Journal of Physics: Conference Series

PAPER • **OPEN ACCESS**

Application of Weyl Module In The Case Of Two Rows

To cite this article: Haytham R. Hassan and Niran Sabah Jasim 2018 *J. Phys.: Conf. Ser.* **1003** 012051

IOP Conf. Series: Journal of Physics: Conf. Series **1003** (2018) 012051 doi:10.1088/1742-6596/1003/1/012051

Application of Weyl Module In The Case Of Two Rows

Haytham R. Hassan 1, Niran Sabah Jasim 2

1 Department of Mathematic, College of Science, Al-Mustansiriyah University
2 Department of Mathematic, College of Education for Pure Science/ Ibn Al-Haitham, University of Baghdad

1 haythamhassaan@yahoo.com , 2 sabahniran @gmail.com

Abstract. The target of this work is to study the two rows resolution of Weyl module, and locate the terms and the exactness of the Weyl Resolution in the Case of Partition (8,7).

1. Introduction

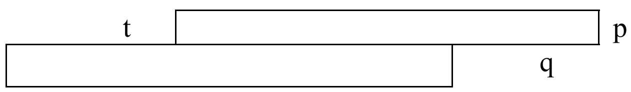
Let R be a commutative ring with identity (1) and $\hat{o}$ be a free R-module. The divided power algebra $D\hat{o} = \sum_{i\geq 0} D_i\hat{o}$ can be defined as the graded commutative algebra generated by x^{i} where $x \in \hat{o}$ and i is a non-negative integer, and $D_i\hat{o}$ is the divided power algebra of degree i.

A partition of length $\ell(\lambda) = n$ is a sequence $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ of non-negative integers in non-increasing order $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n > 0$. The weight of a partition λ is $|\lambda| = \lambda_1 + \lambda_2 + ... + \lambda_n$.

A relative sequence is a pair ((λ, μ) of sequence such that $\mu \leq \lambda$ and denoted by λ / μ.

If both λ and μ are partitions then the relative sequence λ / μ is called skew partition.

The authors in [1] remodel the resolution for the two rowed Weyl module $K_{\lambda/\mu}\hat{o}$ as:

$\lambda / \mu =$ 

For $K_{\lambda/\mu}\hat{o} = \mathrm{Im}(d'_{\lambda/\mu})$ where $d'_{\lambda/\mu} : D\hat{o} \longrightarrow \wedge\hat{o}$ (Weyl map), so we have

$$\Sigma D_{p+\kappa} \otimes D_{q-\kappa} \xrightarrow{\square} D_p \otimes D_q \xrightarrow{d'_{\lambda/\mu}} K_{\lambda/\mu \to 0}$$

And by using letter place the maps will be

$$\begin{pmatrix} w & 1^{(p+\kappa)} \\ w' & 2^{(q-\kappa)} \end{pmatrix} \xrightarrow{\partial_{21}^{(\kappa)}} \begin{pmatrix} w & 1^{(p)} \quad 2^{(\kappa)} \\ w' & 2^{(q-\kappa)} \end{pmatrix} \longrightarrow \sum_w \begin{pmatrix} w_{(1)} & (t+1)'(t+2)'...(p+t)' \\ w'w_{(2)} & 1'2'3'...q' \end{pmatrix},$$

where $w \otimes w' \in D_{p+\kappa} \otimes D_{q-\kappa}$, $\square = \sum_{\kappa=t+1}^{q} \partial_{21}^{(\kappa)}$ is the box map and $d'_{\lambda/\mu} = \partial_{q'2} ... \partial_{1'2} \partial_{(p+t)'1} ... \partial_{(t+1)'1}$ is the composition of place polarizations from position place {1,2} to negative place {1', 2',…,(p + t)'}.

In [2] the authors illustrate that □ transmit an element $x \otimes y$ of $D_{p+\kappa} \otimes D_{q-\kappa}$ to $\sum x_p \otimes x'_\kappa y$ where $\sum x_p \otimes x'_\kappa$ is the component of the diagonal of x in $D_p \otimes D_\kappa$.

The author in [3] introduces these notions as follows:

Let Z_{21} be the free generator of divided power algebra $D(Z_{21})$ in one generator. The divided power element $Z_{21}^{(\kappa)}$ of degree κ of the free generator Z_{21} acts on $D_{p+\kappa} \otimes D_{q-\kappa}$ by place polarization of degree κ from place 1 to place 2.

The (graded) algebra. $A = D(Z_{21})$ act on the graded module $M = D_{p+\kappa} \otimes D_{q-\kappa} = \Sigma M_{q-\kappa}$ (the degree of the second factor determines the grading).

M is a (graded) left A-module, where for $w = Z_{21}^{(\kappa)} \in A$ and $\upsilon \in D_{\beta_1} \otimes D_{\beta_2}$, so we have:

$$w(\upsilon) = Z_{21}^{(\kappa)}(\upsilon) = \partial_{21}^{(\kappa)}(\upsilon)$$

If we take (t^+) graded strand of degree q

$$M_\bullet : 0 \longrightarrow M_{q-t} \xrightarrow{\partial_s} ... \longrightarrow M_\ell \xrightarrow{\partial_s} ... M_1 \xrightarrow{\partial_s} M_0$$

Of the bar complex Bar (M, A; S, •) where $S = \{x\}$.

As in [4] we define the maps { S_i } as follows:

$$S_0 : D_p \otimes D_q \longrightarrow \sum_{\kappa>0} Z_{21}^{(t+\kappa)} x D_{p+t+\kappa} \otimes D_{q-t-\kappa}$$

$$\begin{pmatrix} w & 1^{(p)} \quad 2^{(\kappa)} \\ w' & 2^{(q-\kappa)} \end{pmatrix} \longrightarrow \begin{cases} 0 & \text{if } \kappa \le t \\ Z_{21}^{(\kappa)} x \begin{pmatrix} w & 1^{(p+\kappa)} \\ w' & 2^{(q-\kappa)} \end{pmatrix} & \text{if } \kappa > t \end{cases}$$

And for the higher dimensions as

$$S_{\ell-1} : \sum_{\kappa_i>0} Z_{21}^{(t+\kappa_1)} x Z_{21}^{(\kappa_2)} x \; ... \; Z_{21}^{(\kappa_{\ell-1})} x D_{p+t+|\kappa|} \otimes D_{q-t-|\kappa|} \longrightarrow$$
$$Z_{21}^{(t+\kappa_1)} x Z_{21}^{(\kappa_2)} x \; ... \; Z_{21}^{(\kappa_{\ell-1})} x Z_{21}^{(\kappa_\ell)} x D_{p+t+|\kappa|} \otimes D_{q-t-|\kappa|}$$

IOP Conf. Series: Journal of Physics: Conf. Series **1003** (2018) 012051 doi:10.1088/1742-6596/1003/1/012051

$$Z_{21}^{(t+\kappa_1)} x\, Z_{21}^{(\kappa_2)} x \;\ldots\; Z_{21}^{(\kappa_{\ell-1})} x \left(\begin{array}{c|cc} w & 1^{(p+t+|\kappa|)} & 2^{(m)} \\ w' & 2^{(q-t-|\kappa|-m)} & \end{array}\right) \longrightarrow$$

$$\begin{cases} 0 & \text{if } m = 0 \\ Z_{21}^{(t+\kappa_1)} x\, Z_{21}^{(\kappa_2)} x \;\ldots\; Z_{21}^{(\kappa_{\ell-1})} x\, Z_{21}^{(m)} x \left(\begin{array}{c|c} w & 1^{(p+t+|\kappa|+m)} \\ w' & 2^{(q-t-|\kappa|-m)} \end{array}\right) & \text{if } m > 0 \end{cases}$$

The authors in [1], [2] and [5] described the written of the modules of the resolution as:

M_i for i = 0,1,…, q – t, with

$M_0 = D_p \otimes D_q$

and

$M_i = Z_{21}^{(t+\kappa_1)} x\, Z_{21}^{(\kappa_2)} x \;\ldots\; Z_{21}^{(\kappa_i)} x\, D_{p+t+|\kappa|} \otimes D_{q-t-|\kappa|}$ for $i \geq 1$

2. Terms of the Weyl Resolution in the Case of Partition (8,7)

In this section we submit the terms of characteristic free resolution in the case of partition (8,7).

$M_0 = D_8 \otimes D_7$

$M_1 = Z_{21}^{(1)} x\, D_9 \otimes D_6 \oplus Z_{21}^{(2)} x\, D_{10} \otimes D_5 \oplus Z_{21}^{(3)} x\, D_{11} \otimes D_4 \oplus Z_{21}^{(4)} x\, D_{12} \otimes D_3 \oplus Z_{21}^{(5)} x\, D_{13} \otimes D_2 \oplus$
$Z_{21}^{(6)} x\, D_{14} \otimes D_1 \oplus Z_{21}^{(7)} x\, D_{15} \otimes D_0$

$M_2 = Z_{21}^{(1)} x\, Z_{21}^{(1)} x\, D_{10} \otimes D_5 \oplus Z_{21}^{(2)} x\, Z_{21}^{(1)} x\, D_{11} \otimes D_4 \oplus Z_{21}^{(1)} x\, Z_{21}^{(2)} x\, D_{11} \otimes D_4 \oplus Z_{21}^{(2)} x\, Z_{21}^{(2)} x\, D_{12} \otimes D_3 \oplus$
$Z_{21}^{(3)} x\, Z_{21}^{(1)} x\, D_{12} \otimes D_3 \oplus Z_{21}^{(1)} x\, Z_{21}^{(3)} x\, D_{12} \otimes D_3 \oplus Z_{21}^{(4)} x\, Z_{21}^{(1)} x\, D_{13} \otimes D_2 \oplus Z_{21}^{(1)} x\, Z_{21}^{(4)} x\, D_{13} \otimes D_2 \oplus$
$Z_{21}^{(3)} x\, Z_{21}^{(2)} x\, D_{13} \otimes D_2 \oplus Z_{21}^{(2)} x\, Z_{21}^{(3)} x\, D_{13} \otimes D_2 \oplus Z_{21}^{(5)} x\, Z_{21}^{(1)} x\, D_{14} \otimes D_1 \oplus Z_{21}^{(1)} x\, Z_{21}^{(5)} x\, D_{14} \otimes D_1 \oplus$
$Z_{21}^{(3)} x\, Z_{21}^{(3)} x\, D_{14} \otimes D_1 \oplus Z_{21}^{(2)} x\, Z_{21}^{(4)} x\, D_{14} \otimes D_1 \oplus Z_{21}^{(4)} x\, Z_{21}^{(2)} x\, D_{14} \otimes D_1 \oplus Z_{21}^{(6)} x\, Z_{21}^{(1)} x\, D_{15} \otimes D_0 \oplus$
$Z_{21}^{(1)} x\, Z_{21}^{(6)} x\, D_{15} \otimes D_0 \oplus Z_{21}^{(2)} x\, Z_{21}^{(5)} x\, D_{15} \otimes D_0 \oplus Z_{21}^{(5)} x\, Z_{21}^{(2)} x\, D_{15} \otimes D_0 \oplus Z_{21}^{(3)} x\, Z_{21}^{(4)} x\, D_{15} \otimes D_0 \oplus$
$Z_{21}^{(4)} x\, Z_{21}^{(3)} x\, D_{15} \otimes D_0$

$M_3 = Z_{21}^{(1)} x\, Z_{21}^{(1)} x\, Z_{21}^{(1)} x\, D_{11} \otimes D_4 \oplus Z_{21}^{(2)} x\, Z_{21}^{(1)} x\, Z_{21}^{(1)} x\, D_{12} \otimes D_3 \oplus Z_{21}^{(1)} x\, Z_{21}^{(2)} x\, Z_{21}^{(1)} x\, D_{12} \otimes D_3 \oplus$
$Z_{21}^{(1)} x\, Z_{21}^{(1)} x\, Z_{21}^{(2)} x\, D_{12} \otimes D_3 \oplus Z_{21}^{(3)} x\, Z_{21}^{(1)} x\, Z_{21}^{(1)} x\, D_{13} \otimes D_2 \oplus Z_{21}^{(1)} x\, Z_{21}^{(3)} x\, Z_{21}^{(1)} x\, D_{13} \otimes D_2 \oplus$
$Z_{21}^{(1)} x\, Z_{21}^{(1)} x\, Z_{21}^{(3)} x\, D_{13} \otimes D_2 \oplus Z_{21}^{(2)} x\, Z_{21}^{(2)} x\, Z_{21}^{(1)} x\, D_{13} \otimes D_2 \oplus Z_{21}^{(2)} x\, Z_{21}^{(1)} x\, Z_{21}^{(2)} x\, D_{13} \otimes D_2 \oplus$

IOP Conf. Series: Journal of Physics: Conf. Series **1003** (2018) 012051 doi:10.1088/1742-6596/1003/1/012051

$$Z_{21}^{(1)} x Z_{21}^{(2)} x Z_{21}^{(2)} x D_{13} \otimes D_{2} \oplus Z_{21}^{(4)} x Z_{21}^{(1)} x Z_{21}^{(1)} x D_{14} \otimes D_{1} \oplus Z_{21}^{(1)} x Z_{21}^{(4)} x Z_{21}^{(1)} x D_{14} \otimes D_{1} \oplus$$

$$Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(4)} x D_{14} \otimes D_{1} \oplus Z_{21}^{(3)} x Z_{21}^{(2)} x Z_{21}^{(1)} x D_{14} \otimes D_{1} \oplus Z_{21}^{(3)} x Z_{21}^{(1)} x Z_{21}^{(2)} x D_{14} \otimes D_{1} \oplus$$

$$Z_{21}^{(2)} x Z_{21}^{(3)} x Z_{21}^{(1)} x D_{14} \otimes D_{1} \oplus Z_{21}^{(2)} x Z_{21}^{(1)} x Z_{21}^{(3)} x D_{14} \otimes D_{1} \oplus Z_{21}^{(1)} x Z_{21}^{(2)} x Z_{21}^{(3)} x D_{14} \otimes D_{1} \oplus$$

$$Z_{21}^{(1)} x Z_{21}^{(3)} x Z_{21}^{(2)} x D_{14} \otimes D_{1} \oplus Z_{21}^{(2)} x Z_{21}^{(2)} x Z_{21}^{(2)} x D_{14} \otimes D_{1} \oplus Z_{21}^{(5)} x Z_{21}^{(1)} x Z_{21}^{(1)} x D_{15} \otimes D_{0} \oplus$$

$$Z_{21}^{(1)} x Z_{21}^{(5)} x Z_{21}^{(1)} x D_{15} \otimes D_{0} \oplus Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(5)} x D_{15} \otimes D_{0} \oplus Z_{21}^{(2)} x Z_{21}^{(2)} x Z_{21}^{(3)} x D_{15} \otimes D_{0} \oplus$$

$$Z_{21}^{(2)} x Z_{21}^{(3)} x Z_{21}^{(2)} x D_{15} \otimes D_{0} \oplus Z_{21}^{(3)} x Z_{21}^{(2)} x Z_{21}^{(2)} x D_{15} \otimes D_{0} \oplus Z_{21}^{(3)} x Z_{21}^{(3)} x Z_{21}^{(1)} x D_{15} \otimes D_{0} \oplus$$

$$Z_{21}^{(3)} x Z_{21}^{(1)} x Z_{21}^{(3)} x D_{15} \otimes D_{0} \oplus Z_{21}^{(1)} x Z_{21}^{(3)} x Z_{21}^{(3)} x D_{15} \otimes D_{0} \oplus Z_{21}^{(2)} x Z_{21}^{(4)} x Z_{21}^{(1)} x D_{15} \otimes D_{0} \oplus$$

$$Z_{21}^{(2)} x Z_{21}^{(1)} x Z_{21}^{(4)} x D_{15} \otimes D_{0} \oplus Z_{21}^{(4)} x Z_{21}^{(2)} x Z_{21}^{(1)} x D_{15} \otimes D_{0} \oplus Z_{21}^{(4)} x Z_{21}^{(1)} x Z_{21}^{(2)} x D_{15} \otimes D_{0} \oplus$$

$$Z_{21}^{(1)} x Z_{21}^{(4)} x Z_{21}^{(2)} x D_{15} \otimes D_{0} \oplus Z_{21}^{(1)} x Z_{21}^{(2)} x Z_{21}^{(4)} x D_{15} \otimes D_{0}$$

$$M_{4} = Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(1)} x D_{12} \otimes D_{3} \oplus Z_{21}^{(2)} x Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(1)} x D_{13} \otimes D_{2} \oplus$$

$$Z_{21}^{(1)} x Z_{21}^{(2)} x Z_{21}^{(1)} x Z_{21}^{(1)} x D_{13} \otimes D_{2} \oplus Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(2)} x Z_{21}^{(1)} x D_{13} \otimes D_{2} \oplus$$

$$Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(2)} x D_{13} \otimes D_{2} \oplus Z_{21}^{(3)} x Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(1)} x D_{14} \otimes D_{1} \oplus$$

$$Z_{21}^{(1)} x Z_{21}^{(3)} x Z_{21}^{(1)} x Z_{21}^{(1)} x D_{14} \otimes D_{1} \oplus Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(3)} x Z_{21}^{(1)} x D_{14} \otimes D_{1} \oplus$$

$$Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(3)} x D_{14} \otimes D_{1} \oplus Z_{21}^{(2)} x Z_{21}^{(2)} x Z_{21}^{(1)} x Z_{21}^{(1)} x D_{14} \otimes D_{1} \oplus$$

$$Z_{21}^{(2)} x Z_{21}^{(1)} x Z_{21}^{(2)} x Z_{21}^{(1)} x D_{14} \otimes D_{1} \oplus Z_{21}^{(2)} x Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(2)} x D_{14} \otimes D_{1} \oplus$$

$$Z_{21}^{(1)} x Z_{21}^{(2)} x Z_{21}^{(2)} x Z_{21}^{(1)} x D_{14} \otimes D_{1} \oplus Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(2)} x Z_{21}^{(2)} x D_{14} \otimes D_{1} \oplus$$

$$Z_{21}^{(1)} x Z_{21}^{(2)} x Z_{21}^{(1)} x Z_{21}^{(2)} x D_{14} \otimes D_{1} \oplus Z_{21}^{(4)} x Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(1)} x D_{15} \otimes D_{0} \oplus$$

$$Z_{21}^{(1)} x Z_{21}^{(4)} x Z_{21}^{(1)} x Z_{21}^{(1)} x D_{15} \otimes D_{0} \oplus Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(4)} x Z_{21}^{(1)} x D_{15} \otimes D_{0} \oplus$$

$$Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(4)} x D_{15} \otimes D_{0} \oplus Z_{21}^{(2)} x Z_{21}^{(3)} x Z_{21}^{(1)} x Z_{21}^{(1)} x D_{15} \otimes D_{0} \oplus$$

$$Z_{21}^{(2)} x Z_{21}^{(1)} x Z_{21}^{(3)} x Z_{21}^{(1)} x D_{15} \otimes D_{0} \oplus Z_{21}^{(2)} x Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(3)} x D_{15} \otimes D_{0} \oplus$$

$$Z_{21}^{(3)} x Z_{21}^{(2)} x Z_{21}^{(1)} x Z_{21}^{(1)} x D_{15} \otimes D_{0} \oplus Z_{21}^{(3)} x Z_{21}^{(1)} x Z_{21}^{(2)} x Z_{21}^{(1)} x D_{15} \otimes D_{0} \oplus$$

$$Z_{21}^{(3)} x Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(2)} x D_{15} \otimes D_{0} \oplus Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(2)} x Z_{21}^{(3)} x D_{15} \otimes D_{0} \oplus$$

$$Z_{21}^{(1)} x Z_{21}^{(1)} x Z_{21}^{(3)} x Z_{21}^{(2)} x D_{15} \otimes D_{0} \oplus Z_{21}^{(1)} x Z_{21}^{(2)} x Z_{21}^{(1)} x Z_{21}^{(3)} x D_{15} \otimes D_{0} \oplus$$

$$Z_{21}^{(1)} x Z_{21}^{(3)} x Z_{21}^{(1)} x Z_{21}^{(2)} x D_{15} \otimes D_{0} \oplus Z_{21}^{(1)} x Z_{21}^{(3)} x Z_{21}^{(2)} x Z_{21}^{(1)} x D_{15} \otimes D_{0} \oplus$$

$$Z_{21}^{(1)} x Z_{21}^{(2)} x Z_{21}^{(3)} x Z_{21}^{(1)} x D_{15} \otimes D_{0} \oplus Z_{21}^{(2)} x Z_{21}^{(2)} x Z_{21}^{(2)} x Z_{21}^{(1)} x D_{15} \otimes D_{0} \oplus$$

IOP Conf. Series: Journal of Physics: Conf. Series **1003** (2018) 012051 doi:10.1088/1742-6596/1003/1/012051

$Z_{21}^{(2)}xZ_{21}^{(2)}xZ_{21}^{(1)}xZ_{21}^{(2)}xD_{15}\otimes D_0\oplus Z_{21}^{(2)}xZ_{21}^{(1)}xZ_{21}^{(2)}xZ_{21}^{(2)}xD_{15}\otimes D_0\oplus$
$Z_{21}^{(1)}xZ_{21}^{(2)}xZ_{21}^{(2)}xZ_{21}^{(2)}xD_{15}\otimes D_0$

$M_5=Z_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xD_{13}\otimes D_2\oplus Z_{21}^{(2)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xD_{14}\otimes D_1\oplus$
$Z_{21}^{(1)}xZ_{21}^{(2)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xD_{14}\otimes D_1\oplus Z_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(2)}xZ_{21}^{(1)}xZ_{21}^{(1)}xD_{14}\otimes D_1\oplus$
$Z_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(2)}xZ_{21}^{(1)}xD_{14}\otimes D_1\oplus Z_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(2)}xD_{14}\otimes D_1\oplus$
$Z_{21}^{(3)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xD_{15}\otimes D_0\oplus Z_{21}^{(1)}xZ_{21}^{(3)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(2)}xD_{15}\otimes D_0\oplus$
$Z_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(3)}xZ_{21}^{(1)}xZ_{21}^{(1)}xD_{15}\otimes D_0\oplus Z_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(3)}xZ_{21}^{(1)}xD_{15}\otimes D_0\oplus$
$Z_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(3)}xD_{15}\otimes D_0\oplus Z_{21}^{(2)}xZ_{21}^{(2)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xD_{15}\otimes D_0\oplus$
$Z_{21}^{(2)}xZ_{21}^{(1)}xZ_{21}^{(2)}xZ_{21}^{(1)}xZ_{21}^{(1)}xD_{15}\otimes D_0\oplus Z_{21}^{(2)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(2)}xD_{15}\otimes D_0\oplus$
$Z_{21}^{(2)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(2)}xZ_{21}^{(1)}xD_{15}\otimes D_0\oplus Z_{21}^{(1)}xZ_{21}^{(2)}xZ_{21}^{(2)}xZ_{21}^{(1)}xZ_{21}^{(1)}xD_{15}\otimes D_0\oplus$
$Z_{21}^{(1)}xZ_{21}^{(2)}xZ_{21}^{(1)}xZ_{21}^{(2)}xZ_{21}^{(1)}xD_{15}\otimes D_0\oplus Z_{21}^{(1)}xZ_{21}^{(2)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(2)}xD_{15}\otimes D_0\oplus$
$Z_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(2)}xZ_{21}^{(2)}xZ_{21}^{(1)}xD_{15}\otimes D_0\oplus Z_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(2)}xZ_{21}^{(2)}xD_{15}\otimes D_0\oplus$
$Z_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(2)}xZ_{21}^{(1)}xZ_{21}^{(2)}xD_{15}\otimes D_0$

$M_6=Z_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xD_{14}\otimes D_1\oplus Z_{21}^{(2)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xD_{15}\otimes D_0\oplus$
$Z_{21}^{(1)}xZ_{21}^{(2)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xD_{15}\otimes D_0\oplus Z_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(2)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xD_{15}\otimes D_0\oplus$
$Z_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(2)}xZ_{21}^{(1)}xZ_{21}^{(1)}xD_{15}\otimes D_0\oplus Z_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(2)}xZ_{21}^{(1)}xD_{15}\otimes D_0\oplus$
$Z_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(2)}xD_{15}\otimes D_0$

$M_7=Z_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xZ_{21}^{(1)}xD_{15}\otimes D_0$

3. The Exactness of the Weyl Resolution in the Case of Partition (8,7)

This section clarify the construction of a contracting homotopies $\{S_i\}$ in the case of partition (8,7) where i = 1,2, …, 6 as follows:
$S_0: D_8\otimes D_7 \longrightarrow \sum_{\kappa>0} Z_{21}^{(\kappa)}xD_{8+\kappa}\otimes D_{7-\kappa}$ such that

IOP Conf. Series: Journal of Physics: Conf. Series **1003** (2018) 012051 doi:10.1088/1742-6596/1003/1/012051

$$S_0\left(\left(\begin{array}{c|cc} w & 1^{(8)} & 2^{(\kappa)} \\ w' & 2^{(7-\kappa)} & \end{array}\right)\right)=\begin{cases} 0 & \text{if } \kappa \le 0 \\ Z_{21}^{(\kappa)} x \begin{pmatrix} w & 1^{(8+\kappa)} \\ w' & 2^{(7-\kappa)} \end{pmatrix} & \text{if } \kappa = 1,2,\ldots,7 \end{cases}$$

$$S_1 : \sum_{\kappa>0} Z_{21}^{(\kappa)} x\, D_{8+\kappa} \otimes D_{7-\kappa} \longrightarrow Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2)} x\, D_{8+\kappa} \otimes D_{7-\kappa}$$

such that

$$S_1\left(Z_{21}^{(\kappa)} x \left(\begin{array}{c|cc} w & 1^{(8+\kappa)} & 2^{(m)} \\ w' & 2^{(7-\kappa-m)} & \end{array}\right)\right)=\begin{cases} 0 & \text{if } m=0 \\ Z_{21}^{(\kappa)} x\, Z_{21}^{(m)} x \begin{pmatrix} w & 1^{(8+\kappa+m)} \\ w' & 2^{(7-\kappa-m)} \end{pmatrix} & \text{if } m=1,2,\ldots,6 \end{cases}$$

$$S_2 : \sum_{\kappa_i>0} Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2)} x\, D_{8+|\kappa|} \otimes D_{7-|\kappa|} \longrightarrow Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2)} x\, Z_{21}^{(\kappa_3)} x\, D_{8+|\kappa|} \otimes D_{7-|\kappa|}$$

such that

$$S_2\left(Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2)} x \left(\begin{array}{c|cc} w & 1^{(8+|\kappa|)} & 2^{(\wp)} \\ w' & 2^{(7-|\kappa|-m)} & \end{array}\right)\right)=\begin{cases} 0 & \text{if } m=0 \\ Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2)} x\, Z_{21}^{(m)} x \begin{pmatrix} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{pmatrix} & \text{if } m=1,2,\ldots,5 \end{cases},$$

where $|\kappa| = \kappa_1 + \kappa_2$.

$$S_3 : \sum_{\kappa_i>0} Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2)} x\, Z_{21}^{(\kappa_3)} x\, D_{8+|\kappa|} \otimes D_{7-|\kappa|} \longrightarrow Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2)} x\, Z_{21}^{(\kappa_3)} x\, Z_{21}^{(\kappa_4)} x\, D_{8+|\kappa|} \otimes D_{7-|\kappa|}$$

such that

$$S_3\left(Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2)} x\, Z_{21}^{(\kappa_3)} x \left(\begin{array}{c|cc} w & 1^{(8+|\kappa|)} & 2^{(m)} \\ w' & 2^{(7-|\kappa|-m)} & \end{array}\right)\right)=\begin{cases} 0 & \text{if } m=0 \\ Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2)} x\, Z_{21}^{(\kappa_3)} x\, Z_{21}^{(m)} x \begin{pmatrix} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{pmatrix} & \text{if } m=1,2,3,4 \end{cases},$$

where $|\kappa| = \kappa1 + \kappa2 + \kappa3$.

$$S_4 : \sum_{\kappa_i>0} Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2)} x\, Z_{21}^{(\kappa_3)} x\, Z_{21}^{(\kappa_4)} x\, D_{8+|\kappa|} \otimes D_{7-|\kappa|} \longrightarrow Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2)} x\, Z_{21}^{(\kappa_3)} x\, Z_{21}^{(\kappa_4)} x\, Z_{21}^{(\kappa_5)} x\, D_{8+|\kappa|} \otimes D_{7-|\kappa|}$$

such that

$$S_4\left(Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2)} x\, Z_{21}^{(\kappa_3)} x\, Z_{21}^{(\kappa_4)} x \left(\begin{array}{c|cc} w & 1^{(8+|\kappa|)} & 2^{(m)} \\ w' & 2^{(7-|\kappa|-m)} & \end{array}\right)\right)$$

$$=\begin{cases} 0 & \text{if } m=0 \\ Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2)} x\, Z_{21}^{(\kappa_3)} x\, Z_{21}^{(\kappa_4)} x\, Z_{21}^{(m)} x \begin{pmatrix} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{pmatrix} & \text{if } m=1,2,3 \end{cases},$$

where $|\kappa| = \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4$.

$$S_5 : \sum_{\kappa_i > 0} Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5)} x D_{8+|\kappa|} \otimes D_{7-|\kappa|} \rightarrow Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5)} x Z_{21}^{(\kappa_6)} x D_{8+|\kappa|} \otimes D_{7-|\kappa|}$$

such that

$$S_5\left(Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5)} x \begin{pmatrix} w & \left| \begin{matrix} 1^{(8+|\kappa|)} & 2^{(m)} \\ 2^{(7-|\kappa|-m)} & \end{matrix} \right. \\ w' & \end{pmatrix} \right)$$

$$= \begin{cases} 0 & \text{if } m = 0 \\ Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5)} x Z_{21}^{(m)} x \begin{pmatrix} w & \left| 1^{(8+|\kappa|+m)} \right. \\ w' & \left| 2^{(7-|\kappa|-m)} \right. \end{pmatrix} & \text{if } m = 1,2 \end{cases},$$

where $|\kappa| = \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 + \kappa_5$.

$$S_6 : \sum_{\kappa_i > 0} Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5)} x Z_{21}^{(\kappa_6)} x D_{8+|\kappa|} \otimes D_{7-|\kappa|} \longrightarrow$$
$$Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5)} x Z_{21}^{(\kappa_6)} x Z_{21}^{(\kappa_7)} x D_{8+|\kappa|} \otimes D_{7-|\kappa|}$$

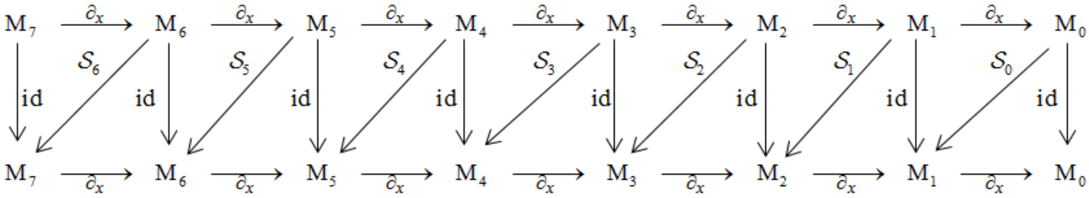
such that

$$S_6\left(Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5)} x Z_{21}^{(\kappa_6)} x \begin{pmatrix} w & \left| \begin{matrix} 1^{(8+|\kappa|)} & 2^{(m)} \\ 2^{(7-|\kappa|-m)} & \end{matrix} \right. \\ w' & \end{pmatrix} \right)$$

$$= \begin{cases} 0 & \text{if } m = 0 \\ Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5)} x Z_{21}^{(\kappa_6)} x Z_{21}^{(m)} x \begin{pmatrix} w & \left| 1^{(8+|\kappa|+m)} \right. \\ w' & \left| 2^{(7-|\kappa|-m)} \right. \end{pmatrix} & \text{if } m = 1 \end{cases},$$

where $|\kappa| = \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 + \kappa_5 + \kappa_6$.

So we possess the posterior scheme:-



Now we possess

$$S_0 \partial_x \left(Z_{21}^{(\kappa)} x \begin{pmatrix} w & \left| \begin{matrix} 1^{(8+\kappa)} & 2^{(m)} \\ 2^{(7-\kappa-m)} & \end{matrix} \right. \\ w' & \end{pmatrix} \right) = S_0 \partial_{21}^{(\kappa)} \begin{pmatrix} w & \left| \begin{matrix} 1^{(8+\kappa)} & 2^{(m)} \\ 2^{(7-\kappa-m)} & \end{matrix} \right. \\ w' & \end{pmatrix} = \binom{\kappa + m}{m} Z_{21}^{(\kappa+m)} x \begin{pmatrix} w & \left| 1^{(8+\kappa+m)} \right. \\ w' & \left| 2^{(7-\kappa-m)} \right. \end{pmatrix},$$

and

$$\partial_x S_1\left(Z_{21}^{(\kappa)} x \left(\begin{array}{c|cc} w & 1^{(8+\kappa)} & 2^{(m)} \\ w' & 2^{(7-\kappa-m)} & \end{array}\right)\right) = \partial_x \left(Z_{21}^{(\kappa)} x\, Z_{21}^{(m)} x \left(\begin{array}{c|c} w & 1^{(8+\kappa+m)} \\ w' & 2^{(7-\kappa-m)} \end{array}\right)\right)$$

$$= -\binom{\kappa+m}{m} Z_{21}^{(\kappa+m)} \left(\begin{array}{c|c} w & 1^{(8+\kappa+m)} \\ w' & 2^{(7-\kappa-m)} \end{array}\right) + Z_{21}^{(\kappa)} x \left(\begin{array}{c|cc} w & 1^{(8+\kappa)} & 2^{(m)} \\ w' & 2^{(7-\kappa-m)} & \end{array}\right)$$

It is lucid that $S_0\partial_x + \partial_x S_1 = \mathrm{id}$.

$$S_1\partial_x \left(Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2)} x \left(\begin{array}{c|cc} w & 1^{(8+|\kappa|)} & 2^{(m)} \\ w' & 2^{(7-|\kappa|-m)} & \end{array}\right)\right)$$

$$= S_1\left[-\binom{|\kappa|}{\kappa_2} Z_{21}^{(|\kappa|)} x \left(\begin{array}{c|cc} w & 1^{(8+|\kappa|)} & 2^{(m)} \\ w' & 2^{(7-|\kappa|-m)} & \end{array}\right) + Z_{21}^{(\kappa_1)} x\, \partial_{21}^{(\kappa_2)} \left(\begin{array}{c|cc} w & 1^{(8+|\kappa|)} & 2^{(m)} \\ w' & 2^{(7-|\kappa|-m)} & \end{array}\right)\right]$$

$$= -\binom{|\kappa|}{\kappa_2} Z_{21}^{(|\kappa|)} x\, Z_{21}^{(m)} x \left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right) + \binom{\kappa_2+m}{m} Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2+m)} x \left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right),$$

and

$$\partial_x S_2\left(Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2)} x \left(\begin{array}{c|cc} w & 1^{(8+|\kappa|)} & 2^{(m)} \\ w' & 2^{(7-|\kappa|-m)} & \end{array}\right)\right) = \partial_x \left(Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2)} x\, Z_{21}^{(m)} x \left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right)\right)$$

$$= \binom{|\kappa|}{\kappa_2} Z_{21}^{(|\kappa|)} x\, Z_{21}^{(m)} x \left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right) - \binom{\kappa_2+m}{m} Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2+m)} x \left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right) + Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2)} x$$
$$\left(\begin{array}{c|cc} w & 1^{(8+|\kappa|)} & 2^{(m)} \\ w' & 2^{(7-|\kappa|-m)} & \end{array}\right),$$

where $|\kappa| = \kappa_1 + \kappa_2$.

It is lucid that $S_1\partial_x + \partial_x S_2 = \mathrm{id}$.

$$S_2\partial_x\left(Z_{21}^{(\kappa_1)}x\,Z_{21}^{(\kappa_2)}x\,Z_{21}^{(\kappa_3)}x\left(\begin{array}{c|cc} w & 1^{(8+|\kappa|)} & 2^{(m)} \\ w' & 2^{(7-|\kappa|-m)} & \end{array}\right)\right)$$

$$=S_2\left[\binom{\kappa_1+\kappa_2}{\kappa_2}Z_{21}^{(\kappa_1+\kappa_2)}x\,Z_{21}^{(\kappa_3)}x\left(\begin{array}{c|cc} w & 1^{(8+|\kappa|)} & 2^{(m)} \\ w' & 2^{(7-|\kappa|-m)} & \end{array}\right)-\binom{\kappa_2+\kappa_3}{\kappa_3}Z_{21}^{(\kappa_1)}x\,Z_{21}^{(\kappa_2+\kappa_3)}x\left(\begin{array}{c|cc} w & 1^{(8+|\kappa|)} & 2^{(m)} \\ w' & 2^{(7-|\kappa|-m)} & \end{array}\right)+\right.$$

$$\left.Z_{21}^{(\kappa_1)}x\,Z_{21}^{(\kappa_2)}x\ \partial_{21}^{(\kappa_3)}\left(\begin{array}{c|cc} w & 1^{(8+|\kappa|)} & 2^{(m)} \\ w' & 2^{(7-|\kappa|-m)} & \end{array}\right)\right]$$

$$=\binom{\kappa_1+\kappa_2}{\kappa_2}Z_{21}^{(\kappa_1+\kappa_2)}x\,Z_{21}^{(\kappa_3)}x\,Z_{21}^{(m)}x\left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right)-\binom{\kappa_2+\kappa_3}{\kappa_3}Z_{21}^{(\kappa_1)}x\,Z_{21}^{(\kappa_2+\kappa_3)}x\,Z_{21}^{(m)}x\left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right)+$$

$$\binom{\kappa_3+m}{m}Z_{21}^{(\kappa_1)}x\,Z_{21}^{(\kappa_2)}x\,Z_{21}^{(\kappa_3+m)}x\left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right),$$

and

$$\partial_x S_3\left(Z_{21}^{(\kappa_1)}x\,Z_{21}^{(\kappa_2)}x\,Z_{21}^{(\kappa_3)}x\left(\begin{array}{c|cc} w & 1^{(8+|\kappa|)} & 2^{(m)} \\ w' & 2^{(7-|\kappa|-m)} & \end{array}\right)\right)=\partial_x\left(Z_{21}^{(\kappa_1)}x\,Z_{21}^{(\kappa_2)}x\,Z_{21}^{(\kappa_3)}x\,Z_{21}^{(m)}x\left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right)\right)$$

$$=-\binom{\kappa_1+\kappa_2}{\kappa_2}Z_{21}^{(\kappa_1+\kappa_2)}x\,Z_{21}^{(\kappa_3)}x\,Z_{21}^{(m)}x\left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right)+\binom{\kappa_2+\kappa_3}{\kappa_3}Z_{21}^{(\kappa_1)}x\,Z_{21}^{(\kappa_2+\kappa_3)}x\,Z_{21}^{(m)}x\left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right)-$$

$$\binom{\kappa_3+m}{m}Z_{21}^{(\kappa_1)}x\,Z_{21}^{(\kappa_2)}x\,Z_{21}^{(\kappa_3+m)}x\left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right)+Z_{21}^{(\kappa_1)}x\,Z_{21}^{(\kappa_2)}x\,Z_{21}^{(\kappa_3)}x\ \partial_{21}^{(m)}\left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right)$$

$$=-\binom{\kappa_1+\kappa_2}{\kappa_2}Z_{21}^{(\kappa_1+\kappa_2)}x\,Z_{21}^{(\kappa_3)}x\,Z_{21}^{(m)}x\left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right)+\binom{\kappa_2+\kappa_3}{\kappa_3}Z_{21}^{(\kappa_1)}x\,Z_{21}^{(\kappa_2+\kappa_3)}x\,Z_{21}^{(m)}x\left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right)-$$

$$\binom{\kappa_3+m}{m}Z_{21}^{(\kappa_1)}x\,Z_{21}^{(\kappa_2)}x\,Z_{21}^{(\kappa_3+m)}x\left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right)+Z_{21}^{(\kappa_1)}x\,Z_{21}^{(\kappa_2)}x\,Z_{21}^{(\kappa_3)}x\left(\begin{array}{c|cc} w & 1^{(8+|\kappa|)} & 2^{(m)} \\ w' & 2^{(7-|\kappa|-m)} & \end{array}\right),$$

where $|\kappa| = \kappa_1 + \kappa_2 + \kappa_3$.

It is lucid that $S_2\partial_x + \partial_x S_3 = \mathrm{id}$.

$$S_3\partial_x\left(Z_{21}^{(\kappa_1)}x\,Z_{21}^{(\kappa_2)}x\,Z_{21}^{(\kappa_3)}x\,Z_{21}^{(\kappa_4)}x\left(\begin{array}{c|cc} w & 1^{(8+|\kappa|)} & 2^{(m)} \\ w' & 2^{(7-|\kappa|-m)} & \end{array}\right)\right)$$

$$= S_3\left[-\binom{\kappa_1+\kappa_2}{\kappa_2} Z_{21}^{(\kappa_1+\kappa_2)} x\, Z_{21}^{(\kappa_3)} x\, Z_{21}^{(\kappa_4)} x \left(\begin{array}{c|cc} w & 1^{(8+|\kappa|)} & 2^{(m)} \\ w' & 2^{(7-|\kappa|-m)} & \end{array} \right) + \binom{\kappa_2+\kappa_3}{\kappa_3} Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2+\kappa_3)} x \right.$$

$$Z_{21}^{(\kappa_4)} x \left(\begin{array}{c|cc} w & 1^{(8+|\kappa|)} & 2^{(m)} \\ w' & 2^{(7-|\kappa|-m)} & \end{array} \right) - \binom{\kappa_3+\kappa_4}{\kappa_4} Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2)} x\, Z_{21}^{(\kappa_3+\kappa_4)} x \left(\begin{array}{c|cc} w & 1^{(8+|\kappa|)} & 2^{(m)} \\ w' & 2^{(7-|\kappa|-m)} & \end{array} \right) +$$

$$\left. Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2)} \mathrm{x} Z_{21}^{(\kappa_3)} x\ \partial_{21}^{(\kappa_4)} \left(\begin{array}{c|cc} w & 1^{(8+|\kappa|)} & 2^{(m)} \\ w' & 2^{(7-|\kappa|-m)} & \end{array} \right) \right]$$

$$= -\binom{\kappa_1+\kappa_2}{\kappa_2} Z_{21}^{(\kappa_1+\kappa_2)} x\, Z_{21}^{(\kappa_3)} x\, Z_{21}^{(\kappa_4)} x\, Z_{21}^{(m)} x \left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array} \right) + \binom{\kappa_2+\kappa_3}{\kappa_3} Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2+\kappa_3)} x$$

$$Z_{21}^{(\kappa_4)} x\, Z_{21}^{(m)} x \left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array} \right) - \binom{\kappa_3+\kappa_4}{\kappa_4} Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2)} x\, Z_{21}^{(\kappa_3+\kappa_4)} x\, Z_{21}^{(m)} x \left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array} \right) +$$

$$\binom{\kappa_4+m}{m} Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2)} \mathrm{x} Z_{21}^{(\kappa_3)} x\, Z_{21}^{(\kappa_4+m)} x \left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array} \right),$$

and

$$\partial_x S_4 \left(Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2)} x\, Z_{21}^{(\kappa_3)} x\, Z_{21}^{(\kappa_4)} x \left(\begin{array}{c|cc} w & 1^{(8+|\kappa|)} & 2^{(m)} \\ w' & 2^{(7-|\kappa|-m)} & \end{array} \right) \right)$$

$$= \partial_x \left(Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2)} x\, Z_{21}^{(\kappa_3)} x\, Z_{21}^{(\kappa_4)} x\, Z_{21}^{(m)} x \left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array} \right) \right)$$

$$= \binom{\kappa_1+\kappa_2}{\kappa_2} Z_{21}^{(\kappa_1+\kappa_2)} x\, Z_{21}^{(\kappa_3)} x\, Z_{21}^{(\kappa_4)} x\, Z_{21}^{(m)} x \left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array} \right) - \binom{\kappa_2+\kappa_3}{\kappa_3} Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2+\kappa_3)} x\, Z_{21}^{(\kappa_4)} x$$

$$Z_{21}^{(m)} x \left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array} \right) + \binom{\kappa_3+\kappa_4}{\kappa_4} Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2)} x\, Z_{21}^{(\kappa_3+\kappa_4)} x\, Z_{21}^{(m)} x \left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array} \right) - \binom{\kappa_4+m}{m}$$

$$Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2)} x\, Z_{21}^{(\kappa_3)} x\, Z_{21}^{(\kappa_4+m)} x \left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array} \right) + Z_{21}^{(\kappa_1)} x\, Z_{21}^{(\kappa_2)} x\, Z_{21}^{(\kappa_3)} x\, Z_{21}^{(\kappa_4)} x\, \partial_{21}^{(m)} \left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array} \right)$$

$$=\binom{\kappa_1+\kappa_2}{\kappa_2}Z_{21}^{(\kappa_1+\kappa_2)}x\,Z_{21}^{(\kappa_3)}x\,Z_{21}^{(\kappa_4)}x\,Z_{21}^{(m)}x\left(\begin{matrix}w\\w'\end{matrix}\left|\begin{matrix}1^{(8+|\kappa|+m)}\\2^{(7-|\kappa|-m)}\end{matrix}\right.\right)-\binom{\kappa_2+\kappa_3}{\kappa_3}Z_{21}^{(\kappa_1)}x\,Z_{21}^{(\kappa_2+\kappa_3)}x\,Z_{21}^{(\kappa_4)}x\,Z_{21}^{(m)}x$$
$$\left(\begin{matrix}w\\w'\end{matrix}\left|\begin{matrix}1^{(8+|\kappa|+m)}\\2^{(7-|\kappa|-m)}\end{matrix}\right.\right)+\binom{\kappa_3+\kappa_4}{\kappa_4}Z_{21}^{(\kappa_1)}x\,Z_{21}^{(\kappa_2)}x\,Z_{21}^{(\kappa_3+\kappa_4)}x\,Z_{21}^{(m)}x\left(\begin{matrix}w\\w'\end{matrix}\left|\begin{matrix}1^{(8+|\kappa|+m)}\\2^{(7-|\kappa|-m)}\end{matrix}\right.\right)-\binom{\kappa_4+m}{m}Z_{21}^{(\kappa_1)}x\,Z_{21}^{(\kappa_2)}x$$
$$Z_{21}^{(\kappa_3)}x\,Z_{21}^{(\kappa_4+m)}x\left(\begin{matrix}w\\w'\end{matrix}\left|\begin{matrix}1^{(8+|\kappa|+m)}\\2^{(7-|\kappa|-m)}\end{matrix}\right.\right)+Z_{21}^{(\kappa_1)}x\,Z_{21}^{(\kappa_2)}x\,Z_{21}^{(\kappa_3)}x\,Z_{21}^{(\kappa_4)}x\left(\begin{matrix}w\\w'\end{matrix}\left|\begin{matrix}1^{(8+|\kappa|)} & 2^{(m)}\\2^{(7-|\kappa|-m)} & \end{matrix}\right.\right),$$

where $|\kappa|=\kappa_1+\kappa_2+\kappa_3+\kappa_4$.

It is lucid that $S_3\partial_x+\partial_x S_4=\mathrm{id}$.

$$S_4\partial_x\left(Z_{21}^{(\kappa_1)}x\,Z_{21}^{(\kappa_2)}x\,Z_{21}^{(\kappa_3)}x\,Z_{21}^{(\kappa_4)}x\,Z_{21}^{(\kappa_5)}x\left(\begin{matrix}w\\w'\end{matrix}\left|\begin{matrix}1^{(8+|\kappa|)} & 2^{(m)}\\2^{(7-|\kappa|-m)} & \end{matrix}\right.\right)\right)$$
$$=S_4\left[\binom{\kappa_1+\kappa_2}{\kappa_2}Z_{21}^{(\kappa_1+\kappa_2)}x\,Z_{21}^{(\kappa_3)}x\,Z_{21}^{(\kappa_4)}x\,Z_{21}^{(\kappa_5)}x\left(\begin{matrix}w\\w'\end{matrix}\left|\begin{matrix}1^{(8+|\kappa|)} & 2^{(m)}\\2^{(7-|\kappa|-m)} & \end{matrix}\right.\right)-\binom{\kappa_2+\kappa_3}{\kappa_3}Z_{21}^{(\kappa_1)}x\,Z_{21}^{(\kappa_2+\kappa_3)}x\,Z_{21}^{(\kappa_4)}x\,Z_{21}^{(\kappa_5)}x\right.$$
$$\left(\begin{matrix}w\\w'\end{matrix}\left|\begin{matrix}1^{(8+|\kappa|)} & 2^{(m)}\\2^{(7-|\kappa|-m)} & \end{matrix}\right.\right)+\binom{\kappa_3+\kappa_4}{\kappa_4}Z_{21}^{(\kappa_1)}x\,Z_{21}^{(\kappa_2)}x\,Z_{21}^{(\kappa_3+\kappa_4)}x\,Z_{21}^{(\kappa_5)}x\left(\begin{matrix}w\\w'\end{matrix}\left|\begin{matrix}1^{(8+|\kappa|)} & 2^{(m)}\\2^{(7-|\kappa|-m)} & \end{matrix}\right.\right)-\binom{\kappa_4+\kappa_5}{\kappa_5}Z_{21}^{(\kappa_1)}x$$
$$\left.Z_{21}^{(\kappa_2)}x\,Z_{21}^{(\kappa_3)}x\,Z_{21}^{(\kappa_4+\kappa_5)}x\left(\begin{matrix}w\\w'\end{matrix}\left|\begin{matrix}1^{(8+|\kappa|)} & 2^{(m)}\\2^{(7-|\kappa|-m)} & \end{matrix}\right.\right)+Z_{21}^{(\kappa_1)}x\,Z_{21}^{(\kappa_2)}x\,Z_{21}^{(\kappa_3)}x\,Z_{21}^{(\kappa_4)}x\;\partial_{21}^{(\kappa_5)}\left(\begin{matrix}w\\w'\end{matrix}\left|\begin{matrix}1^{(8+|\kappa|)} & 2^{(m)}\\2^{(7-|\kappa|-m)} & \end{matrix}\right.\right)\right]$$

$$=\binom{\kappa_1+\kappa_2}{\kappa_2}Z_{21}^{(\kappa_1+\kappa_2)}x\,Z_{21}^{(\kappa_3)}x\,Z_{21}^{(\kappa_4)}x\,Z_{21}^{(\kappa_5)}x\,Z_{21}^{(m)}x\left(\begin{matrix}w\\w'\end{matrix}\left|\begin{matrix}1^{(8+|\kappa|+m)}\\2^{(7-|\kappa|-m)}\end{matrix}\right.\right)-\binom{\kappa_2+\kappa_3}{\kappa_3}Z_{21}^{(\kappa_1)}x\,Z_{21}^{(\kappa_2+\kappa_3)}x\;Z_{21}^{(\kappa_4)}x$$
$$Z_{21}^{(\kappa_5)}x\,Z_{21}^{(m)}x\left(\begin{matrix}w\\w'\end{matrix}\left|\begin{matrix}1^{(8+|\kappa|+m)}\\2^{(7-|\kappa|-m)}\end{matrix}\right.\right)+\binom{\kappa_3+\kappa_4}{\kappa_4}Z_{21}^{(\kappa_1)}x\,Z_{21}^{(\kappa_2)}x\,Z_{21}^{(\kappa_3+\kappa_4)}x\,Z_{21}^{(\kappa_5)}x\,Z_{21}^{(m)}x\left(\begin{matrix}w\\w'\end{matrix}\left|\begin{matrix}1^{(8+|\kappa|+m)}\\2^{(7-|\kappa|-m)}\end{matrix}\right.\right)-$$
$$\binom{\kappa_4+\kappa_5}{\kappa_5}Z_{21}^{(\kappa_1)}x\,Z_{21}^{(\kappa_2)}x\,Z_{21}^{(\kappa_3)}x\,Z_{21}^{(\kappa_4+\kappa_5)}x\,Z_{21}^{(m)}x\left(\begin{matrix}w\\w'\end{matrix}\left|\begin{matrix}1^{(8+|\kappa|+m)}\\2^{(7-|\kappa|-m)}\end{matrix}\right.\right)+\binom{\kappa_5+m}{m}Z_{21}^{(\kappa_1)}x\,Z_{21}^{(\kappa_2)}x\,Z_{21}^{(\kappa_3)}x$$
$$Z_{21}^{(\kappa_4)}x\,Z_{21}^{(\kappa_5+m)}x\left(\begin{matrix}w\\w'\end{matrix}\left|\begin{matrix}1^{(8+|\kappa|+m)}\\2^{(7-|\kappa|-m)}\end{matrix}\right.\right),$$

and

$$\partial_x S_5\left(Z_{21}^{(\kappa_1)}x\,Z_{21}^{(\kappa_2)}x\,Z_{21}^{(\kappa_3)}x\,Z_{21}^{(\kappa_4)}x\,Z_{21}^{(\kappa_5)}x\left(\begin{matrix}w\\w'\end{matrix}\left|\begin{matrix}1^{(8+|\kappa|)} & 2^{(m)}\\2^{(7-|\kappa|-m)} & \end{matrix}\right.\right)\right)$$
$$=\partial_x\left(Z_{21}^{(\kappa_1)}x\,Z_{21}^{(\kappa_2)}x\,Z_{21}^{(\kappa_3)}x\,Z_{21}^{(\kappa_4)}x\,Z_{21}^{(\kappa_5)}x\,Z_{21}^{(m)}x\left(\begin{matrix}w\\w'\end{matrix}\left|\begin{matrix}1^{(8+|\kappa|+m)}\\2^{(7-|\kappa|-m)}\end{matrix}\right.\right)\right)$$

$$= -\binom{\kappa_1+\kappa_2}{\kappa_2} Z_{21}^{(\kappa_1+\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|\kappa|+m)} \\ 2^{(7-|\kappa|-m)} \end{matrix} \right) + \binom{\kappa_2+\kappa_3}{\kappa_3} Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2+\kappa_3)} x Z_{21}^{(\kappa_4)} x$$
$$Z_{21}^{(\kappa_5)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|\kappa|+m)} \\ 2^{(7-|\kappa|-m)} \end{matrix} \right) - \binom{\kappa_3+\kappa_4}{\kappa_4} Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3+\kappa_4)} x Z_{21}^{(\kappa_5)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|\kappa|+m)} \\ 2^{(7-|\kappa|-m)} \end{matrix} \right) +$$
$$\binom{\kappa_4+\kappa_5}{\kappa_4} Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4+\kappa_5)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|\kappa|+m)} \\ 2^{(7-|\kappa|-m)} \end{matrix} \right) - \binom{\kappa_5+m}{m} Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x$$
$$Z_{21}^{(\kappa_5+m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|\kappa|+m)} \\ 2^{(7-|\kappa|-m)} \end{matrix} \right) + Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5)} x \ \partial_{21}^{(m)} \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|\kappa|+m)} \\ 2^{(7-|\kappa|-m)} \end{matrix} \right)$$

$$= -\binom{\kappa_1+\kappa_2}{\kappa_2} Z_{21}^{(\kappa_1+\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5)} x \ Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|\kappa|+m)} \\ 2^{(7-|\kappa|-m)} \end{matrix} \right) + \binom{\kappa_2+\kappa_3}{\kappa_3} Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2+\kappa_3)} x Z_{21}^{(\kappa_4)} x$$
$$Z_{21}^{(\kappa_5)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|\kappa|+m)} \\ 2^{(7-|\kappa|-m)} \end{matrix} \right) - \binom{\kappa_3+\kappa_4}{\kappa_4} Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3+\kappa_4)} x Z_{21}^{(\kappa_5)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|\kappa|+m)} \\ 2^{(7-|\kappa|-m)} \end{matrix} \right) +$$
$$\binom{\kappa_4+\kappa_5}{\kappa_4} Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4+\kappa_5)} x Z_{21}^{(m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|\kappa|+m)} \\ 2^{(7-|\kappa|-m)} \end{matrix} \right) - \binom{\kappa_5+m}{m} Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x$$
$$Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5+m)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|\kappa|+m)} \\ 2^{(7-|\kappa|-m)} \end{matrix} \right) + Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|\kappa|)} & 2^{(m)} \\ 2^{(7-|\kappa|-m)} & \end{matrix} \right) ,$$

where $|\kappa| = \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 + \kappa_5$.

It is lucid that $S_4\partial_x + \partial_x S_5 = \mathrm{id}$.

$$S_5\partial_x \left(Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5)} x Z_{21}^{(\kappa_6)} x \left(\begin{matrix} w \\ w' \end{matrix} \middle| \begin{matrix} 1^{(8+|\kappa|)} & 2^{(m)} \\ 2^{(7-|\kappa|-m)} & \end{matrix} \right) \right)$$

$$= S_5\Bigg[-\binom{\kappa_1+\kappa_2}{\kappa_2} Z_{21}^{(\kappa_1+\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5)} x Z_{21}^{(\kappa_5)} x Z_{21}^{(\kappa_6)} x \left(\begin{array}{c|cc} w & 1^{(8+|\kappa|)} & 2^{(m)} \\ w' & 2^{(7-|\kappa|-m)} & \end{array}\right) + \binom{\kappa_2+\kappa_3}{\kappa_3} Z_{21}^{(\kappa_1)} x$$

$$Z_{21}^{(\kappa_2+\kappa_3)} x Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5)} x Z_{21}^{(\kappa_6)} x \left(\begin{array}{c|cc} w & 1^{(8+|\kappa|)} & 2^{(m)} \\ w' & 2^{(7-|\kappa|-m)} & \end{array}\right) - \binom{\kappa_3+\kappa_4}{\kappa_4} Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3+\kappa_4)} x Z_{21}^{(\kappa_5)} x Z_{21}^{(\kappa_6)} x$$

$$\left(\begin{array}{c|cc} w & 1^{(8+|\kappa|)} & 2^{(m)} \\ w' & 2^{(7-|\kappa|-m)} & \end{array}\right) + \binom{\kappa_4+\kappa_5}{\kappa_5} Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4+\kappa_5)} x Z_{21}^{(\kappa_6)} x \left(\begin{array}{c|cc} w & 1^{(8+|\kappa|)} & 2^{(m)} \\ w' & 2^{(7-|\kappa|-m)} & \end{array}\right) - \binom{\kappa_5+\kappa_6}{\kappa_6} Z_{21}^{(\kappa_1)} x$$

$$Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5+\kappa_6)} x \left(\begin{array}{c|cc} w & 1^{(8+|\kappa|)} & 2^{(m)} \\ w' & 2^{(7-|\kappa|-\wp)} & \end{array}\right) + Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x \;\; Z_{21}^{(\kappa_5)} x\, \partial_{21}^{(\kappa_6)} \left(\begin{array}{c|cc} w & 1^{(8+|\kappa|)} & 2^{(m)} \\ w' & 2^{(7-|\kappa|-m)} & \end{array}\right)\Bigg]$$

$$= -\binom{\kappa_1+\kappa_2}{\kappa_2} Z_{21}^{(\kappa_1+\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5)} x Z_{21}^{(\kappa_6)} x Z_{21}^{(m)} x \left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right) + \binom{\kappa_2+\kappa_3}{\kappa_3} Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2+\kappa_3)} x$$

$$Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5)} x Z_{21}^{(\kappa_6)} x Z_{21}^{(m)} x \left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right) - \binom{\kappa_3+\kappa_4}{\kappa_4} Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3+\kappa_4)} x Z_{21}^{(\kappa_5)} x Z_{21}^{(\kappa_6)} x Z_{21}^{(m)} x$$

$$\left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right) + \binom{\kappa_4+\kappa_5}{\kappa_5} Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4+\kappa_5)} x Z_{21}^{(\kappa_6)} x Z_{21}^{(m)} x \left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right) - \binom{\kappa_5+\kappa_6}{\kappa_6}$$

$$Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5+\kappa_6)} x Z_{21}^{(m)} x \left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right) + \binom{\kappa_6+m}{m} Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x$$

$$Z_{21}^{(\kappa_5)} x Z_{21}^{(\kappa_6+m)} x \left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right) ,$$

and

$$\partial_x S_6\left(Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5)} x Z_{21}^{(\kappa_6)} x \left(\begin{array}{c|cc} w & 1^{(8+|\kappa|)} & 2^{(m)} \\ w' & 2^{(7-|\kappa|-m)} & \end{array}\right)\right)$$

$$= \partial_x \left(Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5)} x Z_{21}^{(\kappa_6)} x Z_{21}^{(m)} x \left(\begin{array}{c|c} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right)\right)$$

$$=\binom{\kappa_1+\kappa_2}{\kappa_2} Z_{21}^{(\kappa_1+\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5)} x Z_{21}^{(\kappa_6)} x Z_{21}^{(m)} x \left(\begin{array}{c|l} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right) - \binom{\kappa_2+\kappa_3}{\kappa_3} Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2+\kappa_3)} x$$

$$Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5)} x Z_{21}^{(\kappa_6)} x Z_{21}^{(m)} x \left(\begin{array}{c|l} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right) + \binom{\kappa_3+\kappa_4}{\kappa_4} Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3+\kappa_4)} x Z_{21}^{(\kappa_5)} x Z_{21}^{(\kappa_6)} x Z_{21}^{(m)} x$$

$$\left(\begin{array}{c|l} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right) - \binom{\kappa_4+\kappa_5}{\kappa_4} Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4+\kappa_5)} x Z_{21}^{(\kappa_6)} x Z_{21}^{(m)} x \left(\begin{array}{c|l} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right) + \binom{\kappa_5+\kappa_6}{\kappa_6}$$

$$Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5+\kappa_6)} x Z_{21}^{(m)} x \left(\begin{array}{c|l} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right) - \binom{\kappa_6+m}{m} Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x$$

$$Z_{21}^{(\kappa_5)} x Z_{21}^{(\kappa_6+m)} x \left(\begin{array}{c|l} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right) + Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5)} x Z_{21}^{(\kappa_6)} x\ \partial_{21}^{(m)} \left(\begin{array}{c|l} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right)$$

$$=\binom{\kappa_1+\kappa_2}{\kappa_2} Z_{21}^{(\kappa_1+\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5)} x Z_{21}^{(\kappa_6)} x Z_{21}^{(m)} x \left(\begin{array}{c|l} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right) - \binom{\kappa_2+\kappa_3}{\kappa_3} Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2+\kappa_3)} x$$

$$Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5)} x Z_{21}^{(\kappa_6)} x Z_{21}^{(m)} x \left(\begin{array}{c|l} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right) + \binom{\kappa_3+\kappa_4}{\kappa_4} Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3+\kappa_4)} x Z_{21}^{(\kappa_5)} x Z_{21}^{(\kappa_6)} x Z_{21}^{(m)} x$$

$$\left(\begin{array}{c|l} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right) - \binom{\kappa_4+\kappa_5}{\kappa_4} Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4+\kappa_5)} x Z_{21}^{(\kappa_6)} x Z_{21}^{(m)} x \left(\begin{array}{c|l} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right) + \binom{\kappa_5+\kappa_6}{\kappa_6}$$

$$Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5+\kappa_6)} x Z_{21}^{(m)} x \left(\begin{array}{c|l} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right) - \binom{\kappa_6+m}{m} Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x$$

$$Z_{21}^{(\kappa_5)} x Z_{21}^{(\kappa_6+m)} x \left(\begin{array}{c|l} w & 1^{(8+|\kappa|+m)} \\ w' & 2^{(7-|\kappa|-m)} \end{array}\right) + Z_{21}^{(\kappa_1)} x Z_{21}^{(\kappa_2)} x Z_{21}^{(\kappa_3)} x Z_{21}^{(\kappa_4)} x Z_{21}^{(\kappa_5)} x Z_{21}^{(\kappa_6)} x \left(\begin{array}{c|ll} w & 1^{(8+|\kappa|)} & 2^{(m)} \\ w' & 2^{(7-|\kappa|-m)} & \end{array}\right),$$

where $|\kappa| = \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4 + \kappa_5 + \kappa_6$.

It is lucid that $S_5\partial_x + \partial_x S_6 = \mathrm{id}$.

From above we get $\{S_0, S_1, S_2, S_3, S_4, S_5, S_6\}$ is a contracting homotopy [6] which mean our complex is exact.

References

[1] David A.Buchsbaum and Gian C.Rota 1993 Projective Resolution of Weyl Modules, *Natl.Acad.Sci.USA* Vol. 90 pp.2448-2450.
[2] David A.Buchsbaum and Brian D.Taylor 2003 Homotopies for Resolution of Skew-Hook Shapes, *Adv. In Applied Math*. Vol.30 pp.26-43.
[3] David A.Buchsbaum 2004 A Characteristic Free Example of Lascoux Resolution, and Letter Place Mathods for Intertwining Numbers, *European Journal of Gombinatorics* Vol.25, (), pp.1169-1179.
[4] David A.Buchsbaum 2001 Resolution of Weyl Modules: The Rota Touch, *Algebraic Combinatorics and Computer Science* pp.97-109.

[5] David A.Buchsbaum and Gian C.Rota 1994, A new Construction in Homological Algebra, *Natl.Acad.Sci., USA* Vol.91 pp.4115-4119.
[6] Vermani L.R. 2003 An Elementary Approach to Homological Algebra, Chapman and Hall/CRC, *Monographs and Surveys in Pure and Applied Mathematics*, 130.