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V-E-INVEXITY IN E-DIFFERENTIABLE MULTIOBJECTIVE PROGRAMMING

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ABSTRACT. In this paper, a new concept of generalized convexity is introduced for not necessarily differentiable vector optimization problems with E-differentiable functions. Namely, for an E-differentiable vector-valued function, the concept of V-E-invexity is defined as a generalization of the E-differentiable E-invexity notion and the concept of V-invexity. Further, the sufficiency of the so-called E-Karush-Kuhn-Tucker optimality conditions are established for the considered E-differentiable vector optimization problems with both inequality and equality constraints under V-E-invexity hypotheses. Furthermore, the so-called vector E-dual problem in the sense of Mond-Weir is defined for the considered E-differentiable multiobjective programming problem and several E-duality theorems are derived also under appropriate V-E-invexity assumptions.

1. **Introduction.** In recent years, several authors have been defined various classes of differentiable and nondifferentiable generalized convex functions in optimization theory. Optimality conditions and duality theorems for differentiable and nondifferentiable multiobjective programming problems have been studied extensively in the literature (see, for example, [2–8], [10–19], [22–31], [33], and others). One of such important generalizations of the convexity notion is the concept of invexity introduced by Hanson [17] for scalar optimization problems. Jeyakumar and Mond [18] introduced a new class of nonconvex differentiable vector-valued functions, namely V-invex functions, in order to resolve the difficulty of demanding the same function η for objective and constraint functions in extremum problems dealing with the concept of invexity introduced by Hanson [17] for scalar optimization problems. They established sufficient optimality criteria and duality results in the multiobjective static case for weak minima solutions under V-invexity. Kuk et al. [20] defined the concept of V-ρ-invexity for vector-valued functions, which is a generalization of the V-invex function [18]. Antczak [10] introduced the concept of V-r-invexity for differentiable multiobjective programming problems, which is a generalization of the concept of differentiable r-invex functions [9] and V-invex functions [18]. In [12], Antczak introduced a new class of nondifferentiable generalized invex functions called V-r-invex functions. He established sufficient optimality conditions

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By assumption, the objective function f_i, $i \in I$, is α_i-E-invex with respect to η at $\overline{y}$ on $\Omega_E \cup Y_E$. Then, by Definition 2.3, the inequality

$$f_i\left(E\left(z\right)\right) - f_i\left(E\left(\overline{y}\right)\right) \geqq \alpha_i\left(E\left(z\right), E\left(\overline{y}\right)\right) \nabla (f_i \circ E)\left(\overline{y}\right) \eta\left(E\left(z\right), E\left(\overline{y}\right)\right), \quad i \in I \quad (41)$$

holds for $z \in \Omega_E \cup Y_E$. Thus, it is also fulfilled for $z = \tilde{x} \in \Omega_E$. Hence, (41) yield

$$f_i\left(E\left(\tilde{x}\right)\right) - f_i\left(E\left(\overline{y}\right)\right) \geqq \alpha_i\left(E\left(\tilde{x}\right), E\left(\overline{y}\right)\right) \nabla (f_i \circ E)\left(\overline{y}\right) \eta\left(E\left(\tilde{x}\right), E\left(\overline{y}\right)\right), \quad i \in I, \quad (42)$$

By the feasibility of $\left(\overline{y}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ in (MWD$_E$), it follows that

$$\overline{\lambda}_i f_i\left(E\left(\widetilde{x}\right)\right) - \overline{\lambda}_i f_i\left(E\left(\overline{y}\right)\right) \geqq \alpha_i\left(E\left(\widetilde{x}\right), E\left(\overline{y}\right)\right) \overline{\lambda}_i \nabla (f_i \circ E)\left(\overline{y}\right) \eta\left(E\left(\widetilde{x}\right), E\left(\overline{y}\right)\right), \quad i \in I. \quad (43)$$

Combining (40) and (43), we have

$$\alpha_i(E(\widetilde{x}), E(\overline{y}))\overline{\lambda}_i \nabla\left(f_i \circ E\right)\left(\overline{y}\right) \eta\left(E\left(\widetilde{x}\right), E\left(\overline{y}\right)\right) < 0, \ i \in I. \quad (44)$$

Since $\alpha_i(E(\widetilde{x}), E(\overline{y})) > 0$, $i \in I$, the above inequalities yield

$$\left[\sum_{i=1}^{p} \overline{\lambda}_i \nabla\left(f_i \circ E\right)\left(\overline{y}\right)\right] \eta\left(E\left(\widetilde{x}\right), E\left(\overline{y}\right)\right) < 0. \quad (45)$$

By assumption, the function g_j, $j \in J$, is β_i-E-invex with respect to η at $\overline{y}$ on $\Omega_E \cup Y_E$, the functions h_t, $t \in T^+(E(\overline{y}))$, $-h_t$, $t \in T^-(E(\overline{y}))$, are γ_t-E-invex at $\overline{y}$ on $\Omega_E \cup Y_E$. Then, by Lemma 4.1, the inequality (25) holds. After adding both sides of (45) and (25), we obtain that the inequality

$$\left[\sum_{i=1}^{p} \overline{\lambda}_i \nabla (f_i \circ E)\left(\overline{y}\right) + \sum_{j=1}^{m} \overline{\mu}_j \nabla (g_j \circ E)\left(\overline{y}\right) + \sum_{t=1}^{s} \overline{\xi}_t \nabla (h_t \circ E)\left(\overline{y}\right)\right] \eta\left(E\left(\widetilde{x}\right), E\left(\overline{y}\right)\right) < 0$$

holds, contradicting the first constraint of (MWD$_E$). Then, this means that $\overline{x}$ is a (weak) Pareto solution of (VP$_E$). Hence, by the strong duality theorem (Theorem 4.6), we get that $\left(\overline{y}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is a weakly efficient solution of a maximum type in (MWD$_E$). Hence, Mond-Weir restricted converse duality holds between (VP$_E$) and (MWD$_E$). □

Based on the above result, we are able to prove the following result.

Theorem 4.10. *(Mond-Weir restricted converse E-duality between (MOP) and (MWD$_E$)). Let $E\left(\overline{x}\right)$ and $\left(\overline{y}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ be a feasible solution of (MOP) and (MWD$_E$). respectively, such that*

$$f_i(E(\overline{x})) = f_i(E(\overline{y})), \qquad i = 1, 2, ..., p.$$

Further, we assume that all hypotheses of Theorem 4.9 are fulfilled. Then $E\left(\overline{x}\right)$ is a weak E-Pareto solution (an E-Pareto solution) of (MOP) and $\left(\overline{y}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is a weakly efficient solution (an efficient solution) of a maximum type in (MWD$_E$).

Proof. The proof of this theorem follows directly from Lemma 3.10 and Theorem 4.9. □

5. **Concluding remarks.** In this paper, a new class of nondifferentiable multiobjective programming problems with both inequality and equality constraint has been considered. Namely, for an E-differentiable vector-valued function, the concept of V-E-invexity has been defined as a generalization of the E-differentiable E-invexity notion and the concept of V-invexity. Sufficient E-optimality conditions and various Mond-Weir E-duality results have been proved for E-differentiable multiobjective programming problems with both inequality and equality constraint

under V-E-invexity hypotheses. These results have been illustrated in the paper by suitable examples.

However, some interesting topics for further research remain. It would be of interest to investigate whether it is possible to prove similar results under E-V-invexity hypotheses for other classes of E-differentiable vector optimization problems. We shall investigate these questions in subsequent papers.

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