any $u \in M$, then f is E-invex with respect to η on M .

Remark 6 From Definition 5, there are the following special cases:

- a) If f is a differentiable function and $E(x) \equiv x$ (E is an identity map), then the definition of an E-invex function reduces to the definition of an invex function introduced by Hanson [12].

- b) If $\eta: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is defined by $\eta(x, u) = x - u$, then we obtain the definition of an E-differentiable E-convex vector-valued function introduced by Megahed et al. [15].
- c) If f is differentiable, $E(x) = x$ and $\eta(x, u) = x - u$, then the definition of an E-invex function reduces to the definition of a differentiable convex vector-valued function.
- d) If f is differentiable and $\eta(x, u) = x - u$, then we obtain the definition of a differentiable E-convex function introduced by Youness [21].

Definition 7 [1] Let $E: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $M \subseteq \mathbb{R}^n$ be an open E-invex set with respect to the vector-valued function $\eta: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $f: \mathbb{R}^n \rightarrow \mathbb{R}^k$ be an E-differentiable function on M . It is said that f is a vector-valued pseudo E-invex function with respect to η at u on M if, for all $x \in M$ and $i = 1, \dots, k$,

$$f_i(E(x)) < f_i(E(u)) \Rightarrow \nabla f_i(E(u))\eta(E(x), E(u)) < 0. \quad (3)$$

If (3) holds for any $u \in M$, then f is pseudo E-invex with respect to η on M .

Definition 8 [1] Let $E: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $M \subseteq \mathbb{R}^n$ be an open E-invex set with respect to the vector-valued function $\eta: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $f: \mathbb{R}^n \rightarrow \mathbb{R}^k$ be an E-differentiable function on M . It is said that f is a vector-valued quasi E-invex function with respect to η at u on M if, for all $x \in M$ and $i = 1, \dots, k$,

$$f_i(E(x)) - f_i(E(u)) \leq 0 \Rightarrow \nabla f_i(E(u))\eta(E(x), E(u)) \leq 0. \quad (4)$$

If (4) holds for any $u \in M$, then f is quasi E-invex with respect to η on M .

In this paper, we consider the following (not necessarily differentiable) multiobjective programming problem (MOP) with both inequality and equality constraints:

$$\begin{aligned} & \text{minimize} && f(x) = (f_1(x), \dots, f_p(x)) \\ & \text{subject to} && g_j(x) \leq 0, \quad j \in J = \{1, \dots, m\}, \\ & && h_t(x) = 0, \quad t \in T = \{1, \dots, q\}, \\ & && x \in \mathbb{R}^n, \end{aligned} \quad (\text{MOP})$$

where $f_i: \mathbb{R}^n \rightarrow \mathbb{R}$, $i \in I = \{1, \dots, p\}$, $g_j: \mathbb{R}^n \rightarrow \mathbb{R}$, $j \in J$, $h_t: \mathbb{R}^n \rightarrow \mathbb{R}$, $t \in T$, are real-valued functions defined on \mathbb{R}^n . We shall write $g := (g_1, \dots, g_m): \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $h := (h_1, \dots, h_q): \mathbb{R}^n \rightarrow \mathbb{R}^q$ for convenience.

For the purpose of simplifying our presentation, we will next introduce some notations which will be used frequently throughout this paper. Let

$$\Omega := \{x \in X: g_j(x) \leq 0, \quad j \in J, h_t(x) = 0, \quad t \in T\}$$

be the set of all feasible solutions of (MOP). Further, we denote by $J(x)$ the set of inequality constraint indices that are active at a feasible solution x , that is, $J(x) = \{j \in J: g_j(x) = 0\}$.

For such multicriterion optimization problems, the following concepts of (weak) Pareto optimal solutions are defined as follows:

Definition 9 A feasible point \bar{x} is said to be a weak Pareto (weakly efficient) solution of (MOP) if and

only if there exists no other feasible point x such that

$$f(x) < f(\bar{x}).$$

Definition 10 A feasible point \bar{x} is said to be a Pareto (efficient) solution of (MOP) if and only if there exists no other feasible point x such that

$$f(x) \leq f(\bar{x}).$$

Let $E: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a given one-to-one and onto operator. Throughout the paper, we shall assume that the functions constituting the considered multiobjective programming problem (MOP) are E -differentiable at any feasible solution.

Now, for the considered multiobjective programming problem (MOP), we define its associated differentiable vector optimization problem as follows:

$$\begin{aligned} & \text{minimize} && f(E(x)) = (f_1(E(x)), \dots, f_p(E(x))) \\ & \text{subject to} && g_j(E(x)) \leq 0, \quad j \in J = \{1, \dots, m\}, \\ & && h_t(E(x)) = 0, \quad t \in T = \{1, \dots, q\}, \\ & && x \in \mathbb{R}^n. \end{aligned} \quad (\text{VP}_E)$$

We call the problem (VP_E) an E -vector optimization problem associated to (MOP). Let

$$\Omega_E := \{x \in \mathbb{R}^n : g_j(E(x)) \leq 0, \quad j \in J, h_t(E(x)) = 0, \quad t \in T\}$$

be the set of all feasible solutions of (VP_E) . Since the functions constituting the problem (MOP) are assumed to be E -differentiable at any feasible solution of (MOP), by Definition 4, the functions constituting the E -vector optimization problem (VP_E) are differentiable at any its feasible solution (in the usual sense). Further, by $J_E(x)$, the set of inequality constraint indices that are active at a feasible solution $x \in \Omega_E$, that is, $J_E(x) = \{j \in J : (g_j \circ E)(x) = 0\}$.

Now, we give the definitions of a weak Pareto (weakly efficient) solution and a Pareto (efficient) solution of the vector optimization problem (VP_E) , which are, at the same time, a weak E -Pareto solution (weakly E -efficient solution) and an E -Pareto solution (E -efficient solution) of the considered multiobjective programming problem (MOP).

Definition 11 A feasible point $E(\bar{x})$ is said to be a weak E -Pareto solution (weakly E -efficient solution) of (MOP) if and only if there exists no other feasible point $E(x)$ such that

$$f(E(x)) < f(E(\bar{x})).$$

Definition 12 A feasible point $E(\bar{x})$ is said to be an E -Pareto solution (E -efficient solution) of (MOP) if and only if there exists no other feasible point $E(x)$ such that

$$f(E(x)) \leq f(E(\bar{x})).$$

Lemma 13 [3] Let $E: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a one-to-one and onto. Then $E(\Omega_E) = \Omega$.

Lemma 14 [3] Let $\bar{x} \in \Omega$ be a weak Pareto solution (Pareto solution) of the considered multiobjective

programming problem (MOP). Then, there exists $\bar{z} \in \Omega_E$ such that $\bar{x} = E(\bar{z})$ and \bar{z} is a weak Pareto (Pareto) solution of the E-vector optimization problem (VP_E).

Lemma 15 [3] Let $\bar{z} \in \Omega_E$ be a weak Pareto (Pareto) solution of the E-vector optimization problem (VP_E). Then $E(\bar{z})$ is a weak Pareto solution (Pareto solution) of the considered multiobjective programming problem (MOP).

Remark 16 As it follows from Lemma 15, if $\bar{z} \in \Omega_E$ is a weak Pareto (Pareto) solution of the E-vector optimization problem (VP_E), then $E(\bar{z})$ is a weak Pareto solution (Pareto solution) of the considered multiobjective programming problem (MOP). We call $E(\bar{z})$ a weak E-Pareto (E-Pareto) solution of the problem (MOP).

As it follows from the above lemmas, there is some equivalence between the vector optimization problem (MOP) and (VP_E). Therefore, if we prove optimality results for the differentiable E-vector optimization problem (VP_E), they will be applicable also for the original nondifferentiable multiobjective programming problem (MOP) in which the involved functions are E-differentiable.

Definition 17 The tangent cone (also called contingent cone or Bouligand cone) of Ω_E at $\bar{x} \in \text{cl } \Omega_E$ is defined by

$$T_{\Omega_E}(\bar{x}) = \{d \in \mathbb{R}^n : \exists \{d_n\} \subset \mathbb{R}^n \downarrow d, \exists \{t_n\} \subset \mathbb{R}^+ \downarrow 0 \text{ s.t. } \bar{x} + t_n d_n \in \Omega_E\}.$$

Definition 18 For the constrained E-vector optimization problem (VP_E), the E-linearized cone at $\bar{x} \in \Omega_E$, denoted by $L_E(\bar{x})$, is defined by

$$L_E(\bar{x}) = \{d \in \mathbb{R}^n : \nabla g_j(E(\bar{x}))d \leq 0, j \in J_E(\bar{x}), \nabla h_t(E(\bar{x}))d = 0, t \in T\}.$$

Now, we present the E-Guignard constraint qualification which were derived for E-differentiable multiobjective programming problems with both inequality and equality constraints by Abdulaleem [1].

Definition 19 [1] It is said that the so-called E-Guignard constraint qualification (GCQ_E) holds at $\bar{x} \in \Omega_E$ for the differentiable constrained E-vector optimization problem (VP_E) with both inequality and equality constraints if

$$\text{cl conv } T_{\Omega_E}(\bar{x}) = L_E(\bar{x}). \quad (5)$$

Now, we present the Karush-Kuhn-Tucker necessary optimality conditions for $\bar{x} \in \Omega_E$ to be a weak Pareto solution of the E-vector optimization problem (VP_E). These conditions are, at the same time, the E-Karush-Kuhn-Tucker necessary optimality conditions for $E(\bar{x}) \in \Omega$ to be a weak E-Pareto solution of the considered E-differentiable multiobjective programming problem (MOP).

Theorem 20 [1] (E-Karush-Kuhn-Tucker necessary optimality conditions). Let $\bar{x} \in \Omega_E$ be a weak Pareto solution of the E-vector optimization problem (VP_E) (and, thus, $E(\bar{x})$ be a weak E-Pareto solution of the considered multiobjective programming problem (MOP)). Further, f, g, h be E-differentiable at \bar{x} and the E-Guignard constraint qualification be satisfied at \bar{x} . Then there exist Lagrange multipliers $\bar{\lambda} \in \mathbb{R}^p, \bar{\mu} \in \mathbb{R}^m, \bar{\xi} \in \mathbb{R}^s$ such that

$$\sum_{i=1}^p \bar{\lambda}_i \nabla f_i(E(\bar{x})) + \sum_{j=1}^m \bar{\mu}_j \nabla g_j(E(\bar{x})) + \sum_{t=1}^s \bar{\xi}_t \nabla h_t(E(\bar{x})) = 0, \quad (6)$$

$$\bar{\mu}_j g_j(E(\bar{x})) = 0, \quad j \in J(E(\bar{x})), \quad (7)$$

$$\bar{\lambda} \geq 0, \bar{\mu} \geq 0. \quad (8)$$

3. Mond-Weir E-duality

In this section, for the differentiable vector E-optimization problem (VP_E), we define its vector Mond-Weir dual problem. In other words, for the considered E-differentiable multiobjective programming problem (MOP), we define its vector E-dual problem (MWD_E) in the sense of Mond-Weir [16]. Then we prove several E-duality results between vector optimization problems (MOP) and (MWD_E) under appropriate E-invexity hypotheses.

Let $E: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a given one-to-one and onto operator. We define the following vector dual problem in the sense of Mond-Weir related for the differentiable multicriteria E-optimization problem (VP_E):

$$\begin{aligned} (f \circ E)(y) &= \left(f_1(E(y)), \dots, f_p(E(y)) \right) \rightarrow V - \max \\ \text{s. t. } \sum_{i=1}^p \lambda_i \nabla(f_i \circ E)(y) + \sum_{j=1}^m \mu_j \nabla(g_j \circ E)(y) + \sum_{t=1}^q \xi_t \nabla(h_t \circ E)(y) &= 0, \\ \sum_{j=1}^m \mu_j (g_j \circ E)(y) + \sum_{t=1}^q \xi_t (h_t \circ E)(y) &\geq 0, \quad (\text{MWD}_E) \\ \lambda \in \mathbb{R}^p, \lambda \geq 0, \mu \in \mathbb{R}^m, \mu \geq 0, \xi \in \mathbb{R}^q, & \end{aligned}$$

where all functions are defined in the similar way as for the considered E-vector optimization problem (VP_E). Further, let

$$\begin{aligned} \Gamma_E &= \{(y, \lambda, \mu, \xi) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^q : \\ \sum_{i=1}^p \lambda_i \nabla(f_i \circ E)(y) + \sum_{j=1}^m \mu_j \nabla(g_j \circ E)(y) + \sum_{t=1}^q \xi_t \nabla(h_t \circ E)(y) &= 0, \\ \sum_{j=1}^m \mu_j (g_j \circ E)(y) + \sum_{t=1}^q \xi_t (h_t \circ E)(y) &\geq 0, \lambda \geq 0, \mu \geq 0\} \end{aligned}$$

be the set of all feasible solutions of the problem (MWD_E). Let us denote, $Y_E = \{y \in \mathbb{R}^n : (y, \lambda, \mu, \xi) \in \Gamma_E\}$. The formulated vector dual problem (MWD_E) is the vector Mond-Weir dual problem for the vector E-optimization problem (VP_E). At the same time, we call (MWD_E) vector Mond-Weir E-dual problem or vector E-dual problem in the sense of Mond-Weir for the considered E-differentiable multiobjective programming problem (MOP).

Now, under E-invexity hypotheses, we prove duality results in the sense of Mond-Weir between the E-vector problems (VP_E) and (MWD_E) and, thus, E-duality results in the sense of Mond-Weir between the problems (MOP) and (MWD_E).

Theorem 21 (Mond-Weir weak duality between (VP_E) and (MWD_E)). Let x and (y, λ, μ, ξ) be any feasible solutions of the problems (VP_E) and (MWD_E), respectively. Further, assume that at least one of the following hypotheses is fulfilled:

- A) each objective function f_i , $i \in I$, is E-invex at y on $\Omega_E \cup Y_E$, each constraint function g_j , $j \in J$, is an E-invex function at y on $\Omega_E \cup Y_E$, the functions h_t , $t \in T^*(E(y))$ and the functions $-h_t$,

$t \in T^-(E(y))$, are E -invex at y on $\Omega_E \cup Y_E$.

- B) $(f \circ E)(y)$ is pseudo E -invex at y on $\Omega_E \cup Y_E$, $\mu_j(g_j \circ E)(y)$ is quasi E -invex at y on $\Omega_E \cup Y_E$, $\xi_t(h_t \circ E)(y)$ is quasi E -invex at y on $\Omega_E \cup Y_E$.

Then

$$(f \circ E)(x) \preccurlyeq (f \circ E)(y). \quad (9)$$

Proof. Let x and (y, λ, μ, ξ) be any feasible solutions of the problems (VP_E) and (VMD_E) , respectively.

The proof of this theorem under hypothesis A). If $x = y$, then the weak duality trivially holds. Now, we prove the weak duality theorem when $x \neq y$. We proceed by contradiction. Suppose, contrary to the result, that the inequality

$$(f \circ E)(x) < (f \circ E)(y) \quad (10)$$

holds. By the feasibility of (y, λ, μ, ξ) in the problem (MWD_E) , the above inequality yields

$$\sum_{i=1}^p \lambda_i \nabla f_i(E(x)) < \sum_{i=1}^p \lambda_i \nabla f_i(E(y)). \quad (11)$$

By assumption, x and (y, λ, μ, ξ) are feasible solutions for the problems (VP_E) and (MWD_E) , respectively. Since the functions f_i , $i \in I$, g_j , $j \in J$, h_t , $t \in T^+$, $-h_t$, $t \in T^-$, are E -invex at y on $\Omega_E \cup Y_E$, by Definition 5, the following inequalities

$$f_i(E(x)) - f_i(E(y)) \geq \nabla f_i(E(y)) \eta(E(x), E(y)), i \in I, \quad (12)$$

$$g_j(E(x)) - g_j(E(y)) \geq \nabla g_j(E(y)) \eta(E(x), E(y)), j \in J_E(y), \quad (13)$$

$$h_t(E(x)) - h_t(E(y)) \geq \nabla h_t(E(y)) \eta(E(x), E(y)), t \in T^+(E(y)), \quad (14)$$

$$-h_t(E(x)) + h_t(E(y)) \geq -\nabla h_t(E(y)) \eta(E(x), E(y)), t \in T^-(E(y)) \quad (15)$$

hold, respectively. Multiplying inequalities (12)-(15) by the corresponding Lagrange multiplier, respectively, we obtain that the inequality

$$\lambda_i f_i(E(x)) - \lambda_i f_i(E(y)) \geq \lambda_i \nabla f_i(E(y)) \eta(E(x), E(y)), i \in I, \quad (16)$$

$$\mu_j g_j(E(x)) - \mu_j g_j(E(y)) \geq \mu_j \nabla g_j(E(y)) \eta(E(x), E(y)), j \in J_E(y), \quad (17)$$

$$\xi_t h_t(E(x)) - \xi_t h_t(E(y)) \geq \xi_t \nabla h_t(E(y)) \eta(E(x), E(y)), t \in T^+(E(y)), \quad (18)$$

$$-\xi_t h_t(E(x)) + \xi_t h_t(E(y)) \geq -\xi_t \nabla h_t(E(y)) \eta(E(x), E(y)), t \in T^-(E(y)) \quad (19)$$

hold, respectively. Then adding both sides of (16)-(19), we obtain that the inequality

$$\begin{aligned} & \sum_{i=1}^p \lambda_i f_i(E(x)) - \sum_{i=1}^p \lambda_i f_i(E(y)) + \sum_{j=1}^m \mu_j g_j(E(x)) - \\ & \sum_{j=1}^m \mu_j g_j(E(y)) + \sum_{\tau=1}^q \xi_\tau h_\tau(E(x)) - \sum_{\tau=1}^q \xi_\tau h_\tau(E(y)) \geq \\ & \left[\sum_{i=1}^p \lambda_i \nabla f_i(E(y)) + \sum_{j=1}^m \mu_j \nabla g_j(E(y)) + \sum_{\tau=1}^q \xi_\tau \nabla h_\tau(E(y)) \right] \eta(E(x), E(y)) \end{aligned}$$

holds. Thus, by $x \in \Omega_E$, it follows that the following inequality

$$\begin{aligned} & \sum_{i=1}^p \lambda_i f_i(E(x)) - \sum_{i=1}^p \lambda_i f_i(E(y)) - \sum_{j=1}^m \mu_j g_j(E(y)) - \sum_{\tau=1}^q \xi_\tau h_\tau(E(y)) \geq \\ & \left[\sum_{i=1}^p \lambda_i \nabla f_i(E(y)) + \sum_{j=1}^m \mu_j \nabla g_j(E(y)) + \sum_{\tau=1}^q \xi_\tau \nabla h_\tau(E(y)) \right] \eta(E(x), E(y)) \end{aligned} \quad (20)$$

holds. Thus, by (20) and the first constraint of (MWD_E) , we have

$$\sum_{i=1}^p \lambda_i f_i(E(x)) - \sum_{i=1}^p \lambda_i f_i(E(y)) - \sum_{j=1}^m \mu_j g_j(E(y)) - \sum_{\tau=1}^q \xi_\tau h_\tau(E(y)) \geq 0. \quad (21)$$

Since (y, λ, μ, ξ) is a feasible solutions of (MWD_E) , the inequality

$$\sum_{i=1}^p \lambda_i \nabla f_i(E(x)) \geq \sum_{i=1}^p \lambda_i \nabla f_i(E(y)) \quad (22)$$

holds. Hence, by (22) and $\lambda_i \geq 0$, $i = 1, 2, \dots, p$, $\sum_{i=1}^p \lambda_i = 1$, it follows that the inequality (11) cannot hold which means that the proof of the Mond-Weir weak duality theorem between the E-vector optimization problems (VP_E) and (MWD_E) is completed under hypothesis A).

The proof of this theorem under hypothesis B). We proceed by contradiction. Suppose, contrary to the result, that (10) holds. Since the function $(f \circ E)(\cdot)$ is pseudo E-invex at y on $\Omega_E \cup Y_E$, by Definition 7, the inequality

$$\sum_{i=1}^p \lambda_i \nabla (f_i \circ E)(y) \eta(E(x), E(y)) < 0 \quad (23)$$

holds. Since $\mu_j (g_j \circ E)(y)$ and $\xi_\tau \nabla (h_\tau \circ E)(y)$ are quasi E-invex at y on $\Omega_E \cup Y_E$, by the foregoing above relations, Definition 8 implies that the inequalities

$$\sum_{j=1}^m \mu_j \nabla (g_j \circ E)(y) \eta(E(x), E(y)) \leq 0, \quad (24)$$

$$\sum_{\tau=1}^q \xi_\tau \nabla (h_\tau \circ E)(y) \eta(E(x), E(y)) \leq 0. \quad (25)$$

hold, respectively. Combining (23), (24) and (25), it follows that the inequality

$$\left[\sum_{i=1}^p \lambda_i \nabla f_i(E(y)) + \sum_{j=1}^m \mu_j \nabla g_j(E(y)) + \sum_{\tau=1}^q \xi_\tau \nabla h_\tau(E(y)) \right] \eta(E(x), E(y)) < 0 \quad (26)$$

holds, contradicting the first constraint of the vector Mond-Weir E-dual problem (MWD_E) . This means that the proof of the Mond-Weir weak duality theorem between the E-vector optimization problems (VP_E) and (MWD_E) is completed under hypothesis B).

Theorem 22 (Mond-Weir weak E-duality between (MOP) and (MWD_E)). Let $E(x)$ and (y, λ, μ, ξ) be any feasible solutions of the problems (MOP) and (MWD_E) , respectively. Further, assume that all hypotheses

of Theorem 21 are fulfilled. Then, Mond-Weir weak E-duality between (MOP) and (MWD_E) holds, that is,

$$(f \circ E)(x) \prec (f \circ E)(y).$$

Proof. Let $E(x)$ and (y, λ, μ, ξ) be any feasible solutions of the problems (MOP) and (MWD_E), respectively. Then, by Lemma 13, it follows that x is any feasible solution of (VP_E). Since all hypotheses of Theorem 21 are fulfilled, the Mond-Weir weak E-duality theorem between the problems (MOP) and (MWD_E) follows directly from Theorem 21.

If some stronger E-invexity hypotheses are imposed on the functions constituting the considered E-differentiable multiobjective programming problem, then the stronger result is true.

Theorem 23 (Mond-Weir weak duality between (VP_E) and (MWD_E)). Let x and (y, λ, μ, ξ) be any feasible solutions of the problems (VP_E) and (MWD_E), respectively. Further, assume that at least one of the following hypotheses is fulfilled:

- A) each objective function f_i , $i \in I$, is strictly E-invex at y on $\Omega_E \cup Y_E$, each constraint function g_j , $j \in J$, is an E-invex function at y on $\Omega_E \cup Y_E$, the functions h_t , $t \in T^+(E(y))$ and the functions $-h_t$, $t \in T^-(E(y))$, are E-invex at y on $\Omega_E \cup Y_E$.
 - B) $(f \circ E)(y)$ is strictly pseudo E-invex at y on $\Omega_E \cup Y_E$, $\mu_j(g_j \circ E)(y)$ is quasi E-invex at y on $\Omega_E \cup Y_E$, $\xi_t(h_t \circ E)(y)$ is quasi E-invex at y on $\Omega_E \cup Y_E$.
- Then

$$(f \circ E)(x) \preceq (f \circ E)(y). \quad (27)$$

Theorem 24 (Mond-Weir weak E-duality between (MOP) and (MWD_E)). Let $E(x)$ and (y, λ, μ, ξ) be any feasible solutions of the problems (MOP) and (MWD_E), respectively. Further, assume that all hypotheses of Theorem 23 are fulfilled. Then, weak E-duality between (MOP) and (VMD_E) holds, that is,

$$(f \circ E)(x) \preceq (f \circ E)(y).$$

Theorem 25 (Mond-Weir strong duality between (VP_E) and (MWD_E) and also strong E-duality between (MOP) and (MWD_E)). Let $\bar{x} \in \Omega_E$ be a weak Pareto solution (Pareto solution) of the E-vector optimization problem (VP_E) (and, at thus, $E(\bar{x})$ be a weak E-Pareto solution (E-Pareto solution) of the E-vector optimization problem (MOP)). Further, assume that the E-Guignard constraint qualification (GCQ_E) be satisfied at \bar{x} . Then there exist $\bar{\lambda} \in R^p$, $\bar{\mu} \in R^m$, $\bar{\mu} \geq 0$, $\bar{\xi} \in R^q$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ is feasible for the problem (MWD_E) and the objective functions of (VP_E) and (MWD_E) are equal at these points. If also all hypotheses of the Mond-Weir weak duality (Theorem 21) Theorem 23 are satisfied, then $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ is a (weak) efficient solution of a maximum type in the problem (MWD_E).

In other words, if $E(\bar{x}) \in \Omega$ is a (weak) E-Pareto solution of the multiobjective programming problem (MOP), then $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ is a (weak) efficient solution of a maximum type in the dual problem (MWD_E) in the sense of Mond-Weir. This means that the Mond-Weir strong E-duality holds between the problems (MOP) and (MWD_E).

Proof. Since $\bar{x} \in \Omega_E$ is a weak Pareto solution of the problem (VP_E) and the E-Guignard constraint qualification (GCQ_E) is satisfied at \bar{x} , by Theorem 20, there exist $\bar{\lambda} \in R^p$, $\bar{\mu} \in R^m$, $\bar{\mu} \geq 0$, $\bar{\xi} \in R^q$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ is a feasible solution of the problem (MWD_E). This means that the objective functions of

(VP_E) and (MWD_E) are equal. If we assume that all hypotheses of the Mond-Weir weak duality (Theorem 21) are fulfilled, $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ is a (weak) efficient solution of a maximum type in the dual problem (MWD_E) in the sense of Mond-Weir.

Moreover, we have, by Lemma 13, that $E(\bar{x}) \in \Omega$. Since $\bar{x} \in \Omega_E$ is a weak Pareto solution of the problem (VP_E), by Lemma 15, it follows that $E(\bar{x})$ is a weak E-Pareto solution in the problem (MOP). Then, by the Mond-Weir strong duality between (VP_E) and (MWD_E), we conclude that also the Mond-Weir strong E-duality holds between the problems (MOP) and (MWD_E). This means that if $E(\bar{x}) \in \Omega$ is a weak E-Pareto solution of the problem (MOP), there exist $\bar{\lambda} \in R^p$, $\bar{\mu} \in R^m$, $\bar{\mu} \geq 0$, $\bar{\xi} \in R^q$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ is a weakly efficient solution of a maximum type in the Mond-Weir dual problem (MWD_E).

Theorem 26 (Mond-Weir converse duality between (VP_E) and (MWD_E)). Let $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ be a (weakly) efficient solution of a maximum type in Mond-Weir dual problem (MWD_E) such that $\bar{x} \in \Omega_E$. Moreover, assume that the objective functions f_i , $i \in I$, are (E-invex) strictly E-invex at \bar{x} on $\Omega_E \cup Y_E$, the constraint functions g_j , $j \in J$, are E-invex at \bar{x} on $\Omega_E \cup Y_E$, the functions h_t , $t \in T^+(E(\bar{x}))$ and the functions $-h_t$, $t \in T^-(E(\bar{x}))$, are E-invex at \bar{x} on $\Omega_E \cup Y_E$. Then \bar{x} is a (weak) Pareto solution of the problem (VP_E).

Proof. Let $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ be a (weakly) efficient solution of a maximum type in Mond-Weir dual problem (MWD_E) such that $\bar{x} \in \Omega_E$. By means of contradiction, we suppose that there exists $\bar{x} \in \Omega_E$ such that the inequality

$$(f \circ E)(\bar{x}) < (f \circ E)(\bar{x}) \quad (28)$$

holds. By the feasibility of $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ in the problem (MWD_E), the above inequality yields

$$\sum_{i=1}^p \bar{\lambda}_i \nabla f_i(E(\bar{x})) < \sum_{i=1}^p \bar{\lambda}_i \nabla f_i(E(\bar{x})). \quad (29)$$

Since the functions f_i , $i \in I$, g_j , $j \in J$, h_t , $t \in T^+$, $-h_t$, $t \in T^-$, are E-invex at \bar{x} on $\Omega_E \cup Y_E$, by Definition 5, the following inequalities

$$f_i(E(\bar{x})) - f_i(E(\bar{x})) \geq \nabla f_i(E(\bar{x})) \eta(E(\bar{x}), E(\bar{x})), i \in I, \quad (30)$$

$$g_j(E(\bar{x})) - g_j(E(\bar{x})) \geq \nabla g_j(E(\bar{x})) \eta(E(\bar{x}), E(\bar{x})), j \in J_E(\bar{x}), \quad (31)$$

$$h_t(E(\bar{x})) - h_t(E(\bar{x})) \geq \nabla h_t(E(\bar{x})) \eta(E(\bar{x}), E(\bar{x})), t \in T^+(E(\bar{x})), \quad (32)$$

$$-h_t(E(\bar{x})) + h_t(E(\bar{x})) \geq -\nabla h_t(E(\bar{x})) \eta(E(\bar{x}), E(\bar{x})), t \in T^-(E(\bar{x})) \quad (33)$$

hold, respectively. Multiplying inequalities (30)-(33) by corresponding Lagrange multipliers, respectively, we obtain that the inequality

$$\bar{\lambda}_i f_i(E(\bar{x})) - \bar{\lambda}_i f_i(E(\bar{x})) \geq \bar{\lambda}_i \nabla f_i(E(\bar{x})) \eta(E(\bar{x}), E(\bar{x})), i \in I, \quad (34)$$

$$\bar{\mu}_j g_j(E(\bar{x})) - \bar{\mu}_j g_j(E(\bar{x})) \geq \bar{\mu}_j \nabla g_j(E(\bar{x})) \eta(E(\bar{x}), E(\bar{x})), j \in J_E(\bar{x}), \quad (35)$$

$$\bar{\xi}_t h_t(E(\bar{x})) - \bar{\xi}_t h_t(E(\bar{x})) \geq \bar{\xi}_t \nabla h_t(E(\bar{x})) \eta(E(\bar{x}), E(\bar{x})), t \in T^+(E(\bar{x})), \quad (36)$$

$$-\bar{\xi}_t h_t(E(\bar{x})) + \bar{\xi}_t h_t(E(\bar{x})) \geq -\bar{\xi}_t \nabla h_t(E(\bar{x})) \eta(E(\bar{x}), E(\bar{x})), t \in T^-(E(\bar{x})) \quad (37)$$

hold, respectively. Then, adding both sides of (34)-(37), we get that the inequality

$$\begin{aligned} & \sum_{i=1}^p \bar{\lambda}_i f_i(E(\bar{x})) - \sum_{i=1}^p \bar{\lambda}_i f_i(E(\bar{x})) + \sum_{j=1}^m \bar{\mu}_j g_j(E(\bar{x})) - \\ & \sum_{j=1}^m \bar{\mu}_j g_j(E(\bar{x})) + \sum_{t=1}^q \bar{\xi}_t h_t(E(\bar{x})) - \sum_{t=1}^q \bar{\xi}_t h_t(E(\bar{x})) \geq \\ & \left[\sum_{i=1}^p \bar{\lambda}_i \nabla f_i(E(\bar{x})) + \sum_{j=1}^m \bar{\mu}_j \nabla g_j(E(\bar{x})) + \sum_{t=1}^q \bar{\xi}_t \nabla h_t(E(\bar{x})) \right] \eta(E(\bar{x}), E(\bar{x})) \end{aligned}$$

holds. Thus, by $\bar{x} \in \Omega_E$, $\bar{x} \in \Omega_E$ and (29), the following inequality

$$\left[\sum_{i=1}^p \bar{\lambda}_i \nabla f_i(E(\bar{x})) + \sum_{j=1}^m \bar{\mu}_j \nabla g_j(E(\bar{x})) + \sum_{t=1}^q \bar{\xi}_t \nabla h_t(E(\bar{x})) \right] \eta(E(\bar{x}), E(\bar{x})) < 0 \quad (38)$$

holds, contradicting the feasibility of $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ in (MWD_E) . This means that the proof of the converse duality theorem between the E-vector optimization problems (VP_E) and (MWD_E) is completed.

Theorem 27 (Mond-Weir converse E-duality between (MOP) and (MWD_E)). Let $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ be a (weakly) efficient solution of a maximum type in Mond-Weir dual problem (MWD_E) . Further, assume that all hypotheses of Theorem 26 are fulfilled. Then $E(\bar{x}) \in \Omega$ is a (weak) E-Pareto solution of the problem (MOP) .

Proof. The proof of this theorem follows directly from Lemma 15 and Theorem 26.

Theorem 28 (Mond-Weir restricted converse duality between (VP_E) and (MWD_E)). Let \bar{x} be feasible of the considered E-differentiable multiobjective programming problem (VP_E) and $(\bar{y}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ be feasible of its vector Mond-Weir dual problem (MWD_E) . Moreover, assume that the functions f_i , $i \in I$, are strictly E-invex at \bar{y} on $\Omega_E \cup Y_E$, the constraint functions g_j , $j \in J$, are E-invex at \bar{y} on $\Omega_E \cup Y_E$, the functions h_t , $t \in T^+(E(\bar{y}))$ and functions $-h_t$, $t \in T^-(E(\bar{y}))$, are E-invex at \bar{y} on $\Omega_E \cup Y_E$ such that $f(E(\bar{x})) = f(E(\bar{y}))$. Then \bar{x} is a (weak) Pareto solution of the problem (VP_E) and $(\bar{y}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ is a (weakly) efficient point of a maximum type for the problem (MWD_E) .

Proof. By means of contradiction, suppose that \bar{x} is not a (weak) Pareto solution of the problem (VP_E) . This means, by Definition 11, that there exists $\bar{x} \in \Omega_E$ such that

$$f(E(\bar{x})) < f(E(\bar{x})). \quad (39)$$

By assumption, $f(E(\bar{x})) = f(E(\bar{y}))$. Hence, (39) yields

$$f(E(\bar{x})) < f(E(\bar{y})). \quad (40)$$

By $(\bar{y}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ is a feasible solution for (MWD_E). Then, it follows that $\bar{\lambda} \geq 0$. Hence, the above inequality yields

$$\sum_{i=1}^p \bar{\lambda}_i f_i(E(\bar{x})) < \sum_{i=1}^p \bar{\lambda}_i f_i(E(\bar{y})). \quad (41)$$

By assumption, the functions f_i , $i \in I$, g_j , $j \in J$, h_t , $t \in T^+$, $-h_t$, $t \in T^-$, are E-invex at \bar{y} on $\Omega_E \cup Y_E$. Then by Definition 5, the following inequalities

$$f_i(E(z)) - f_i(E(\bar{y})) \geq \nabla f_i(E(\bar{y})) \eta(E(z), E(\bar{y})), \quad i \in I, \quad (42)$$

$$g_j(E(z)) - g_j(E(\bar{y})) \geq \nabla g_j(E(\bar{y})) \eta(E(z), E(\bar{y})), \quad j \in J(E(\bar{y})), \quad (43)$$

$$h_t(E(z)) - h_t(E(\bar{y})) \geq \nabla h_t(E(\bar{y})) \eta(E(z), E(\bar{y})), \quad t \in T^+(E(\bar{y})), \quad (44)$$

$$-h_t(E(z)) + h_t(E(\bar{y})) \geq -\nabla h_t(E(\bar{y})) \eta(E(z), E(\bar{y})), \quad t \in T^-(E(\bar{y})) \quad (45)$$

hold for $z \in \Omega_E \cup Y_E$. Thus, they are also fulfilled for $z = \bar{x} \in \Omega_E$. Hence, (42)-(45) yield, respectively,

$$f_i(E(\bar{x})) - f_i(E(\bar{y})) \geq \nabla f_i(E(\bar{y})) \eta(E(\bar{x}), E(\bar{y})), \quad i \in I, \quad (46)$$

$$g_j(E(\bar{x})) - g_j(E(\bar{y})) \geq \nabla g_j(E(\bar{y})) \eta(E(\bar{x}), E(\bar{y})), \quad j \in J(E(\bar{y})), \quad (47)$$

$$h_t(E(\bar{x})) - h_t(E(\bar{y})) \geq \nabla h_t(E(\bar{y})) \eta(E(\bar{x}), E(\bar{y})), \quad t \in T^+(E(\bar{y})), \quad (48)$$

$$-h_t(E(\bar{x})) + h_t(E(\bar{y})) \geq -\nabla h_t(E(\bar{y})) \eta(E(\bar{x}), E(\bar{y})), \quad t \in T^-(E(\bar{y})). \quad (49)$$

By the feasibility of $(\bar{y}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ in (MWD_E), it follows that

$$\bar{\lambda}_i f_i(E(\bar{x})) - \bar{\lambda}_i f_i(E(\bar{y})) \geq \bar{\lambda}_i \nabla f_i(E(\bar{y})) \eta(E(\bar{x}), E(\bar{y})), \quad i \in I, \quad (50)$$

$$\bar{\mu}_j g_j(E(\bar{x})) - \bar{\mu}_j g_j(E(\bar{y})) \geq \bar{\mu}_j \nabla g_j(E(\bar{y})) \eta(E(\bar{x}), E(\bar{y})), \quad j \in J(E(\bar{y})), \quad (51)$$

$$\bar{\xi}_t h_t(E(\bar{x})) - \bar{\xi}_t h_t(E(\bar{y})) \geq \bar{\xi}_t \nabla h_t(E(\bar{y})) \eta(E(\bar{x}), E(\bar{y})), \quad t \in T^+(E(\bar{y})), \quad (52)$$

$$-\bar{\xi}_t h_t(E(\bar{x})) + \bar{\xi}_t h_t(E(\bar{y})) \geq -\bar{\xi}_t \nabla h_t(E(\bar{y})) \eta(E(\bar{x}), E(\bar{y})), \quad t \in T^-(E(\bar{y})). \quad (53)$$

Adding both sides of (50)-(53), we obtain

$$\begin{aligned} & \sum_{i=1}^p \bar{\lambda}_i f_i(E(\bar{x})) - \sum_{i=1}^p \bar{\lambda}_i f_i(E(\bar{y})) + \sum_{j=1}^m \bar{\mu}_j g_j(E(\bar{x})) - \\ & \sum_{j=1}^m \bar{\mu}_j g_j(E(\bar{y})) + \sum_{t=1}^q \bar{\xi}_t h_t(E(\bar{x})) - \sum_{t=1}^q \bar{\xi}_t h_t(E(\bar{y})) \geq \\ & [\sum_{i=1}^p \bar{\lambda}_i \nabla f_i(E(\bar{y})) + \sum_{j=1}^m \bar{\mu}_j \nabla g_j(E(\bar{y})) + \sum_{t=1}^q \bar{\xi}_t \nabla h_t(E(\bar{y}))] \eta(E(\bar{x}), E(\bar{y})) \end{aligned} \quad (54)$$

By (54), the first constraint of (MWD_E) and $\bar{x} \in \Omega_E$ imply that the following inequality

$$\sum_{i=1}^p f_i(E(\bar{x})) \geq \sum_{i=1}^p f_i(E(\bar{y})) \quad (55)$$

holds, contradicting (41). Then, $\bar{x} = \bar{y}$ and this means by Mond-Weir weak duality (Theorem 21) that \bar{x} is a weak Pareto solution of the problem (VP_E) and $(\bar{y}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ is a weakly efficient solution of a maximum type for the problem (MWD_E) . Thus, the proof of this theorem is completed.

Theorem 29 (Mond-Weir restricted converse E-duality between (MOP) and (MWD_E)). Let $(\bar{y}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ be a feasible solution of the problem (MWD_E) . Further, assume that there exist $E(\bar{x}) \in \Omega$ such that $\bar{x} = \bar{y}$. If all hypotheses of Theorem 28 are fulfilled, then $E(\bar{x})$ is an E-Pareto solution of the problem (MOP) and $(\bar{y}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ is a weakly efficient solution of maximum type for the problem (MWD_E) .

Proof. The proof of this theorem follows directly from Lemma 15 and Theorem 28.

4. Mixed E-duality

In this section, a vector mixed E-dual problem is defined for the considered E-differentiable multiobjective programming problem (MOP) with inequality and equality constraints.

Before we define the foregoing vector dual problem, we introduce some notations which will be helpful in presenting its formulation.

Let the index set J be partitioned into two disjoint subset J_1 and J_2 such that $J = J_1 \cup J_2$ and the index set T be partitioned into two disjoint subset T_1 and T_2 such that $T = T_1 \cup T_2$. Let J_1 be an index set such that $J_1 \subseteq J$ and $J_2 = J \setminus J_1$ and, moreover, $|J_1|$ and $|J_2|$ denote the cardinality of the index sets J_1 and J_2 , respectively. Further, let T_1 be an index set such that $T_1 \subseteq T$ and $T_2 = T \setminus T_1$ and, moreover, $|T_1|$ and $|T_2|$ denote the cardinality of the index sets T_1 and T_2 , respectively. Let us denote the set

$$\Omega^2 = \{x \in X: g_j(x) \leq 0, j \in J_2, h_t(x) = 0, t \in T_2\}.$$

Now, for the considered E-differentiable multiobjective programming problem (MOP) , we introduce the definition of the mixed scalar Lagrange function $L_E: \Omega_E \times R_+^p \times R_+^{|J_1|} \times R^{|T_1|} \rightarrow R$ as follows

$$L(x, \lambda, \mu_{J_1}, \xi_{T_1}) := \sum_{i=1}^p \lambda_i f_i(x) + \sum_{j \in J_1} \mu_j g_j(x) + \sum_{t \in T_1} \xi_t h_t(x). \quad (56)$$

Let us denote by e the vector of R^p whose components are all ones. Further, let $E: R^n \rightarrow R^n$ be a given one-to-one and onto operator. Further, let us define the following set

$$\Omega_E^2 = \{x \in X: (g_j \circ E)(x) \leq 0, j \in J_2, (h_t \circ E)(x) = 0, t \in T_2\}.$$

Now, we consider the following vector E-dual problem for the considered E-differentiable multiobjective programming problem (MOP) :

$$\begin{aligned}
& \text{maximize} && (f \circ E)(y) + [\mu_{J_1}(g_{J_1} \circ E)(y) + \xi_{T_1}(h_{T_1} \circ E)(y)]e \\
& \text{s. t.} && \sum_{i=1}^p \lambda_i \nabla(f_i \circ E)(y) + \sum_{j=1}^m \mu_j \nabla(g_j \circ E)(y) + \sum_{t=1}^q \xi_t \nabla(h_t \circ E)(y) = 0, \\
& && \mu_{J_2}(g_{J_2} \circ E)(y) \geq 0, && (\text{VMD}_E) \\
& && \xi_{T_2} \nabla(h_{T_2} \circ E)(y) = 0, \\
& && \lambda \in \mathbb{R}^p, \lambda \geq 0, \lambda e = 1, \mu \in \mathbb{R}^m, \mu \geq 0, \xi \in \mathbb{R}^q,
\end{aligned}$$

where all functions are defined in the similar way as for the considered vector optimization problem (MOP). Further, let Γ_E denote the set of all feasible solutions of (VMD_E), that is,

$$\begin{aligned}
\Gamma_E = \{ & (y, \lambda, \mu, \xi) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^q: \sum_{i=1}^p \lambda_i \nabla(f_i \circ E)(y) + \\
& \sum_{j=1}^m \mu_j \nabla(g_j \circ E)(y) + \sum_{t=1}^q \xi_t \nabla(h_t \circ E)(y) = 0, \sum_{j \in J_2} \mu_j (g_j \circ E)(y) \geq 0, \\
& \sum_{t \in T_2} \xi_t \nabla(h_t \circ E)(y) = 0, \lambda \geq 0, \lambda e = 1, \mu \geq 0\}.
\end{aligned}$$

Further, $Y_E = \{y \in X: (y, \lambda, \mu, \xi) \in \Gamma_E\}$. We call (VMD_E) the vector mixed E-dual problem for the E-differentiable multiobjective optimization problem (MOP).

Note that if set $J_1 = \emptyset$ and $T_1 = \emptyset$ in (VMD_E), then we get a vector Mond-Weir E-dual problem for (MOP) and, moreover, if we set $J_2 = \emptyset$ and $T_2 = \emptyset$ in (VMD_E), then we obtain a vector Wolfe E-dual problem for (MOP).

Now, we shall prove several mixed E-duality results between E-vector optimization problems (VP_E) and (VMD_E) under E-invexity assumptions. Then, we use these duality results in proving several mixed E-duality results between vector optimization problems (MOP) and (VMD_E).

Theorem 30 (Mixed weak duality between (VP_E) and (VMD_E)). Let x and (y, λ, μ, ξ) be any feasible solutions of the problems (VP_E) and (VMD_E), respectively. Further, assume that at least one of the following hypotheses is fulfilled:

- A) each objective function f_i , $i \in I$, is E-invex at y on $\Omega_E \cup Y_E$, each constraint function g_j , $j \in J$, is an E-invex function at y on $\Omega_E \cup Y_E$, the functions h_t , $t \in T^+(E(y))$ and the functions $-h_t$, $t \in T^-(E(y))$, are E-invex at y on $\Omega_E \cup Y_E$.
- B) $(f \circ E)(y) + [\mu_{J_1}(g_{J_1} \circ E)(y) + \xi_{T_1}(h_{T_1} \circ E)(y)]e$ is pseudo E-invex at y on $\Omega_E \cup Y_E$, $\mu_{J_2}(g_{J_2} \circ E)(y)$ is quasi E-invex at y on $\Omega_E \cup Y_E$, $\xi_{T_2} \nabla(h_{T_2} \circ E)(y)$ is quasi E-invex at y on $\Omega_E \cup Y_E$.

Then

$$(f \circ E)(x) \prec (f \circ E)(y) + [\mu_{J_1}(g_{J_1} \circ E)(y) + \xi_{T_1}(h_{T_1} \circ E)(y)]e. \quad (57)$$

Proof. Let x and (y, λ, μ, ξ) be any feasible solutions of the problems (VP_E) and (VMD_E), respectively. The proof of this theorem under hypothesis A). By means of contradiction, suppose that

$$(f \circ E)(x) < (f \circ E)(y) + [\mu_{J_1}(g_{J_1} \circ E)(y) + \xi_{T_1}(h_{T_1} \circ E)(y)]e.$$

Thus,

$$(f_i \circ E)(x) < (f_i \circ E)(y) + [\sum_{j \in J_1} \mu_j (g_j \circ E)(y) + \sum_{t \in T_1} \xi_t (h_t \circ E)(y)], i \in I. \quad (58)$$

Multiplying each inequality (58) by λ_i and then adding both sides of the resulting inequalities, we get

$$\sum_{i=1}^p \lambda_i (f_i \circ E)(x) < \sum_{i=1}^p \lambda_i (f_i \circ E)(y) + [\sum_{j \in J_1} \mu_j (g_j \circ E)(y) + \sum_{t \in T_1} \xi_t (h_t \circ E)(y)] \sum_{i=1}^p \lambda_i.$$

Since $\sum_{i=1}^p \lambda_i = 1$, the following inequality

$$\sum_{i=1}^p \lambda_i (f_i \circ E)(x) < \sum_{i=1}^p \lambda_i (f_i \circ E)(y) + \sum_{j \in J_1} \mu_j (g_j \circ E)(y) + \sum_{t \in T_1} \xi_t (h_t \circ E)(y)$$

holds. By $x \in \Omega_E$ and $(y, \lambda, \mu, \xi) \in \Gamma_E$, we have

$$\sum_{i=1}^p \lambda_i (f_i \circ E)(x) + \sum_{j \in J_1} \mu_j (g_j \circ E)(x) + \sum_{t \in T_1} \xi_t (h_t \circ E)(x) < \quad (59)$$

$$\sum_{i=1}^p \lambda_i (f_i \circ E)(y) + \sum_{j \in J_1} \mu_j (g_j \circ E)(y) + \sum_{t \in T_1} \xi_t (h_t \circ E)(y),$$

$$\sum_{j \in J_2} \mu_j (g_j \circ E)(x) \leq \sum_{j \in J_2} \mu_j (g_j \circ E)(y), \quad (60)$$

$$\sum_{t \in T_2} \xi_t (h_t \circ E)(x) = \sum_{t \in T_2} \xi_t (h_t \circ E)(y). \quad (61)$$

Combining (59), (60) and (61), we get

$$\sum_{i=1}^p \lambda_i (f_i \circ E)(x) + \sum_{j=1}^m \mu_j (g_j \circ E)(x) + \sum_{t=1}^q \xi_t (h_t \circ E)(x) < \quad (62)$$

$$\sum_{i=1}^p \lambda_i (f_i \circ E)(y) + \sum_{j=1}^m \mu_j (g_j \circ E)(y) + \sum_{t=1}^q \xi_t (h_t \circ E)(y).$$

Since the functions f_i , $i \in I$, g_j , $j \in J$, h_t , $t \in T^+$, $-h_t$, $t \in T^-$, are E -invex on $\Omega_E \cup Y_E$, by Definition 5, the following inequalities

$$(f_i \circ E)(x) - (f_i \circ E)(y) \geq \nabla(f_i \circ E)(y) \eta(E(x), E(y)), i \in I, \quad (63)$$

$$(g_j \circ E)(x) - (g_j \circ E)(y) \geq \nabla(g_j \circ E)(y) \eta(E(x), E(y)), j \in J_E(y), \quad (64)$$

$$(h_t \circ E)(x) - (h_t \circ E)(y) \geq \nabla(h_t \circ E)(y) \eta(E(x), E(y)), t \in T^+(E(y)), \quad (65)$$

$$-(h_t \circ E)(x) + (h_t \circ E)(y) \geq -\nabla(h_t \circ E)(y) \eta(E(x), E(y)), t \in T^-(E(y)) \quad (66)$$

hold, respectively. Multiplying inequalities (60)-(63) by corresponding Lagrange multipliers, respectively, and then adding both sides of resulting inequalities, we obtain

$$\begin{aligned} & \sum_{i=1}^p \lambda_i (f_i \circ E)(x) - \sum_{i=1}^p \lambda_i (f_i \circ E)(y) + \sum_{j=1}^m \mu_j (g_j \circ E)(x) - \sum_{j=1}^m \mu_j (g_j \circ E)(y) \\ & + \sum_{t=1}^q \xi_t (h_t \circ E)(x) - \sum_{t=1}^q \xi_t (h_t \circ E)(y) \geq [\sum_{i=1}^p \lambda_i \nabla (f_i \circ E)(y) + \\ & \sum_{j=1}^m \mu_j \nabla (g_j \circ E)(y) + \sum_{t=1}^q \xi_t \nabla (h_t \circ E)(y)] \eta(E(x), E(y)). \end{aligned}$$

Hence, by (62), the inequality above implies that the following inequality

$$[\sum_{i=1}^p \lambda_i \nabla (f_i \circ E)(y) + \sum_{j=1}^m \mu_j \nabla (g_j \circ E)(y) + \sum_{t=1}^q \xi_t \nabla (h_t \circ E)(y)] \eta(E(x), E(y)) < 0 \quad (67)$$

holds, contradicts the first constraint of the vector mixed E-dual problem (VMD_E). This means that the proof of the mixed weak duality theorem between the E-vector optimization problems (VP_E) and (VMD_E) is completed under hypothesis A).

The proof of this theorem under hypothesis B). We proceed by contradiction. Suppose, contrary to the result, that (58) holds. Since the function $(f \circ E)(\cdot) + [\mu_{j_1} (g_{j_1} \circ E)(\cdot) + \xi_{T_1} (h_{T_1} \circ E)(\cdot)]e$ is pseudo E-invex at y on $\Omega_E \cup Y_E$, by Definition 7, the inequality

$$[\sum_{i=1}^p \lambda_i \nabla (f_i \circ E)(y) + \sum_{j \in J_1} \mu_j \nabla (g_j \circ E)(y) + \sum_{t \in T_1} \xi_t \nabla (h_t \circ E)(y)] \eta(E(x), E(y)) < 0 \quad (68)$$

holds. From $x \in \Omega_E$ and $(y, \lambda, \mu, \xi) \in \Gamma_E$, the relations (60) and (61) are fulfilled. Since $\mu_{j_1} (g_{j_1} \circ E)(y)$ and $\xi_{T_1} (h_{T_1} \circ E)(y)$ are quasi E-invex at y on $\Omega_E \cup Y_E$, by the foregoing above relations, Definition 8 implies that the inequalities

$$\sum_{j \in J_1} \mu_j \nabla (g_j \circ E)(y) \eta(E(x), E(y)) \leq 0, \quad (69)$$

$$\sum_{t \in T_1} \xi_t \nabla (h_t \circ E)(y) \eta(E(x), E(y)) \leq 0. \quad (70)$$

hold, respectively. Combining (68), (69) and (70), it follows that the inequality (67) is fulfilled, contradicting the first constraint of the vector mixed E-dual problem (VMD_E). This means that the proof of the mixed weak duality theorem between the E-vector optimization problems (VP_E) and (VMD_E) is completed under hypothesis B).

Theorem 31 (Mixed weak E-duality between (MOP) and (VMD_E)). Let $E(x)$ and (y, λ, μ, ξ) be any feasible solutions of the problems (MOP) and (VMD_E), respectively. Further, assume that all hypotheses of Theorem 30 are fulfilled. Then, mixed weak E-duality between (MOP) and (VMD_E) holds, that is,

$$(f \circ E)(x) \preceq (f \circ E)(y) + [\mu_{j_1} (g_{j_1} \circ E)(y) + \xi_{T_1} (h_{T_1} \circ E)(y)]e.$$

Proof. Let $E(x)$ and (y, λ, μ, ξ) be any feasible solutions of the problems (MOP) and (VMD_E), respectively. Then, by Lemma 13, it follows that x is any feasible solution of (VP_E). Since all hypotheses of Theorem 30 are fulfilled, the mixed weak E-duality theorem between the problems (MOP) and (VMD_E) follows directly from Theorem 30.

If some stronger E-invexity hypotheses are imposed on the functions constituting the considered E-differentiable multiobjective programming problem, then the stronger result is true.

Theorem 32 (Mixed weak duality between (VP_E) and (VMD_E)). Let x and (y, λ, μ, ξ) be any feasible

solutions of the problems $(VP_{\mathbb{E}})$ and $(VMD_{\mathbb{E}})$, respectively. Further, assume that at least one of the following hypotheses is fulfilled:

- A) each objective function f_i , $i \in I$, is strictly E-invex at y on $\Omega_{\mathbb{E}} \cup Y_{\mathbb{E}}$, each constraint function g_j , $j \in J$, is an E-invex function at y on $\Omega_{\mathbb{E}} \cup Y_{\mathbb{E}}$, the functions h_t , $t \in T^+(E(y))$ and the functions $-h_t$, $t \in T^-(E(y))$, are E-invex at y on $\Omega_{\mathbb{E}} \cup Y_{\mathbb{E}}$.
- B) $(f \circ E)(y) + [\mu_{I_1}(g_{I_1} \circ E)(y) + \xi_{T_1}(h_{T_1} \circ E)(y)]e$ is strictly pseudo E-invex at y on $\Omega_{\mathbb{E}} \cup Y_{\mathbb{E}}$, $\mu_{I_2}(g_{I_2} \circ E)(y)$ is quasi E-invex at y on $\Omega_{\mathbb{E}} \cup Y_{\mathbb{E}}$, $\xi_{T_2}(h_{T_2} \circ E)(y)$ is quasi E-invex at y on $\Omega_{\mathbb{E}} \cup Y_{\mathbb{E}}$.

Then

$$(f \circ E)(x) \not\leq (f \circ E)(y) + [\mu_{I_1}(g_{I_1} \circ E)(y) + \xi_{T_1}(h_{T_1} \circ E)(y)]e. \quad (71)$$

Theorem 33 (Mixed weak E-duality between (MOP) and $(VMD_{\mathbb{E}})$). Let $E(x)$ and (y, λ, μ, ξ) be any feasible solutions of the problems (MOP) and $(VMD_{\mathbb{E}})$, respectively. Further, assume that all hypotheses of Theorem 32 are fulfilled. Then, mixed weak E-duality between (MOP) and $(VMD_{\mathbb{E}})$ holds, that is,

$$(F \circ E)(x) \not\leq (f \circ E)(y) + [\mu_{I_1}(g_{I_1} \circ E)(y) + \xi_{T_1}(h_{T_1} \circ E)(y)]e.$$

Theorem 34 (Mixed strong duality between $(VP_{\mathbb{E}})$ and $(VMD_{\mathbb{E}})$ and also strong E-duality between (MOP) and $(VMD_{\mathbb{E}})$). Let $\bar{x} \in \Omega_{\mathbb{E}}$ be a weak Pareto solution (Pareto solution) of the E-vector optimization problem $(VP_{\mathbb{E}})$ and the E-Guignard constraint qualification $(GCQ_{\mathbb{E}})$ be satisfied at \bar{x} . Then there exist $\bar{\lambda} \in R^p$, $\bar{\lambda} \neq 0$, $\bar{\mu} \in R^m$, $\bar{\mu} \geq 0$, $\bar{\xi} \in R^q$, $\bar{\xi} \geq 0$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ is feasible for the problem $(VMD_{\mathbb{E}})$ and the objective functions of $(VP_{\mathbb{E}})$ and $(VMD_{\mathbb{E}})$ are equal at these points. If also all hypotheses of the mixed weak duality (Theorem 30) Theorem 32 are satisfied, then $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ is a (weak) efficient solution of a maximum type in problem $(VMD_{\mathbb{E}})$.

In other words, if $E(\bar{x}) \in \Omega$ is a (weak) E-Pareto solution of the multiobjective programming problem (MOP), then $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ is a (weak) efficient solution of a maximum type in the vector mixed dual problem $(VMD_{\mathbb{E}})$. This means that the mixed strong E-duality holds between the problems (MOP) and $(VMD_{\mathbb{E}})$.

Proof. Since $\bar{x} \in \Omega_{\mathbb{E}}$ is a weak Pareto solution of the problem $(VP_{\mathbb{E}})$ and the E-Guignard constraint qualification $(GCQ_{\mathbb{E}})$ is satisfied at \bar{x} , by Theorem 20, there exist $\bar{\lambda} \in R^p$, $\bar{\lambda} \neq 0$, $\bar{\mu} \in R^m$, $\bar{\mu} \geq 0$, $\bar{\xi} \in R^q$, $\bar{\xi} \geq 0$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ is a feasible solution of the problem $(VMD_{\mathbb{E}})$. This means that the objective functions of $(VP_{\mathbb{E}})$ and $(VMD_{\mathbb{E}})$ are equal. If we assume that all hypotheses of mixed weak duality (Theorem 30) are fulfilled, $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ is a (weak) efficient solution of a maximum type in the dual problem $(VMD_{\mathbb{E}})$ in the sense of mixed.

Moreover, we have by Lemma 13, that $E(\bar{x}) \in \Omega$. Since $\bar{x} \in \Omega_{\mathbb{E}}$ is a weak Pareto solution of the problem $(VP_{\mathbb{E}})$, by Lemma 15, it follows that $E(\bar{x})$ is a weak E-Pareto solution in the problem (MOP). Then, by the strong duality between $(VP_{\mathbb{E}})$ and $(VMD_{\mathbb{E}})$, we conclude that also the mixed strong E-duality holds between the problems (MOP) and $(VMD_{\mathbb{E}})$. This means that if $E(\bar{x}) \in \Omega$ is a weak E-Pareto solution of the problem (MOP), there exist $\bar{\lambda} \in R^p$, $\bar{\lambda} \neq 0$, $\bar{\mu} \in R^m$, $\bar{\mu} \geq 0$, $\bar{\xi} \in R^q$, $\bar{\xi} \geq 0$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ is a weakly efficient solution of a maximum type in the mixed dual problem $(VMD_{\mathbb{E}})$.

Theorem 35 (Mixed converse duality between $(VP_{\mathbb{E}})$ and $(VMD_{\mathbb{E}})$). Let $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ be a (weakly) efficient solution of a maximum type in mixed dual problem $(VMD_{\mathbb{E}})$ such that $\bar{x} \in \Omega_{\mathbb{E}}$. Moreover, that the objective functions f_i , $i \in I$, are E-invex at \bar{x} on $\Omega_{\mathbb{E}} \cup Y_{\mathbb{E}}$, the constraint functions g_j , $j \in J$, are E-invex at \bar{x} on $\Omega_{\mathbb{E}} \cup Y_{\mathbb{E}}$, the functions h_t , $t \in T^*(E(\bar{x}))$ and the functions $-h_t$, $t \in T^-(E(\bar{x}))$, are E-invex at \bar{x} on $\Omega_{\mathbb{E}} \cup Y_{\mathbb{E}}$. Then \bar{x} is a (weak) Pareto solution of the problem $(VP_{\mathbb{E}})$.

Proof. Proof of this theorem follows directly from Theorem 30.

Theorem 36 (Mixed converse E-duality between (MOP) and $(VMD_{\mathbb{E}})$). Let $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\xi})$ be a (weakly) efficient solution of a maximum type in mixed dual problem $(VMD_{\mathbb{E}})$. Further, assume that all hypotheses of Theorem 35 are fulfilled. Then $E(\bar{x}) \in \Omega$ is a (weak) E-Pareto solution of the problem (MOP) .

Proof. The proof of this theorem follows directly from Lemma 15 and Theorem 35.

5. Conclusion

In this paper, the class of E-differentiable vector optimization problems with both inequality and equality constraints has been considered. The so-called vector Mond-Weir E-dual problem and vector mixed E-dual problem have been formulated for such E-differentiable multiobjective programming problems. Then, various E-duality theorems between the considered E-differentiable E-invex vector optimization problem and its Mond-Weir and mixed vector dual problems have been proved under (generalized) E-invexity hypotheses.

However, some interesting topics for further research remain. It would be of interest to investigate whether it is possible to prove similar results for other classes of E-differentiable vector optimization problems. We shall investigate these questions in subsequent papers.

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