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Some fuzzy topological operators via fuzzy ideals

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Abstract

Fuzzy ideals, fuzzy local function and the notion of compatibility of fuzzy ideals with fuzzy topologies were introduced and studied by D. Sarker (Fuzzy Sets and Systems 87 (1997) 117). The purpose of this paper deals with new sort of fuzzy local function namely fuzzy α-local function. Many of its characterizations, properties and connections between it and other corresponding fuzzy notions are studied. Also we introduce and study two types of fuzzy compatibility namely fuzzy α-compatibility and fuzzy weak α-compatibility of fuzzy topologies via fuzzy ideals. Possible application to superstrings and $\mathscr{E}^{(\infty)}$ space-time are touched upon. © 2001 Elsevier Science Ltd. All rights reserved.

1. Introduction

The concept of fuzzy sets and fuzzy set operations was first introduced by Zadeh [5]. Subsequently, Chang defined the notion of fuzzy topology [2]. Since then various aspects of general topology were investigated and carried out in fuzzy sense by several authors of this field. The local properties of a fuzzy topological space which may also be in certain cases the properties of the whole space are an important field for study in fuzzy topology by introducing the notion of fuzzy ideal and fuzzy local function [4]. The notion of fuzzy point and α-q-neighbourhood (in short α-q-nbd) of a fuzzy point introduces the scope of such analysis in fuzzy topology [1,3]. The concept of fuzzy topology may have very important applications in quantum particles physics particularly in connection with string theory and $\mathscr{E}^{(\infty)}$ theory [6,7].

Our aim in this paper is to introduce and study the new sort of fuzzy local functions called fuzzy α-local function. Many of its characterizations and connections between it and other corresponding fuzzy notions are studied. Utlizing on these previous new concepts, the class $*^{\alpha}$-fuzzy topology has been constructed as a generalization of the fuzzy topology. Finally we define quasi-α-cover of fuzzy set and introduce the notions of fuzzy compatibility namely fuzzy α-compatibility and fuzzy weak α-compatibility of fuzzy topologies via fuzzy ideals.

1.1. Preliminaries

Throughout this paper, by (X,τ) we mean a fits in the sense of Chang [2]. A fuzzy point in X with support $x \in X$ and value ε $(0 < \varepsilon \leqslant 1)$ is denoted by x_ε. A fuzzy point x_ε is said to be contained in a fuzzy set μ in X iff $\varepsilon \leqslant \mu\ (x)$ and this will denoted by $x_\varepsilon \in \mu$ [1]. For a fuzzy set μ in X, cl(μ) and μ^c will, respectively, denote closure and complement of μ. The constant fuzzy sets taking values 0 and 1 on X are denoted by

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0_x, 1_x, respectively. A fuzzy set μ is said to be quasi-concident with a fuzzy set η, denoted by $\mu q \eta$, if there exists $x \in X$ such that $\mu(x) + \eta(x) > 1$ [3]. Obviously, for any two fuzzy sets μ and η, $\mu q \eta$ will imply $\eta q \mu$. A fuzzy set ρ in a fits (X, τ) is called a q-nbd (resp. α-q-nbd) of a fuzzy point x_ε iff there exists a fuzzy open (resp. α-open) set v such that $x_\varepsilon q v \subseteq \rho$ (resp. $x_\varepsilon q v \subseteq \delta$) [1,3]. We will denote the set of all q-nbd (resp. α-q-nbd) of x_ε in (X, τ) by $N(x_\varepsilon)$ (resp. $\alpha N(x_\varepsilon)$). A non-empty collection of fuzzy sets L of a set X is called a fuzzy ideal [4] iff (i) $\mu \in L$ and $\eta \subseteq \mu \Rightarrow \eta \in L$ (heredity) and (ii) $\mu \in L$ and $\eta \in L \Rightarrow \mu \cup \eta \in L$ (finite additivity). Fuzzy closure operator of a fuzzy set μ (in short $\mathrm{cl}^*(\mu)$) is defined as $\mathrm{cl}^*(\mu) = \mu \vee \mu^*$, and $\tau^*(L)$ be the fuzzy topology generated by cl^*, i.e., $\tau^*(L) = \{\mu{:}\mathrm{cl}^*(\mu)^{\mathrm{c}} = \mu^{\mathrm{c}}\}$ [4]. The fuzzy local function $\mu^*(L, \tau)$ of a fuzzy set μ is the union of all fuzzy points x_ε such that if $\rho \in N(x_\varepsilon)$ and $\lambda \in L$, then there is at least one $r \in X$ for which $\rho(r) + \mu(r) - 1 > \lambda(r)$ [4]. The definition of quasi-cover and the notion of compatibility of fuzzy ideal with fuzzy topological space were introduced in [4].

2. Fuzzy α-local function

Definition 2.1. Let (X, τ) be a fts with fuzzy ideal L on X and let μ be any fuzzy set of X. Then the fuzzy α-local function $\mu^{*\alpha}(L, \tau)$ of μ is the union of all fuzzy points x_ε such that if $\rho \in \alpha N(x_\varepsilon)$ and $\lambda \in L$ then there is at least one $r \in X$ for which $\rho(r) + \mu(r) - 1 > \lambda(r)$.

In other words, we say that a fuzzy set μ is fuzzy α-locally in L at x_ε, if there exists $\rho \in \alpha N(x_\varepsilon)$ such that for every $r \in X, \rho(r) + \mu(r) - 1 \leqslant \lambda(r)$ for some $\lambda \in L$. $\mu^*(\tau, L)$ is the set of fuzzy points at which μ does not have the property fuzzy α-locally.

We will occasionally write $\mu^{*\alpha}$ or $\mu^{*\alpha}(L)$ for $\mu^{*\alpha}(L, \tau)$. We define $*^\alpha$-fuzzy closure operator, denoted by $\mathrm{cl}^{*\alpha}$ for fuzzy topology $\tau^{*\alpha}(L)$ finer than τ^α as follows: $\mathrm{cl}^{*\alpha}(\mu) = \mu \vee \mu^{*\alpha}$ for every fuzzy set μ on X. When there is no ambiguity, we will simply write $\mu^{*\alpha}$ and $\tau^{*\alpha}$ for $\mu^{*\alpha}(L, \tau)$ and $\tau^{*\alpha}(L)$, respectively.

Remark 2.1.

(i) The class of fuzzy α-local functions is contained in the class of fuzzy local functions in the sense of Sarker, that is $\mu^{*\alpha}(L) \leqslant \mu^*(L)$ for every fuzzy set μ on X and the converse is not true in general (Example 2.1).
(ii) $\mu^{*\alpha}(L, \tau) = \mu^*(L, \tau^\alpha)$.
(iii) $\tau^*(L) \leqslant \tau^{*\alpha}$.

Example 2.1. Let $l_x = \{(x, l)\}$ with fuzzy topology $\tau = \{l_x,\ O_x,\ \mu,\ \rho,\ \eta\}$, $\mu = 0.8$, $\rho = 0.6$, $\eta = 0.2$ and fuzzy ideal $L = \{O_x\}$, $\tau^\alpha = \{l_x,\ O_x,\ \mu,\ \rho,\ \eta,\ \mu_1,\ \rho_1\}$, $\mu_1 = 0.1$, $\rho_1 = 0.7$.

If $\xi = 0.3$, then one can deduce that $\xi^*(L) = 0.4$, $\xi^{*\alpha} = 0.3$ and consequently $\xi^{*\alpha}(L) \leqslant \xi^*(L) = \alpha\text{-cl}(\xi^*)$.

The following theorem gives some general properties of fuzzy α-local function.

Theorem 2.1. *Let (X, τ) be a fts with fuzzy ideal L and μ, η be two fuzzy subsets of X. Then*:

(i) *$\mu^{*\alpha}$ is a fuzzy α-closed set.*
(ii) *If $\mu \leqslant \eta$, then $\mu^{*\alpha} \leqslant \eta^{*\alpha}$.*
(iii) *If $L_1 \leqslant L_2$, then $\mu^{*\alpha}(L_2, \tau) \leqslant \mu^{*\alpha}(L_1, \tau)$.*
(iv) *$\mu^{*\alpha} = \alpha\text{-cl}(\mu^{*\alpha}) \leqslant \alpha\text{-cl}(\mu)$.*
(v) *$(\mu^{*\alpha})^{*\alpha} \leqslant \mu^{*\alpha}$.*
(vi) *$(\mu \vee \eta)^{*\alpha} = \mu^{*\alpha} \vee \eta^{*\alpha}$.*
(vii) *$(\mu \wedge \eta)^{*\alpha} \leqslant \mu^{*\alpha} \wedge \eta^{*\alpha}$.*
(viii) *If $\rho \in L$, then $(\mu \vee \rho)^{*\alpha} = \mu^{*\alpha} = (\mu - \rho)^{*\alpha}$.*
(ix) *If $\lambda \in L$, then $(1x - \lambda)^{*\alpha} = 1^{*\alpha}$.*
(x) *If $U \in \tau^\alpha$, then $U \wedge \mu^{*\alpha} = U \wedge (U \wedge \mu)^{*\alpha} \leqslant (U \wedge \mu)^{*\alpha}$.*

Proof. We prove only (i) and (vi).

(i) Let $x_\varepsilon \notin \mu^{*\alpha}(L,\tau)$. Then there is at least one $\rho \in \alpha N(x_\varepsilon)$ such that for every $r \in X$, $\rho(r) + \mu(r) - 1 \leqslant \lambda(r)$ for some $\lambda \in L$ and $x_\varepsilon \in \rho$ implies $\rho \leqslant l_x - \mu^{*\alpha}(L)$ and we have $l_x - \mu^{*\alpha}(L)$ is fuzzy α-open set. Therefore, $\mu^{*\alpha}$ is fuzzy α-closed.

(vi) Suppose that $x_\varepsilon \notin \mu^{*\alpha} \vee \eta^{*\alpha}$, that is $\varepsilon > (\mu^{*\alpha} \vee \eta^{*\alpha})(x) = \max\{\mu^{*\alpha}(x), \eta^{*\alpha}(x)\}$. So x_ε is not contained in both $\mu^{*\alpha}$ and $\eta^{*\alpha}$. This implies that there is at least one α-q-nbd μ of x_ε such that for every $r \in X$, $\mu_1(r) + \mu(r) - 1 \leqslant \lambda_1(r)$ for some $\lambda_1 \in L$ and similarly, there is at least one α-q-nbd μ_2 of x_ε such that for every $r \in X$, $\mu_2(r) + \eta(r) - 1 \leqslant \lambda_2(x)$ for some $\lambda_2 \in L$. Let $V = \mu_1 \vee \mu_2$. So V is also α-q-nbd of x_ε and $V(r) + (\mu \vee \eta)(r) - 1 \leqslant \lambda_1 \vee \lambda_2 \ (r)$ for every $r \in X$. Therefore, by finite additivity of fuzzy ideal as $\lambda_1 \vee \lambda_2 \in L$, $x_\varepsilon \notin (\mu \vee \eta)^{*\alpha}$. Hence $(\mu \vee \eta)^{*\alpha} \leqslant \mu^{*\alpha} \vee \eta^{*\alpha}$. And by using (ii) we get the equality. □

Corollary 2.1. *For a fts (X,τ) with fuzzy ideal L and μ fuzzy subset of X, we have*

(i) $(\mu^{*\alpha})^{*\alpha} \leqslant \mu^{*\alpha} = \alpha\text{-cl}(\mu^{*\alpha}) \leqslant \alpha\text{-cl}(\mu)$,
(ii) $(\mu^{*\alpha})^{*\alpha} \leqslant \alpha\text{-cl}(\mu^{*\alpha}) \leqslant \alpha\text{-cl}(\mu)$,
(iii) $\text{cl}^{*\alpha}(\mu) = \mu \vee \alpha\text{-cl}(\mu^{*\alpha}) \leqslant \alpha\text{-cl}(\mu)$,
(iv) $\mu^{*\alpha} \leqslant \mu^{*} \leqslant \text{cl}(\mu)$.

3. $*^\alpha$-Fuzzy topology

The concept of fuzzy ideal and the fuzzy α-local function has been presented in the last article. So, this one is devoted to construct the fuzzy topology concerning with the previous notions. In what follows, we return to the statements of Theorem 2.1 and Corollary 2.1 which show that the fuzzy Kuratwski α-closure operator $\text{cl}^{*\alpha}(\mu) = \mu \vee \mu^{*\alpha}$ is verified, and it is not difficult to define here the $*^\alpha$-fuzzy topology in terms of the previous α-closure operator. However, a simple basis for the fuzzy α-open sets of $\tau^{*\alpha}$ can be described in the following result.

Lemma 3.1. *For a fts (X,τ) with fuzzy ideal L, the class $\beta(L,\tau^\alpha) = \{\mu - \lambda : \mu \in \tau^\alpha,\ \lambda \in L\}$ is the base for the fuzzy topology $\tau^{*\alpha}$.*

Proof. Since $O_x \in L$, then $\tau^\alpha \leqslant \beta$ from which it follows that $X = \vee\beta$. Also, for every $\beta_1,\ \beta_2 \in \beta$, we have, $\beta_1 = \mu_1 - \lambda_1$, $\beta_2 = \mu_2 - \lambda_2$, where $\mu_1,\ \mu_2 \in \tau^\alpha$, $\lambda_1,\ \lambda_2 \in L$. Then

$$\begin{aligned}\beta_1 \wedge \beta_2 &= (\mu_1 - \lambda_1) \wedge (\mu_1 - \lambda_2) = (\mu_1 \wedge \lambda_1^{\mathrm{c}}) \wedge (\mu_2 \wedge \lambda_2^{\mathrm{c}}) \\ &= (\mu_1 - \mu_2) \wedge (\lambda_1 - \lambda_2)^{\mathrm{c}} \\ &= (\mu_1 - \mu_2) - (\lambda_1 \vee \lambda_2) \in \beta.\end{aligned}$$

Therefore, β is a base for $\tau^{*\alpha}$. □

Proposition 3.1. *For any fts (X,τ) with fuzzy ideal L it is clear that $\tau^{*\alpha}(L)$ is finer than both τ^α and τ.*

Example 3.1. For a fts (X,τ) with fuzzy ideal L and μ fuzzy subset of X we have:

(i) If $L = \{o_x\}$, then $\mu^{*\alpha} = \alpha\text{-cl}(\mu)$, $\text{cl}^{*\alpha}(\mu) = \alpha\text{-cl}(\mu)$ and hence in this case $\tau^\alpha = \tau^{*\alpha}$.
(ii) If $L = \{I^x\}$, then $\mu^{*\alpha} = o_x$, $\text{cl}^{*\alpha}(\mu) = \mu$ and hence in this case $\tau^{*\alpha}$ is the discrete fuzzy topology (D).

Remark 3.1. For a fts (X,τ) with fuzzy ideal L, since $\{o_x\} \leqslant L \leqslant I^x$, then by Theorem 2.1, we have $\tau^\alpha \leqslant \tau^{*\alpha} \leqslant D$.

Theorem 3.1. *If L_1 and L_2 are two fuzzy ideals on (X,τ) such that $L_1 \leqslant L_2$, then*

(i) $\mu^{*\alpha}(L_1) \geqslant \mu^{*\alpha}(L_2)$ *for every* $\mu \in I^x$.
(ii) $\tau^{*\alpha}(L_1) \leqslant \tau^{*\alpha}(L_2)$.

Proof. Obvious. □

Example 3.2. Let T be the fuzzy indiscrete topology on X. So l_x is the only α-q-nbd of every fuzzy point x_ε. This implies, for each $\lambda \in L$, $\mu(r) > \lambda(r)$ for at least one $r \in X$. So $\mu \notin L$. Therefore, $\mu^{*\alpha} = l_x$ if $\mu \notin L$ and $\mu^{*\alpha} = o_x$ if $\mu \in L$. So, we have $\text{cl}^{*\alpha}(\mu) = l_x$ if $\mu \notin L$ and $\text{cl}^{*\alpha}(\mu) = \mu$ if $\mu \in L$ for any fuzzy set μ of X. Hence $\tau^{*\alpha} = \{\mu : \mu^c \in L\}$.

Let $\tau^\alpha \vee T^{*\alpha}(L)$ be the largest fuzzy topology of τ^α and $T^{*\alpha}(L)$, i.e., the smallest fuzzy topology generated by $\tau^\alpha \cup T^{*\alpha}(L)$. Then we have the following theorem.

Theorem 3.2. $\tau^{*\alpha}(L) = \tau^a \vee T^{*\alpha}(L)$.

Proof. Follows from the fact that β forms a basis for $\tau^{*\alpha}$. □

Theorem 3.3. *Let (X, τ) be a fts with L_1 and L_2 are two fuzzy ideals on X and μ be a fuzzy subset of X, then*
(i) $\mu^{*\alpha}(L_1 \wedge L_2) = \mu^{*\alpha}(L_1) \vee \mu^{*\alpha}(L_2)$,
(ii) $\mu^{*\alpha}(L_1 \vee L_2, \tau) = \mu^{*\alpha}(L_1, \tau^{*\alpha}(L_2)) \wedge \mu^{*\alpha}(L_2, \tau^{*\alpha}(L_1))$.

Proof. (i) Let $x_\varepsilon \notin \mu^{*\alpha}\ (L_1) \cup \mu^{*\alpha}(L_2)$. Then $x_\varepsilon \notin$ both $\mu^{*\alpha}(L_1)$ and $\mu^{*\alpha}(L_2)$. Now $x_\varepsilon \notin \mu^{*\alpha}(L_1)$ implies there is at least one α-q-nbd η_1 of x_ε (in τ^α) such that for every $r \in X$, $\eta_1(r) + \mu(r) - 1 > \lambda_1(r)$ for some $\lambda_1 \in L$. Again, $x_\varepsilon \notin \mu^{*\alpha}(L_2)$ implies that there is at least one α-q-nbd η_2 of x_ε (in τ^α) such that for every $r \in X$, $\eta_2(r) + \mu(r) - 1 \leqslant \lambda_2(r)$ for some $\lambda_2 \in L$. Therefore, we have $(\eta_1 \cap \eta_2)(r) + \mu(r) - 1 \leqslant \lambda_1 \wedge \lambda_2(r)$ for every $r \in X$. Since $\eta_1 \cap \eta_2$ is also a α-q-nbd of x_ε (in τ^α) and $\lambda_1 \cap \lambda_2 \in L_1 \cap L_2$, hence $x_\varepsilon \notin \mu^*(L_1 \cap L_2)$, so that $\mu^{*\alpha}(L_1 \cap L_2) \leqslant \mu^{*\alpha}(L_1) \cup \mu^{*\alpha}(L_2)$. Also $L_1 \cap L_2$ is included in both L_1 and L_2, so by Theorem 3.2, reverse inclusion is obvious, which completes the proof of (i).

(ii) Since $x_\varepsilon \in \mu^{*\alpha}(L_1 \vee L_2, \tau^\alpha)$ implies there is at least one α-q-nbd η of x_ε (in τ^α) such that, for every $r \in X$, $\eta(r) + \mu(r) - 1 \leqslant \lambda'(r)$ for some $\lambda' \in L_1 \vee L_2$. Therefore, by heredity of fuzzy ideals and considering the structure of fuzzy α-open sets in generated fuzzy topology, we can find $\eta_1,\ \eta_2$ the α-q-nbds of x_ε in $\tau^{*\alpha}(L_1)$ or $\tau^{*\alpha}(L_2)$, respectively, such that for every $r \in X$, $\eta_1(r) + \mu(r) - 1 \leqslant \lambda_1(r)$ or $\eta_2(r) + \mu(r)\ - 1 \leqslant \lambda_2(r)$ for some $\lambda_1 \in L_1$ or $\lambda_2 \in L_2$. This implies $x_\varepsilon \notin \mu^*\ (L_2, \tau^{*\alpha}(L_1))$ or $x_\varepsilon \notin \mu^{*\alpha}\ (L_2, \tau^{*\alpha}(L_1))$. Thus, we have $\mu^{*\alpha}(L_1, \tau^{*\alpha}(L_2)) \cap \mu^{*\alpha}(L_2, \tau^{*\alpha}(L_1)) \subseteq \mu^*(L_1 \vee L_2, \tau^\alpha)$. Conversely, let $x_\varepsilon \notin \mu^{*\alpha}(L_1, \tau^{*\alpha}(L_2))$. This implies there is at least one α-q-nbd η in $\tau^{*\alpha}(L_2)$ of x_ε such that for every $r \in X$, $\eta(r) + \mu(r) - 1 \leqslant \lambda_1(r)$ for some $\lambda_1 \in L$. Since η is $\tau^{*\alpha}(L_2)$ α-q-nbd of x_ε, by heredity of fuzzy ideals we have a α-q-nbd v of x_ε (in τ^α) such that for every $r \in X$, $v(r) + \mu(r) - 1 \leqslant \lambda_1 \cup \lambda_2(r)$ for some $\lambda_1 \in L_1$, $\lambda_2 \in L_2$, i.e., $x_\varepsilon \notin \mu^{*\alpha}(L_1 \vee L_2, \tau^\alpha)$. Thus, $\mu^{*\alpha}(L_1 \vee L_2,\ \tau^\alpha) \subseteq \mu^{*\alpha}(L_1,\ \tau^{*\alpha}(L_2))$. Similarly, $\mu^{*\alpha}(L_1 \vee L_2,\ \tau^\alpha) \subseteq (L_2,\ \tau^*(L_1))$ and this completes the proof. □

Remark 3.2. By taking $L_1 = L_2$ in the above theorem, the following corollary answers the question about the relationship between $\tau^{*\alpha}$ and $(\tau^{*\alpha})^{*\alpha}$.

Corollary 3.1. *Let (X, τ) be a fts with fuzzy ideal L. Then*:
(i) $\mu^{*\alpha}(L, \tau^*) = \mu^{*\alpha}(L, \tau^{*\alpha})$,
(ii) $\tau^{*\alpha} = (\tau^{*\alpha})^{*\alpha}$.

Proof. (i) Obvious.

(ii) We observe that $\tau^{*\alpha}(L) = \tau^\alpha$ iff every $\lambda \in L$ is fuzzy α-closed in τ^α since every member of L is fuzzy α-closed in $\tau^{*\alpha}$. Hence it follows that $(\tau^{*\alpha})^{*\alpha} = \tau^{*\alpha}$. □

Definition 3.1. A fuzzy subset μ of a fts (X, τ) with fuzzy ideal L is said to be $\tau^{*\alpha}$-closed iff $\mu^{*\alpha} \leqslant \mu$. Equivalently, μ is fuzzy $\tau^{*\alpha}$-closed iff $\text{cl}^{*\alpha}(\mu) = \mu$.

Corollary 3.2. *For a fuzzy subset μ of a fts (X, τ) with fuzzy ideal L the following statements are equivalent*:
(i) $\mu \in \tau^{*\alpha}$,
(ii) $(l_x - \mu)$ is $\tau^{*\alpha}$-*closed*,
(iii) $(l_x - \mu)^{*\alpha} \leqslant (\mu^c)$,
(iv) $\mu \leqslant l_x - (\mu^c)^{*\alpha}$.

Theorem 3.4. *Let (X,τ) be a fts with fuzzy ideal L. Then*:

$$\tau^{*\alpha}(L) = \{\mu \in I^x : \mathrm{cl}^{*\alpha}(\mu^c) = \mu^c\}$$

Proof. This follows from Definition 2.1 and the fact $\mathrm{cl}^{*\alpha}(\mu) = \mu \vee \mu^{*\alpha}$ for every $\mu \in I^x$. □

Corollary 3.3. *Given a fts* (X,τ) with fuzzy ideal L, if (X,τ) *is fuzzy disconnected, then* $(X,\tau^{*\alpha})$ *is fuzzy disconnected.*

4. α-Compatibility of fuzzy ideals with fuzzy topology

In this section, we define and study two types of compatibility of τ with L namely fuzzy α-compatibility and fuzzy weak α-compatibility.

Definition 4.1. For a fts (X,τ) with fuzzy ideal L, τ is said to be fuzzy α-compatibile with L, denoted by $\tau \underset{\sim}{\alpha} L$, if for every fuzzy set μ of X, if for all fuzzy point $x_\varepsilon \in \mu$, there exists an α-q-nbd η of x_ε (in τ^α) such that $\eta(r) + \mu(r) - 1 \leqslant \lambda(r)$ holds for every $r \in X$ and for some $\lambda \in L$, then $\mu \in L$.

Definition 4.2 ([4]). Let $\{\mu_j : j \in J\}$ be indexed family of fuzzy sets of X such that μ_j q μ for each $j \in J$, where μ is a fuzzy set of X. Then $\{\mu_j : j \in J\}$ is said to be a quasi-cover of μ iff $\mu(r) + \bigcup_{j\in J} \mu_j(r) \geqslant 1$ for every $r \in X$.

Further, if each μ_j is fuzzy α-open set, then this quasi cover will be called a fuzzy quasi α-open cover of the fuzzy set μ of X. Therefore, in either case $\mu^c \subseteq \bigcup_{j\in J} \mu_j$.

The significance of fuzzy α-compatibility is shown by the following theorem.

Theorem 4.1. *Let (X,τ) be a fts with fuzzy ideal L, if $L \underset{\sim}{\alpha} \tau$. Then the base $\beta(L,\tau^\alpha)$ for $\tau^{*\alpha}(L)$ is a fuzzy topology and hence $\beta(L,\tau^*) = \tau^{*\alpha}$ and all fuzzy α-open sets in $\tau^{*\alpha}(L)$ are of simple form, i.e., $\tau^{*\alpha} = \{\mu - \lambda : \mu \in \tau^\alpha, \lambda \in L\}$.*

Proof. Follows immediately from the definition. □

Theorem 4.2. *For a fts (X,τ) with fuzzy ideal L the following are equivalent*:

(i) $\tau \underset{\sim}{\alpha} L$.

(ii) *If for every fuzzy set μ of X has a fuzzy quasi α-open cover $\{\mu_j : j \in J\}$ such that for each j, $\mu(r) + \mu_j(r) - 1 \leqslant \lambda(r)$ for some $\lambda \in L$ and for every $r \in X$, then $\mu \in L$.*

(iii) *For every fuzzy set μ of X, $\mu \cap \mu^{*\alpha} = o_x$ implies $\mu \in L$.*

(iv) *For every fuzzy set μ of X, $\mu \in L$, where $\mu = \bigcup x_\varepsilon$ such that $x_\varepsilon \in \mu$ but $x_\varepsilon \notin \mu^{*\alpha}$.*

(v) *For every $\tau^{*\alpha}$-closed fuzzy set μ, $\mu \in L$.*

(vi) *For every fuzzy set μ of X, if μ contains no non-empty fuzzy subset ν with $\nu \subseteq \nu^{*\alpha}$, then $\mu \in L$.*

Proof. (i) ⇒(ii): Let $\{\mu_j : j \in J\}$ be a fuzzy quasi α-open cover of fuzzy set μ of X such that for each $j \in J$, $\mu(r) + \mu_j(r) - 1 \leqslant \lambda(r)$ for some $\lambda \in L$ and for every $r \in X$. Therefore, as $\{\mu_j : j \in J\}$ is a fuzzy quasi α-open cover of μ for each $x_\varepsilon \in \mu$, there exists at least one $\mu_{jo}(r)$ such that x_ε q μ_{jo} and for every $r \in X$, $\mu(r) + \mu_{jo}(r) - 1 \leqslant \lambda(r)$ for some $\lambda \in L$. Obviously μ_{jo} is a α-q-nbd of $x_\in$ (in τ^α). Therefore, as $\tau \underset{\sim}{\alpha} L$, $\mu \in L$.

(ii)⇒(i): Clear from the fact that a collection of fuzzy α-open set which contains at least one α-open q-nbd of each fuzzy point of μ constitutes a fuzzy quasi α-open cover of μ.

(ii)⇒(iii): Let $\mu \cap \mu^* = o_x$, i.e., min $\{\mu(r),\ \mu^*(r)\} = o_x$ for every $r \in X$. So, a fuzzy point $x_\varepsilon \in \mu$ implies $x_\varepsilon \notin \mu^{*\alpha}$. This means there is α-$q$-nbd η of x_ε such that for every $r \in X$, $\eta(r) + \mu(r) - 1 \leqslant \lambda(r)$ for some $\lambda \in L$. If $x_\varepsilon \in \mu$ since η is an α-q-nbd of x_ε, there is a fuzzy α-open set ν (in τ^α) such that $x_\in q\nu \subseteq \eta$ and so the collection of such ν's for each $x_\varepsilon \in \mu$ constitutes a fuzzy quasi-α-open cover of μ, and therefore, by condition (ii), $\mu \in L$.

(iii)⇒(i): Let for every fuzzy point $x_\varepsilon \in \mu$, there is a α-q-nbd η of x_ε (in τ^α) such that for every $r \in X$, $\eta(r) + \mu(r) - 1 \leqslant \lambda(r)$ for some $\lambda \in L$. That means $x_\varepsilon \notin \mu^{*\alpha}$. Now there are two cases: either $\mu^{*\alpha}(x) = o_x$ or, $\mu^{*\alpha}(x) \neq o_x$ but $\varepsilon > \mu^{*\alpha}(x) \neq o_x$. Let, if possible, $x_\varepsilon \in \mu$ be such that $\varepsilon > \mu^{*\alpha}(x) \neq o_x$. Let $\varepsilon' = \mu^{*\alpha}(x)$. Then the fuzzy point $x_\varepsilon \in \mu^{*\alpha}$ and also $x_\varepsilon \in \mu$. This implies for each α-q-nbd v of x_ε and for each $\lambda \in L$, there is at least one $r \in X$ such that $v(r) + \mu(r) - 1 > \lambda(r)$. Since $x_\varepsilon \in \mu$, this contradicts the assumption for every fuzzy point of μ. So $\mu^{*\alpha}(x) = o_x$. That means, $x_\varepsilon \in \mu$ implies $x_\varepsilon \notin \mu^{*\alpha}$. Now this is true for every fuzzy set μ of X. So, for every fuzzy set μ of X, $\mu \wedge \mu^{*\alpha} = o_x$. Hence, by condition (iii), we have $\mu \in L$, which implies $\tau\alpha L$.

(iii)⇒(iv): Let the fuzzy point $x_\varepsilon \in \tilde{\mu}$. This means $x_\varepsilon \in \mu$ but $x_\varepsilon \notin \mu^{*\alpha}$. So, there is a α-q-nbd η of x_ε such that for every $r \in X$, $\eta(r) + \mu(r) - 1 \leqslant \lambda(r)$ for some $\lambda \in L$. Since $\tilde{\mu} \subset \mu$, so for every $r \in X$, $\eta(r) + \tilde{\mu}(r) - 1 \geqslant \lambda(r)$ for some $\lambda \in L$. Therefore, $x_\lambda \notin \tilde{\mu}^{*\alpha}$ so that either $\tilde{\mu}^{*\alpha}(x) = o_x$ or $\tilde{\mu}^{*\alpha}(x) \neq o_x$ but $\varepsilon > \tilde{\mu}^{*\alpha}(x)$. Let $x_{\varepsilon 1}$ be a fuzzy point such that $\varepsilon_1 \leqslant \tilde{\mu}^{*\alpha}(x) < \varepsilon$, i.e., $x_{\varepsilon 1} \in \tilde{\mu}^{*\alpha}$. So for each α-q-nbd for x_ε and for $\lambda \in L$, there is at least one $r \in X$ such that $v(r) + \tilde{\mu}(r) - 1 > \lambda(r)$. Since $\tilde{\mu} \subseteq \mu$ for each α-q-nbd v of $x_{\varepsilon 1}$ and for each $\lambda \in L$, there is at least one $r \in X$ such that $v(r) + \mu(r) - 1 > \lambda(r)$. This implies $x_{\varepsilon_1} \in \mu^{*\alpha}$. But as $\varepsilon_1 < \varepsilon$, $x_\varepsilon \in \tilde{\mu}$ implies $x_{\varepsilon_1} \in \tilde{\mu}$, and therefore $x_{\in 1} \in \mu^{*\alpha}$. This is a contradiction. Hence $\tilde{\mu}^{*\alpha}(x) = o_x$, so that $x_\varepsilon \in \tilde{\mu}$ implies $x_\varepsilon \notin \tilde{\mu}^{*\alpha}$ with $\tilde{\mu}^{*\alpha}(x) = o_x$. Thus, we have $\tilde{\mu} \cap \tilde{\mu}^{*\alpha} = o_x$ for every fuzzy set μ of X. Hence, by condition (iii), $\tilde{\mu} \in L$.

(iv)⇒(v): Straightforward.

(iv)⇒(vi): Let μ be any fuzzy set of X that contains no non-empty (i.e., not o_x) fuzzy subset v with $v \subseteq v^{*\alpha}$. Clearly, for every fuzzy set μ of X, $\mu = \tilde{\mu} \cup (\mu \cap \mu^{*\alpha})$. Therefore, $\mu^{*\alpha} = (\tilde{\mu} \cup (\mu \cap \mu^{*\alpha}))^{*\alpha} = \tilde{\mu}^{*\alpha} \cup (\mu \cap \mu^{*\alpha})^{*\alpha}$ (by Theorem 2.1 (vi)). Now by condition (iv) $\tilde{\mu} \in L$ so that $\tilde{\mu}^{*\alpha} = o_x$. Hence $(\mu \cap \mu^{*\alpha}) = \mu^{*\alpha}$. But $\mu \cap \mu^{*\alpha} \subseteq \mu^{*\alpha}$ so that $\mu \cap \mu^{*\alpha} \subseteq (\mu \cap \mu^{*\alpha})$. This contradicts the hypothesis about fuzzy set μ of X that, it contains no non-empty fuzzy subset v with $v \subseteq v^{*\alpha}$. Therefore, $\mu \cap \mu^{*\alpha} = o_x$ so that $\mu = \tilde{\mu}$ and hence by (iv), $\mu \in L$.

(vi)⇒(iv): Since, for every fuzzy set μ of X, $\tilde{\mu} \cup \tilde{\mu}^* = o_x$, by (vi), as $\tilde{\mu}$ contains no non-empty fuzzy subset v with $v \subseteq v^{*\alpha}$, $\tilde{\mu} \in L$.

(v)⇒(i): Let μ be any fuzzy set of X. Let for every fuzzy point $x_\in \in \mu$, there is an α-q-nbd η of x_ε(in τ^α) such that for every $r \in X$, $\eta(r) + \mu(r) - 1 \leqslant \lambda(r)$, for some $\lambda \in L$. This implies $x_\varepsilon \notin \mu^{*\alpha}$. Let $v = \mu \cup \mu^{*\alpha}$. Then $v^{*\alpha} = (\mu \cup \mu^{*\alpha})^{*\alpha} = \mu^{*\alpha} \cup (\mu^{*\alpha})^{*\alpha} = \mu^{*\alpha}$. So, $\mathrm{cl}^{*\alpha}(v) = v \cup v^{*\alpha} = v$. That means v is fuzzy $\tau^{*\alpha}$-closed set. Therefore, by (v), $\mu \in L$. Again, any fuzzy point $r_t \in \tilde{v}$ implies $r_t \in v$ but $r_t \notin v^{*\alpha} = \mu^{*\alpha}$. So, as $v = \mu \cup \mu^{*\alpha}$, $r_t \in \mu$. Now, by hypothesis about μ, we have for every $x_\varepsilon \notin \mu^{*\alpha}$. So, $\tilde{v} = \mu$. Hence $\mu \in L$, i.e., $\tau\underset{\sim}{\alpha}L$. □

Theorem 4.3. *Let (X, τ) be a fts with fuzzy ideal L. Then the following are equivalent and implied by $\tau\underset{\sim}{\alpha}L$.*
(i) *For every fuzzy set μ of X, $\mu \cap \mu^{*\alpha} = o_x$ implies $\mu^{*\alpha} = o_x$.*
(ii) *For every fuzzy set μ of X, $\tilde{\mu}^* = o_x$ ($\tilde{\mu}$ is defined as in Theorem 4.2 (iv)).*
(iii)*For every fuzzy set μ of X, $(\mu \cap \mu^{*\alpha}) = \mu^{*\alpha}$.*

Proof. Clear from Theorem 4.2. □

Theorem 4.4. *Let (X, τ) be a fts with fuzzy ideal L. Let $\tau\underset{\sim}{\alpha}L$. Then a fuzzy set of X is closed with respect to $\tau^{*\alpha}$ iff it is the union of a fuzzy set which is closed with respect to τ^α and a fuzzy set in L.*

Proof. Let μ be a fuzzy set of X such that it is fuzzy $\tau^{*\alpha}$-closed. That means $\mu^{*\alpha} \subseteq \mu$ and we have $\mu = \tilde{\mu} \cup \mu^{*\alpha}$. Since τ is α-compatible with L, therefore, $\mu \in L$. Also $\mu^{*\alpha}$ is always τ^α-closed by statement (iii) of Theorem 2.1. Conversely, let μ be any fuzzy set of X such that $\mu = v \cup \lambda$, where $\mathrm{cl}(v) = v \subseteq \mu$. This means $\mu^{*\alpha} \subseteq v \subseteq \mu$. So, we have $\mathrm{cl}^{*\alpha}(\mu) = \mu \cup \mu^{*\alpha} = \mu$ and this implies μ is fuzzy $\tau^{*\alpha}$-closed set. □

An important consequence of Theorem 4.4 is the following Corollary.

Corollary 4.1. *The fuzzy topology τ is α-compatible with fuzzy ideal L on X implies $\beta(L, \tau)$, a basis for $\tau^{*\alpha}$, is itself a fuzzy topology and also $\beta = \tau^{*\alpha}$.*

Proof. Clear. □

Definition 4.3. Given a fts (X, τ) with fuzzy ideal L, L is said to be fuzzy weakly α-compatible with respect to τ, denoted by $L\underset{\sim}{\alpha}u$ iff $\mu^{*\alpha}(L) = o_x$ implies $\mu \in L$.

Remark 4.1. One may notice that fuzzy α-compatibility is contained in fuzzy weak α-compatibility.

The following theorem simply states the contrapositive of the condition and defines weak α-compatibility.

Theorem 4.5. *Given a fts (X, τ) with fuzzy ideal $L, L\underset{\sim}{\alpha}\tau$ iff $\mu \notin L$ implies $\mu^{*\alpha} \neq o_x$.*

Theorem 4.6. *Let L_1 and L_2 be fuzzy ideals on a fts (X, τ), with $L_1 \sim \tau$ and $L_2\underset{\sim}{\alpha}\tau$. If $\mu^{*\alpha}(L_1) = \mu^{*\alpha}(L_2)$, then $L_1 = L_2$ for every $\mu \in I^x$.*

Proof. If $L_1 \neq L_2$, then either there exists an $\lambda \in L_1 - L_2$ or $\eta \in L_2 - L_1$. First assume there exists an $\lambda \in L_1 - L_2$. Then $\lambda \in L_1$ implies $\lambda^{*\alpha}(L) = o_x$, but $\lambda \in L_2$ implies $\lambda^{*\alpha}(L_2) = o_x$ (Theorem 4.3). Similarly, if $\eta \in L_2 - L_1$, then $\eta \in L_2$ implies $\eta^{*\alpha}(L_2) = o_x$, but $\eta \notin L_1$ by Theorem 4.3. We have $\eta^{*\alpha}(L_1) \neq o_x$ which is a contradiction. Therefore, $L_1 = L_2$. $\square$

In conclusion, we may stress once more the importance of fuzzy topology as a nontrivial extension of fuzzy sets and fuzzy logic [8] and the possible application in quantum physics [6,7].

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